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One problem related to the Smarandache function

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Abstract For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m:\ n \mid m!,\ n \in N\}$. The main purpose of this paper is using the elementary method to study the number of the solutions of the congruent equation $1^{S(n-1)} + 2^{S(n-1)} + \cdots + (n-1)^{S(n-1)} + 1 \equiv 0 \pmod{n}$, and give its all prime number solutions.
Keywords F. Smarandache function, divisibility, primitive root.

§1. Introduction and result

For any positive integer n, the famous F. Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m:\ n \mid m!,\ n \in N\}$. For example, the first few values of $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $S(11) = 11$, $S(12) = 4$, $\cdots$. About the elementary properties of $S(n)$, many authors had studied it, and obtained a series interesting results, see references [1], [2], [3] and [4]. For example, Xu Zhefeng [2] studied the value distribution problem of $S(n)$, and proved the following conclusion:

Let $P(n)$ denotes the largest prime factor of n, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} (S(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Lu Yaming [3] studied the solutions of an equation involving the F. Smarandache function $S(n)$, and proved that for any positive integer $k \geq 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite group positive integer solutions $(m_1,\ m_2,\ \cdots,\ m_k)$.

Rongji Chen [4] proved that for any fixed $r \in N$ with $r \geq 3$, the positive integer n is a solution of the equation

$$S(n)^r + S(n)^{r-1} + \cdots + S(n) = n$$

if and only if

$$n = p(p^{r-1} + p^{r-2} + \cdots + 1),$$

where p is an odd prime such that

$$p^{r-1} + p^{r-2} + \cdots + 1 \mid (p-1)!.$$

On the other hand, in reference [5], C. Dumitrescu and V. Seleacu asked us to study the solvability of the congruent equation

$$1^{S(n-1)} + 2^{S(n-1)} + \cdots + (n-1)^{S(n-1)} + 1 \equiv 0 \mod n. \tag{1}$$

About this problem, it seems that none had studied it yet, at least we have not seen any related papers before. In this paper, we use the elementary method and the properties of the primitive roots to study the solvability of the congruent equation (1), and obtain its all prime solutions. That is, we shall prove the following conclusion:

Theorem. Let n be a prime, then n satisfy the congruent equation

$$1^{S(n-1)} + 2^{S(n-1)} + \cdots + (n-1)^{S(n-1)} + 1 \equiv 0 \mod n$$

if and only if $n = 2,\ 3$ and 5.

It is clear that our Theorem obtained all prime solutions of the congruent equation (1). About the general solutions of the congruent equation (1) is still an unsolved problem.

2. Proof of the theorem

In this section, we shall complete the proof of our Theorem directly. We only discuss the prime solutions of (1).

(1) For $n = 2$, since $2 \mid 1^{S(1)} + 1 = 2$, so $n = 2$ is a prime solution of (1).

(2) For $n = 3$, since $3 \mid 1^{S(2)} + 2^{S(2)} + 1 = 6$, so $n = 3$ is a prime solution of (1).

(3) For $n = 5$, since $5 \mid 1^{S(4)} + 2^{S(4)} + 3^{S(4)} + 4^{S(4)} + 1 = 355$, so $n = 5$ is also a prime solution of (1).

(4) For prime $n = p \geq 7$, it is clear that p has at least a primitive root. Let g be a primitive root of p, that is to say, $\left(g^i - 1,\ p\right) = 1$ for all $1 \leq i \leq p-2$, the congruences

$$g^{p-1} \equiv 1 \mod p \ \text{ and } \ g^{m(p-1)} \equiv 1 \mod p \tag{2}$$

hold for any positive integer m.

Then from the properties of the primitive root mod p we know that $g^0,\ g^1,\ \cdots,\ g^{p-2}$ is a reduced residue class. So we have the congruent equation

$$\begin{aligned} &1^{S(n-1)} + 2^{S(n-1)} + \cdots + (n-1)^{S(n-1)} \\ &\equiv g^{0\cdot S(p-1)} + g^{1\cdot S(p-1)} + \cdots + g^{(p-2)\cdot S(p-1)} \\ &\equiv \frac{g^{(p-1)\cdot S(p-1)} - 1}{g^{S(p-1)} - 1} \mod p. \end{aligned} \tag{3}$$

It is clear that for any prime $p \geq 7$, we have $S(p-1) \leq p-2$, so $\left(g^{S(p-1)} - 1,\ p\right) = 1$. Therefore, from (2) and (3) we have

$$1^{S(n-1)} + 2^{S(n-1)} + \cdots + (n-1)^{S(n-1)} \equiv \frac{g^{(p-1)\cdot S(p-1)} - 1}{g^{S(p-1)} - 1} \equiv 0 \bmod p.$$

So from this congruence we may immediately get

$$1^{S(n-1)} + 2^{S(n-1)} + \cdots + (n-1)^{S(n-1)} + 1 \equiv 1 \bmod p.$$

Thus, if prime $p \geq 7$, then it is not a solution of (1). This completes the proof of Theorem.

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