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On the Pseudo-Smarandache function

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Abstract The main purpose of this paper is using the elementary method to study the properties of the Pseudo-Smarandache function $Z(n)$, and proved the following two conclusions: The equation $Z(n) = Z(n+1)$ has no positive integer solutions; For any given positive integer M, there exists an integer s such that the absolute value of $Z(s) - Z(s+1)$ is greater than M.

Keywords Pseudo-Smarandache function, equation, positive integer solution.

§1. Introduction and results

For any positive integer n, the Pseudo-Smarandache function $Z(n)$ is defined as the smallest positive integer m such that $[1 + 2 + 3 + \cdots + m]$ is divisible by n. That is,

$$Z(n) = \min\left\{m : \ m \in N : \ n \mid \frac{m(m+1)}{2}\right\},$$

where N denotes the set of all positive integers. For example, the first few values of $Z(n)$ are: $Z(1) = 1,\ Z(2) = 3,\ Z(3) = 2,\ Z(4) = 7,\ Z(5) = 4,\ Z(6) = 3,\ Z(7) = 6,\ Z(8) = 15,\ Z(9) = 8, Z(10) = 4,\ Z(11) = 10,\ Z(12) = 8,\ Z(13) = 12,\ Z(14) = 7,\ Z(15) = 5, \cdots\cdots$.

In reference [1], Kenichiro Kashihara had studied the elementary properties of $Z(n)$, and proved some interesting conclusions. Some of them as follows:

For any prime $p \geq 3$, $Z(p) = p - 1$;

For any prime $p \geq 3$ and any $k \in N, Z(p^k) = p^k - 1$;

For any $k \in N, Z(2^k) = 2^{k+1} - 1$;

If n is not the form 2^k for some integer $k > 0$, then $Z(n) < n$.

On the other hand, Kenichiro Kashihara proposed some problems related to the Pseudo-Smarandache function $Z(n)$, two of them as following:

(A) Show that the equation $Z(n) = Z(n+1)$ has no solutions.

(B) Show that for any given positive number r, there exists an integer s such that the absolute value of $Z(s) - Z(s+1)$ is greater than r.

For these two problems, Kenichiro Kashihara commented that I am not able to solve them, but I guess they are true. I checked it for $1 \leq n \leq 60$.

In this paper, we using the elementary method to study these two problems, and solved them completely. That is, we shall prove the following:

Theorem 1. The equation $Z(n) = Z(n+1)$ has no positive integer solutions.

Theorem 2. For any given positive integer M, there exists a positive integer s such that

$$|Z(s) - Z(s+1)| > M.$$

§2. Proof of the theorems

In this section, we shall prove our theorems directly. First we prove Theorem 1. If there exists some positive integer n such that the equation $Z(n) = Z(n+1)$. Let $Z(n) = Z(n+1) = m$, then from the definition of $Z(n)$ we can deduce that

$$n \mid \frac{m(m+1)}{2}, \quad n+1 \mid \frac{m(m+1)}{2}.$$

Since $(n,\ n+1) = 1$, we also have

$$n(n+1) \mid \frac{m(m+1)}{2} \quad \text{and} \quad \frac{n(n+1)}{2} \mid \frac{m(m+1)}{2}.$$

Therefore,

$$n < m. \tag{1}$$

On the other hand, since one of n and $n+1$ is an odd number, if n is an odd number, then $Z(n) = m \leq n-1 < n$; If $n+1$ is an odd number, then $Z(n+1) = m \leq n$. In any cases, we have

$$m \leq n. \tag{2}$$

Combining (1) and (2) we have $n < m \leq n$, it is not possible. This proves Theorem 1.

Now we prove Theorem 2. For any positive integer M, we taking positive integer α such that $s = 2^\alpha > M+1$. This time we have

$$Z(s) = Z(2^\alpha) = 2^{\alpha+1} - 1.$$

Since $s+1$ is an odd number, so we have

$$Z(s+1) \leq s = 2^\alpha.$$

Therefore, we have

$$|Z(s) - Z(s+1)\ | \geq (2^{\alpha+1} - 1) - 2^\alpha = 2^\alpha - 1 > M + 1 - 1 = M.$$

So there exists a positive integer s such that the absolute value of $Z(s) - Z(s+1)$ is greater than M. This completes the proof of Theorem 2.

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