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On the Pseudo-Smarandache-Squarefree function

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Abstract For any positive integer n, the Pseudo-Smarandache-Squarefree function $Zw(n)$ is defined as the smallest integer m such that m^n is divisible by n. That is, $Zw(n) = \min\{m : n|m^n,\ m \in N\ \}$, where N denotes the set of all positive integers. The main purpose of this paper is using the elementary methods to study a limit problem related to the Pseudo-Smarandache-Squarefree function $Zw(n)$, and give an interesting limit theorem.

Keywords Pseudo-Smarandache-Squarefree function, mean value, limit.

§1. Introduction and Results

For any positive integer n, the Pseudo-Smarandache-Squarefree function $Zw(n)$ is defined as the smallest integer m such that m^n is divisible by n. That is, $Zw(n) = \min\{m : n|m^n, m \in N\ \}$, where N denotes the set of all positive integers. For example, the first few values of $Zw(n)$ are $Zw(1) = 1$, $Zw(2) = 2$, $Zw(3) = 3$, $Zw(4) = 2$, $Zw(5) = 5$, $Zw(6) = 6$, $Zw(7) = 7$, $Zw(8) = 2$, $Zw(9) = 3$, $Zw(10) = 10$, $Zw(11) = 11$, $Zw(12) = 6$, $Zw(13) = 13$, $Zw(14) = 14$, $Zw(15) = 15$, $\cdots$. Obviously, the Pseudo-Smarandache-Squarefree function $Zw(n)$ has the following properties:

(1) If p be a prime, then $Zw(p) = p$.

(2) If m be a square-free number (m¿1, and for any prime p, if $p|m$, then $p^2 \dagger m$), then $Zw(m) = m$.

(3) If p be any prime and $k \geq 1$, then we have $Zw(p^k) = p$.

(4) $Zw(n) \leq n$.

(5) The function $Zw(n)$ is multiplicative. That is, if $(m,\ n) = 1$, then $Zw(mn) = Zw(m)Zw(n)$.

(6) The function $Zw(n)$ is not additive. That is, for some positive integers m and n, $Zw(m + n) \neq Zw(m) + Zw(n)$.

According to the above properties, the $Zw(n)$ function is very similar to the Möbius function:

$$Zw(n) = \begin{cases} n & \text{if } n \text{ is a square free number;} \\ 1 & \text{if and only if } n = 1; \\ \text{Product of distinct prime factors of } \ n & \text{if } n \text{ is not a square-free number.} \end{cases}$$

On the other hand, we can easily deduce that $\sum_{n=1}^{\infty}\frac{1}{Zw(n)}$ is divergent . In fact for any prime p, we have $Zw(p)=p$. So that $\sum_{n=1}^{\infty}\frac{1}{Zw(n)} > \lim_{x\longrightarrow\infty}\sum_{p\le x}\frac{1}{p}=+\infty$.

About the other elementary properties of $Zw(n)$, some authors also had studied it, and obtained some interesting results, see references [1], [2], [5], [7] and [8]. Simultaneously, F. Russo [1] proposed some new problems, two of them as follows:

Problem 1: Evaluate limit $\prod_{n=2}^{\infty}\frac{1}{Zw(n)}$.

Problem 2: Evaluate $\lim_{k\to\infty}\frac{Zw(k)}{\theta(k)}$, where $\theta(k)=\sum_{n\le k}\ln\left(Zw(n)\right)$.

The problem 1 had been solved by Maohua Le [2]. But for the problem 2, it seems that none had studied it yet, at least we have not seen such a paper before. The problem is interesting, because it can help us to obtain some deeply properties of the Pseudo-Smarandache-Squarefree function $Zw(n)$. The main purpose of this paper is using the elementary methods to study this problem, and give an interesting limit theorem for it. That is, we shall prove the following conclusion:

Theorem. For any positive integer $k>1$, let $Zw(n)$ and $\theta(k)$ are defined as the above, then we have the asymptotic formula

$$\frac{Zw(k)}{\theta(k)}=\frac{Zw(k)}{\sum\limits_{n\le k}\ln\left(Zw(n)\right)}=O\left(\frac{1}{\ln k}\right).$$

From this theorem we may immediately deduce the following:

Corollary. For any positive integer k, we have the limit

$$\lim_{k\to\infty}\frac{Zw(k)}{\theta(k)}=0.$$

It is clear that our corollary solved the problem 2.

§2. Proof of the theorem

To complete the proof of the theorem, we need the following an important Lemma.

Lemma. For any real number $x\ge 1$, we have the asymptotic formula

$$\sum_{n\le x}|\mu(n)|=\frac{6}{\pi^2}x+O\left(\sqrt{x}\right),$$

where $\mu(n)$ denotes the Möbius function.

Proof. For any real number $x>1$ and positive integer n, from the properties of the Möbius function $\mu(n)$ (See reference [3]):

$$|\mu(n)|=\sum_{d^2|n}\mu(d),$$

and note that

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}=\frac{1}{\zeta(2)}=\frac{6}{\pi^2}$$

we have

$$\begin{aligned}
\sum_{n\le x}|\mu(n)| &= \sum_{n\le x}\sum_{d^2|n}\mu(d)\\
&= \sum_{md^2\le x}\mu(d)\\
&= \sum_{d\le\sqrt{x}}\mu(d)\sum_{m\le\frac{x}{d^2}}1\\
&= \sum_{d\le\sqrt{x}}\mu(d)\left(\frac{x}{d^2}+O(1)\right)\\
&= x\left(\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}-\sum_{d>\sqrt{x}}\frac{\mu(d)}{d^2}\right)+O\left(\sum_{d\le\sqrt{x}}|\mu(d)|\right)\\
&= x\left(\frac{6}{\pi^2}+O\left(\frac{1}{\sqrt{x}}\right)\right)+O\left(\sqrt{x}\right)\\
&= \frac{6}{\pi^2}x+O\left(\sqrt{x}\right).
\end{aligned}$$

This proves our Lemma.

Now, we shall use this Lemma to complete the proof our theorem. In fact note that for any square-free number n, $Zw(n)=n$, we have

$$\begin{aligned}
\theta(k) &= \sum_{n\le k}\ln\left(Zw(n)\right)\\
&\ge \sum_{n\le k}|\mu(n)|\ln n\\
&\ge \sum_{\sqrt{k}\le n\le k}|\mu(n)|\ln(\sqrt{k})\\
&= \frac{1}{2}\ln k\sum_{\sqrt{k}\le n\le k}|\mu(n)|\\
&= \frac{1}{2}\ln k\left(\sum_{n\le k}|\mu(n)|-\sum_{n\le\sqrt{k}}|\mu(n)|\right). \qquad (1)
\end{aligned}$$

So from Lemma and (1) we have

$$\begin{aligned}
\theta(k) &\ge \frac{1}{2}\ln k\left(\sum_{n\le k}|\mu(n)|-\sum_{n\le\sqrt{k}}|\mu(n)|\right)\\
&\ge \frac{1}{2}\ln k\left(\frac{6}{\pi^2}k+O(\sqrt{k})\right)\\
&= \frac{3}{\pi^2}k\cdot\ln k+O(\sqrt{k}\cdot\ln k). \qquad (2)
\end{aligned}$$

Note that $Zw(n) \leq n$, from (2) we may immediately deduce that

$$0 < \frac{Zw(k)}{\theta(k)} \leq \frac{k}{\frac{3}{\pi^2} k \cdot \ln k + O(\sqrt{k} \cdot \ln k)} = O\left(\frac{1}{\ln k}\right)$$

or

$$\frac{Zw(k)}{\theta(k)} = O\left(\frac{1}{\ln k}\right).$$

This completes the proof of Theorem.

The corollary follows from our theorem as $\longrightarrow \infty$.

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