Pythagorean Neutrosophic Planar Graphs with an Application in Decision-Making

4 authors:

Chellamani .P
Sacred Heart College, Tirupattur
21 PUBLICATIONS   134 CITATIONS

SEE PROFILE

Ajay Devaraj
Sacred Heart College
47 PUBLICATIONS   230 CITATIONS

SEE PROFILE

Mohammed Al-shamiri
King Khalid University and Ibb University
70 PUBLICATIONS   169 CITATIONS

SEE PROFILE

Rashad Ismail
King Khalid University
30 PUBLICATIONS   120 CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:

Fuzzy labelling of graphs View project

Fuzzy Sets View project

Pythagorean Neutrosophic Planar Graphs with an Application in Decision-Making

P. Chellamani[1,2,*], D. Ajay[1], Mohammed M. Al-Shamiri[3,4] and Rashad Ismail[3,4]

[1]Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, 635601, Tamilnadu, India
[2]Department of Mathematics, St. Joseph's College of Engineering, OMR, Chennai, 600119, Tamilnadu, India
[3]Department of Mathematics, Faculty of Science and Arts, King Khalid University, Muhayl Assir, Saudi Arabia
[4]Department of Mathematics and Computer, Faculty of Science, Ibb University, Ibb, Yemen
*Corresponding Author: P. Chellamani. Email: joshmani238@gmail.com

Abstract: Graph theory has a significant impact and is crucial in the structure of many real-life situations. To simulate uncertainty and ambiguity, many extensions of graph theoretical notions were created. Planar graphs play a vital role in modelling which has the property of non-crossing edges. Although crossing edges benefit, they have some drawbacks, which paved the way for the introduction of planar graphs. The overall purpose of the study is to contribute to the conceptual development of the Pythagorean Neutrosophic graph. The basic methodology of our research is the incorporation of the analogous concepts of planar graphs in the Pythagorean Neutrosophic graphs. The significant finding of our research is the introduction of Pythagorean Neutrosophic Planar graphs, a conceptual blending of Pythagorean Neutrosophic and Planar graphs. The idea of Pythagorean Neutrosophic multigraphs and dual graphs are also introduced to deal with the ambiguous situations. This paper investigates the Pythagorean Neutrosophic planar values, which form the edges of the Pythagorean neutrosophic graphs. The concept of Pythagorean Neutrosophic dual graphs, isomorphism, co-weak and weak isomorphism have also been explored for Pythagorean Neutrosophic planar graphs. A decision-making algorithm was proposed with a numerical illustration by using the Pythagorean Neutrosophic fuzzy graph.

Keywords: Pythagorean neutrosophic planar graph; planarity value; isomorphism; dual graphs; multigraph

1 Introduction

Graphs are illustrative representations that express the relation between objects and their data. When the relationships are ambiguous, a graph can be implemented as a fuzzy graph model, which has the same structure as a crisp graph but works with ambiguous data. The fuzzy set theory for dealing with incomplete and vague information originated from the work of Zadeh [1]. Following the

fuzzy sets, Intuitionistic sets [2], in which the elements possess membership (μ) and non-membership (ρ) grades with the condition that $\mu + \rho \leq 1$. By considering the vagueness of information, adding some restrictions leads to the extension and development of the neutrosophic set by Smarandache [3], which assigns a truth, indeterminacy, and false membership grade to the elements, with the condition that the sum of the membership grades is within the range of 0 and 3. To expand this concept, Yager [4] proposed the concept of Pythagorean sets, which have an added relaxation in their condition as $\mu^2 + \rho^2 \leq 1$. The fusion of Pythagorean and Neutrosophic sets resulted in the development of the Pythagorean Neutrosophic set, which allows the element to have the membership (μ), indeterminacy (σ) and non-membership grade (γ) with the constraint that $\mu^2 + \rho^2 + \gamma^2 \leq 2$.

Kaufmann [5], based on fuzzy relation [6], developed the idea of Fuzzy Graphs (FGs). Later, Rosenfeld [7] defined the basic properties of fuzzy relations, which are generalized with a fuzzy set as a base set, and fuzzy analogs of graphic theoretical concepts like bridges and trees were established with their properties. Bhattacharya [8] introduced the notions of eccentricity, and center explored how a fuzzy group can be associated with a fuzzy graph. Fundamental operations on FGs and their properties were discussed by Mordeson et al. in [9].

Shannon et al. introduced intuitionistic fuzzy relations in [10], and intuitionistic fuzzy graphs (IFG) with their properties were investigated in [11]. The operations of IFGs were described by Parvathi et al. in [12]. In [13,14], Akram et al. established ideas such as strong IFGs and IF hypergraphs. Pythagorean Fuzzy Graphs (PFGs) and their applications were explored by Naz et al. in [15], and the energy of a Pythagorean Fuzzy Graph (PFG) was studied by Akram et al. in [16], followed by the assertion of some PFG operations by Akram et al. in [17]. Akram et al. [18] proposed certain graphs using the base Pythagorean and the abstraction of the fuzzy dual graph with the investigation of its properties in [19].

Yager proposed the concept of fuzzy multiset in [20], and fuzzy planar graphs were developed along with their properties in [21] and [22]. Alshehri et al. [23] established some exciting proofs of Intuitionistic fuzzy planar graphs, and the concept of bipolar fuzzy planar graphs was described in [24]. Strong neutrosophic graphs were introduced in [25], and single-valued neutrosophic graphs were instituted by Broumi et al. [26]. Akram et al. introduced neutrosophic graphs and neutrosophic soft graphs along with their applications [27]. Single-valued neutrosophic hypergraphs and intuitionistic neutrosophic soft graphs were studied in [28,29]. The recent development of planar graphs in this area can be seen in [30–34]. The concept of the Pythagorean Neutrosophic graphs [35] was developed using the Pythagorean Neutrosophic set [36] and further into other graph-theoretical concepts in [37–40].

In this study, the graph-theoretical results are applied in the Pythagorean Neutrosophic fuzzy environment. The concept of planar graphs is more captivating because of their complexity. In designing circuits, circuits or lines are arranged so they do not intersect to avoid circuit problems, and planar graphs can be used to tackle this problem. Though all developments in fuzzy graph theory have advantages, the Pythagorean Neutrosophic graphs have their advantage with more fuzzified inputs. This research article elaborates on the abstraction of Pythagorean Neutrosophic Multi Graphs (PNMGs), Pythagorean Neutrosophic Planar Graphs (PNPGs), and Pythagorean Neutrosophic Dual Graphs (PNDGs). The potential applications of these graphs can be used to assess and design a variety of real-world challenges. Planarity is a crucial feature that is investigated in this work. A significant addition to the literature is the development of the Pythagorean neutrosophic planar and multigraphs with their characterizations.

The following is the order in which this article is organized: Section 2 deals with introducing the Pythagorean Neutrosophic Multi Graphs (PNMGs) and investigating their properties. Section 3

proposes the concept of the Pythagorean Neutrosophic Planar Graphs (PNPGs) along with the results and the investigation of its characteristics. An algorithm for decision-making using Pythagorean Neutrosophic fuzzy graphs has been proposed with the examination of a practical numerical illustration in Section 4 and the work is concluded in Section 5.

2 Pythagorean Neutrosophic Multigraphs

Definition 2.1. A Pythagorean Neutrosophic Multi Set (*PNMS*) C of a non-void set H is grouped by functions, '*count M*', '*count NM*' and '*count I*' of C symbolized by CM_C, CNM_C, and CI_C and given as CM_C, CNM_C, $CI_C : H \rightarrow R$ with R, a collection of all multisets from interval $[0, 1]$. A *PNMS* C is represented by $C = \{< z, (\mu_C^1 (z), \mu_C^2 (z), \ldots, \mu_C^g (z)), (\sigma_C^1 (z), \sigma_C^2 (z), \ldots, \sigma_C^g (z)), (\gamma_C^1 (z), \gamma_C^2(z), \ldots, \gamma_C^g (z)) > |z \in H\}$. where the M sequence $(\mu_C^1 (z), \mu_C^2 (z), \ldots, \mu_C^g (z))$, the I sequence $\sigma_C^1 (z), \sigma_C^2 (z), \ldots, \sigma_C^g (z)$ and the NM sequence $(\gamma_C^1 (z), \gamma_C^2 (z), \ldots, \gamma_C^g (z))$ may be increasing (or) decreasing order, and sum of $\mu_C^f (z), \sigma_C^f (z), \gamma_C^f(z) \in [0, 1]$ satisfies the criteria $0 \leq sup\ \mu_C^f(z) + sup\ \sigma_C^f(z) + sup\ \gamma_C^f(z) \leq 2$ for $z \in H$ and $C = \{< z, \mu_C(z)_f, \sigma_C(z)_f, \gamma_C(z)_f > /z \in H, f = 1, 2, \ldots, g\}$.

Definition 2.2. $C = \{< s, \mu_C(s)_k, \sigma_C(s)_k, \gamma_C(s)_k > /w \in H, k = 1, 2, \ldots, q\}$ and $O = \{< s, \mu_O(s)_k, \sigma_O(s)_k, \gamma_O(s)_k > /s \in H, k = 1 - q\}$ be two PNMSs in H. Then,

1. $C \subseteq O$ iff $\mu_C(s)_k \leq \mu_O(s)_k, \sigma_C(s)_k \leq \sigma_O(s)_k, \gamma_C(s)_k \geq \gamma_O(s)_k$ for $k = 1 - q$ and $s \in H$;
2. $C = O$ iff $C \subseteq O$ and $O \subseteq C$.
3. $C^c = \{< s, \gamma_C(s)_k, 1 - \sigma_C(s)_k, \mu_C(s)_k > /s \in H, k = 1 - q\}$.
4. $C \cup O = \{s, \mu_C(s)_k \vee \mu_O(s)_k, \sigma_C(s)_k \vee \sigma_O(s)_k, \gamma_C(s)_k \wedge \gamma_O(s)_k, /s \in H, k = 1 - q\}$.
5. $C \cap O = \{s, \mu_C(s)_k \wedge \mu_O(s)_k, \sigma_C(s)_k \wedge \sigma_O(s)_k, \gamma_C(s)_k \vee \gamma_O(s)_k, /s \in H, k = 1 - q\}$.

Definition 2.3. Let $C = (\mu_C, \sigma_C, \gamma_C)$ be a Pythagorean Neutrosophic (PN) set on V and $O = \{ds, \mu_O(ds)_k, \sigma_O(ds)_k, \gamma_O(ds)_k\ i = 1 - \frac{m}{ds} \in V \times V\}$ a PNMS of $V \times V$ with $\mu_O(ds)_k \leq \min\{\mu_O(d), \mu_O(s)\}$, $\sigma_O(ds)_k \leq \min\{\sigma_O(d), \sigma_O(s)\}, \gamma_O(ds)_k \leq \max\{\gamma_O(d), \gamma_O(s)\}, \forall\ k = 1 - m$. $G = (C, O)$ is a PN Multi Graph (PNMG). $\mu_O(ds)_k, \sigma_O(ds)_k, \gamma_O(ds)_k$ Symbolize the M, I and NM value of ds in G, correspondingly. m represents the count of edges among the vertices. In *PNMG G*, O represents a *PN* Multi Edge set (*PNME*).

Example 2.1. Let the multigraph be $G = (V, E)$ with $V = \{a, b, c, d\}$, $E = \{ab, bc, bc, bc, bd\}$.

Let $C = (\mu_C, \sigma_C, \gamma_C)$ be a *PN* set on V and $O = (\mu_O, \sigma_O, \gamma_O)$ be a *PNME* set on $V \times V$ defined as

$C = \{< a, .5, .3, .3 >, < b, .4, .2, .4 >, < c, .5, .4, .3 >, < d, .4, .3, .4 >\}$,

$O = \{< ab, .3, .2, .3 >< bc, .3, .2, .3 >, < bc, .2, .1, .2 >, < bc, .4, .2, .4 >, < bd, .3, .2, .2 >\}$.

Definition 2.4. Let $O = \{< ds, \mu_O(ds)_k, \sigma_O(ds)_k, \gamma_O(ds)_k >, k = 1 - m|ds \in V \times V\}$, a *PNME* in *PNMG G*. Degree of a vertex $d \in V$ is

$$deg(d) = \left(\sum_{k=1}^{m} \mu_O(ds)_k, \sum_{k=1}^{m} \sigma_O(ds)_k, \sum_{k=1}^{m} \gamma_O(ds)_k \right), \forall\ s \in V.$$

Example 2.2. The vertices $a, b, c\ and\ d$ in *example* 2.1, hold the degrees

$deg(a) = (.3, .2, .3), deg(b) = (1.2, .7, 1.1), deg(c) = (.9, .5, .9), deg(d) = (.3, .2, .2)$.

Definition 2.5. *Let* $O = \{(ds,\ \mu_O(ds)_k,\ \sigma_O(ds)_k,\ \gamma_O(ds)_k,\ k = 1\ to\ m/ds\ \in\ V \times V\}$ *be a PNMS of PNMG G. Multi edge ds of G is strong if*

$$\frac{1}{2}\ min\ [\mu_C(d),\ \mu_C(s)] \le \mu_O(ds)_k, \quad \frac{1}{2}\ min\ [\sigma_C(d),\ \sigma_C(s)] \le \sigma_O(ds)_k,$$

$$\frac{1}{2}\ max\ [\gamma_C(d),\ \gamma_C(s)] \ge\ \gamma_O(ds)_k, \quad for\ all\ k = 1\ to\ m.$$

Definition 2.6. *Let* $O = \left\{(ds,\ \mu_O(ds)_k,\ \sigma_O(ds)_k,\ \gamma_O(ds)_k,\ k = 1\ to\ \dfrac{m}{ds} \in\ V \times V\right\}$ *be a PNMS in PNMG G.*

1. *G has order,*

   $$O(G) = \sum_{d \in V}\mu_C(d),\ \sum_{d \in V}\sigma_C(d),\ \sum_{d \in V}\gamma_C(d),$$

2. *G has size,*

   $$S(G) = \left(\sum_{k=1}^{n}\mu_O(ds)_k,\ \sum_{k=1}^{n}\sigma_O(ds)_k,\ \sum_{k=1}^{n}\gamma_O(ds)_k\right), \forall ds \in V \times V.$$

3. *The total degree of* $d\ \in\ V$ *is,*

   $$td_G(d) = \left(\sum_{k=1}^{n}\mu_O(ds)_k,\ \sum_{k=1}^{n}\sigma_O(ds)_k,\ \sum_{k=1}^{n}\gamma_O(ds)_k\right),\ \forall d\ \in\ V.$$

Definition 2.7. *Let* $O = \{(ds,\ \mu_O(ds)_k,\ \sigma_O(ds)_k,\ \gamma_O(ds)_k,\ k = 1\ to\ m/ds\ \in\ V \times V\}$, *a PNME in PNMG G. G is complete if* $\min\ [\mu_C(d),\ \mu_C(s)] = \mu_O(ds)_k$, $\min\ [\sigma_C(d),\ \sigma_C(s)] = \sigma_O(ds)_k$, $\max\ [\gamma_C(d),\ \gamma_C(s)] = \gamma_O(ds)_k$, $\forall\ k = 1 - m$ *and* $\forall\ d, s\ \in\ V.$

Example 2.3. Consider *PNMG G* in Fig. 1. Using the calculation as defined, it is verified that *G* in Fig. 1 is a complete *PNMG*.
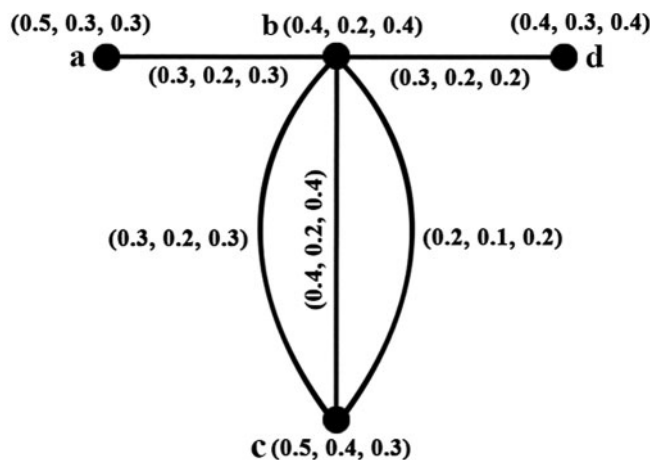

Figure 1: Pythagorean neutrosophic multigraph

Definition 2.8. If each node of *G* has the same degree of *M*, *I* and *NM* values, then *G* is a regular *PNMG*.

Definition 2.9. Let G be a *PNMG* such that $O = \{ds, \mu_O(ds)_i, \sigma_O(ds)_i, \gamma_O(ds)_i, i = 1 \ to \ m/ds \in V \times V\}$.

1. The edge *ds* has degree

$$D_G(ds) = \left((deg_\mu)_G(d) + (deg_\mu)_G(s) - 2\mu_O(ds)_i\right), \left((deg_\sigma)_G(d) + (deg_\sigma)_G(s) - 2\sigma_O(ds)_i\right),$$
$$\left((deg_\sigma)_G(d) + (deg_\sigma)_G(s) - 2\sigma_O(ds)_i\right).$$

2. The edge *ds* has degree
$$tD_G(ds) = \left((deg_\mu)_G(d) + (deg_\mu)_G(s) - 2\mu_O(ds)_i\right), \left((deg_\sigma)_G(d) + (deg_\sigma)_G(s) - 2\sigma_O(ds)_i\right),$$

$$\left((deg_\sigma)_G(d) + (deg_\sigma)_G(s) - 2\sigma_O(ds)_i\right),$$

where $((ds)_i)$ is the ith edge between *d* & *s*.

Definition 2.10. If the degree of *M*, *I* and *NM* of every edge in *PNMG G* are equal, then *G* is Edge Regular (*ER*).

Example 2.4. Degree of the edges in *Example* 1 are $D_G(ab) = (1.2, .7, 1.1), D_G(bd) = (1.2, .7, 1.2),$ $D_G(bc) = (1.2, .7, 1.1), D_G(bc) = (1.1, .7, 1), D_G(bc) = (1.3, .8, 1.2)$ and total degree of edges are $tD_G(ab) = (1.5, 0.9, 1.4), tD_G(bd) = (1.5, 0.9, 1.4), tD_G(bc) = (1.5, 0.9, 1.4), tD_G(bc) = (1.5, 0.9, 1.4),$ $tD_G(bc) = (1.5, 0.9, 1.4).$

Theorem 2.1. Let $G = (C, O)$ be a *PNMG*. If *G* is regular and edge regular *PNMG*, then the *M* $\mu(ds)_k$, *I* $\sigma(ds)_k$, *NM* $\gamma(ds)_k$ for every line $(ds) \in V \times V$ are constants.

Proof. Consider, $G = (C, O)$, a *PNMG*; *G* is regular and edge regular *PNMG*. There exists constants p_1, p_2, p_3 and q_1, q_2, q_3 for regular and edge regular correspondingly so that for every node, $deg_G(d) = \left((deg_\mu)_G(d), (deg_\sigma)_G(d), (deg_\sigma)_G(d)\right) = p_1, p_2, p_3$ for every *edge* $ds \in V \times V$,

$$DG(ds) = ((D\mu)_G(ds), (D\sigma)_G(ds), (D\gamma)_G(ds)),$$

$$= \left((deg_\mu)_G(d) + (deg_\mu)_G(s) - 2\mu_O(ds)_k\right), \left((deg_\sigma)_G(d) + (deg_\sigma)_G(s) - 2\sigma_O(ds)_k\right), \left((deg_\sigma)_G(d)\right.$$

$$\left. + (deg_\sigma)_G(s) - 2\gamma_O(ds)_k\right).$$

$$= (q_1, q_2, q_3).$$

Thus, for the *M*, *I* and *NM* values, $p_1 + p_1 - 2\mu_O(ds)_k = 2q_1, p_1 + p_1 - \mu_O(ds)_k = 2q_1, 2p_1 - 2q_1 = 2\mu_O(ds)_k, p_1 - q_1 = \mu_O(ds)_k$, Similarly, $p_2 - q_2 = \sigma_O(ds)_k$ and $p_3 - q_3 = \gamma_O(ds)_k$.

Thus, the *M*, *I* and *NM* values of a regular *PNMG* with edge regular are constant.

Theorem 2.2. Let $G = (C, O)$ be a *PNMG* on $G^* = (V, E)$. If G^* is *y*–regular multigraph, $\mu_O(ds)_k$, $\sigma_O(ds)_k$ and $\gamma_O(ds)_k$ are constants for every edge $ds \in V \times V$, then *G* is a regular and edge regular *PNMG*.

Proof. Consider $G^* = (V, E)$ as a *y*− regular multigraph. Consider $\mu_O(ds)_k = q_1, \sigma_O(ds)_k = q_2$ and $\gamma_O(ds)_k = q_3$. For every vertex $d \in V$,

$$deg_G(d) = \left((deg_\mu)_G(d), (deg_\sigma)_G(d), (deg_\sigma)_G(d)\right),$$

$$= \left(\sum_{d \neq s} \mu_O(ds)_k, \sum_{d \neq s} \sigma_O(ds)_k, \sum_{d \neq s} \gamma_O(ds)_k\right),$$
$$= (y \times q_1, y \times q_2, y \times q_3),$$

$$= \left(\sum_{d \neq s} \mu_O(yx)_k, \sum_{d \neq s} \sigma_O(yx)_k, \sum_{d \neq s} \gamma_O(yx)_k \right),$$

$$= \left((deg_\mu)_G(s), \ (deg_\sigma)_G(s), \ (deg_\sigma)_G(s) \right) = deg_G(s).$$

For every edge, $ds \in V \times V$,

$$DG(ds) = ((D\mu)_G(ds), \ (D\sigma)_G(ds), \ (D\gamma)_G(ds))$$

$$= \left((deg_\mu)_G(d) + (deg_\mu)_G(s) - 2\mu_O(ds)_k \right), \ ((deg_\sigma)_G(d) + (deg_\sigma)_G(s) - 2\sigma_O(ds)_k),$$

$$((deg_\sigma)_G(d) + (deg_\sigma)_G(s) - 2\gamma_O(ds)_k).$$

$$= ((y \times q_1) + (y \times q_1) - 2(q_1), (y \times q_2) + (y \times q_2)$$

$$- 2(q_2), (y \times q_3) + (y \times ; q_3) - 2(q_3))$$

$$= (2q_1 \ (y - 1), \ 2q_2 \ (y - 1), \ 2q_3 \ (y - 1)).$$

Thus, G is regular and edge regular *PNMG*.

Definition 2.11. The strength of *PN* edge fj is determined by the value

$$S_{fi} = \left((S_\mu)_{fi}, \ (S_\sigma)_{fi}, \ (S_\gamma)_{fi} \right) = \left(\frac{\mu_O(fj)_i}{\min(\mu_C(f), \mu_C(j))}, \ \frac{\sigma_O(fj)_i}{\min(\sigma_C(f), \sigma_C(j))}, \ \frac{\gamma_O(fj)_i}{\min(\gamma_C(f), \gamma_C(j))} \right).$$

An edge fj of a *PNMG* is *PN* strong if $(S_\mu)_{fj} \geq 0.5, (S_\sigma)_{fj} \geq 0.5, (S_\gamma)_{fj} \geq 0.5$.

Definition 2.12. Let G be a *PNMG* such that O has 2 edges $(ab, \mu_o(ab)_k, \sigma_o(ab)_k, \gamma_o(ab)_k)$ and $(cd, \mu_o(ab)_y, \sigma_o(ab)_y, \gamma_o(ab)_y)$, which intersect at P, k and y are fixed integers. At P, the intersecting value is defined by,

$$S_P = \left((S_\mu)_P, \ (S_\sigma)_P, \ (S_\gamma)_P \right) = \left(\frac{(S_\mu)_{ab} + (S_\mu)_{cd}}{2} + \frac{(S_\sigma)_{ab} + (S_\sigma)_{cd}}{2} + \frac{(S_\gamma)_{ab} + (S_\gamma)_{cd}}{2} \right).$$

The planarity decreases when the number of points of intersection in *PNMG* increases.

S_P is inversely proportional to the planarity for *PNMG*.

3 Pythagorean Neutrosophic Planar Graphs

The concept of the Pythagorean Neutrosophic planar graphs has been discussed.

Definition 3.1. Let G be a *PNMG* and the point of intersections among the edges be $P_1, P_2, \ldots, P_m$, G is a *PN* Planar Graph (*PNPG*) with *PN* Planarity Value (*PNPV*) $f = (f_\mu, f_\sigma, f_\gamma)$, where

$$f = (f_\mu, f_\sigma, f_\gamma) = \left(\frac{1}{1 + \{(S_\mu)_{P_1} + (S_\mu)_{P_2} + \ldots + (S_\mu)_{P_m}\}}, \ \frac{1}{1 + \{(S_\sigma)_{P_1} + (S_\sigma)_{P_2} + \ldots + (S_\sigma)_{P_m}\}}, \right.$$

$$\left. \frac{1}{1 + \{(S_\gamma)_{P_1} + (S_\gamma)_{P_2} + \ldots + (S_\gamma)_{P_m}\}} \right).$$

$0 \leq f_\mu \leq 1, 0 \leq f_\sigma \leq 1, 0 \leq f_\gamma \leq 1$. The *PNPV* is $(1, 1, 1)$ for a geometrical representation of *PNPG* if it has no intersecting point.

Example 3.1. Take a multigraph $G^* = (V, E)$ such that $V = \{a, b, c, d, e\}$, $E = \{ab, ac, ad, ad, bc, bd, cd, ce, ae, de, be\}$. Let $C = (\mu_C, \sigma_C, \gamma_C)$ be a *PN* set on V and $O = (\mu_O, \sigma_O, \gamma_O)$ be a *PNME*

set on $V \times V$ as are described as,

$C = \{< a, .5, .5, .2 >, < b, .6, .7, .3 >, < c, .4, .6, .4 >, < d, .7, .5, .3 >, < e, .8, .6, .5 >\}$,
$O = \{< ab, .5, .4, .2 >, < ac, .4, .5, .3 >, < ad, .5, .5, .3 >, < ad, .4, .4, .2 >,$
$\quad < bc, .4, .6, .4 >, < bd, .6, .5, .2 >, < cd, .3, .5, .3 >, < ae, .5, .5, .4 >,$
$\quad < ce, .3, .6, .5 >, < de, .7, .5, .4 >, < be, .6, .6, .4 >\}$.

The *PNMG* has two points of intersection in Fig. 2 (P_1 and P_2). P_1 is a point among the lines $(ad, .5, .5, .3)$ and $(bc, .4, .6, .4)$ and P_2 is a point among the edges $(ad, .4, .4, .2)$ and $(bc, .4, .6, .4)$.
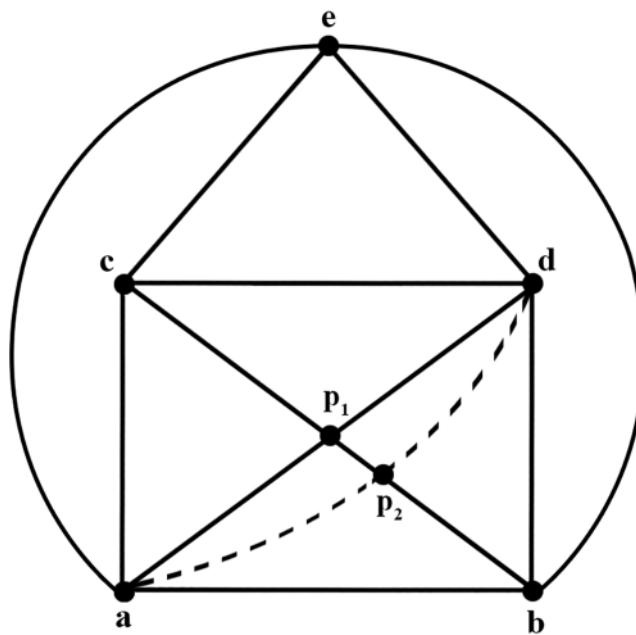

Figure 2: Pythagorean neutrosophic planar graph

The strength for the edges *ab*, *ad* and bc are $S_{ad} = \left(\dfrac{.5}{.5}, \dfrac{.5}{.5}, \dfrac{.3}{.3}\right) = (1, 1, 1)$, $S_{ad} = \left(\dfrac{.4}{.5}, \dfrac{.4}{.5}, \dfrac{.2}{.3}\right) =$ $(.8, .8, .67)$, $S_{bc} = \left(\dfrac{.4}{.4}, \dfrac{.6}{.6}, \dfrac{.4}{.4}\right) = (1, 1, 1)$. For P_1, intersecting value S_{P_1} is $(1, 1, 1)$ and for P_2, S_{P_2} is $(.9, .9, .835)$.

Therefore *PNPV* for the *PNMG* given in Fig. 2 is $(.345, .345, .353)$.

Theorem 3.1. Let *G* be a complete *PNMG*. The *PNPV*, $f = (f_\mu, f_\sigma, f_\gamma)$ of *G* is given by $f_\mu = \dfrac{1}{1 + n_p}$, $f_\sigma = \dfrac{1}{1 + n_p}$ and $f_\gamma = \dfrac{1}{1 + n_p}$ such that $f_\mu + f_\sigma + f_\gamma \leq 3$, where n_p is the count of point of intersection among the lines in *G*.

Definition 3.2. A *PNPG* *G* is called strong (*SPNPG*) if the *PNPV* $f = (f_\mu, f_\sigma, f_\gamma)$ of the graph is $f_\mu \geq 0.5, f_\sigma \geq 0.5, f_\gamma \geq 0.5$.

Theorem 3.2. Let *G* be a *S PNPG*. The number of points of intersections among *S* lines in *G* is utmost one.

Proof. Let G be a $\mathscr{S}$ PNPG. Consider G has at least 2 points of intersections P_1 and P_2 between 2 $\mathscr{S}$ lines in G. For any $\mathscr{S}$ edge $(wq, \mu_o(wq)_i, \sigma_o(wq)_i, \gamma_o(wq)_i)$, $\mu_o(wq)_i \geq \frac{1}{2} \min\{\mu_C(w), \mu_C(q)\}$, $\sigma_o(wq)_i \geq \frac{1}{2} \min\{\sigma_C(w), \sigma_C(q)\}$, $\gamma_o(wq)_i \leq \frac{1}{2} \max\{\gamma_C(w), \gamma_C(q)\}$.

Thus, $(S_\mu)_{wq}, (S_{\mu\sigma})_{wq}, (S_\gamma)_{wq} \geq .5$. Thus, for two intersecting $\mathscr{S}$ edges $(wq, \mu_o(wq)_k, \sigma_o(wq)_k, \gamma_o(wq)_k)$ and $(cd, \mu_o(cd)_j, \sigma_o(cd)_j, \gamma_o(cd)_j)$,

$$\frac{(S_\mu)_{wq} + (S_\mu)_{cd}}{2} + \frac{(S_\sigma)_{wq} + (S_\sigma)_{cd}}{2} + \frac{(S_\gamma)_{wq} + (S_\gamma)_{cd}}{2} \geq .5,$$

(i.e.,) $(S_\mu)_{P_1}, (S_\sigma)_{P_1} \geq .5, (S_\gamma)_{P_1} \leq .5$, Likewise, $(S_\mu)_{P_2}, (S_\sigma)_{P_2} \geq .5, (S_\gamma)_{P_2} \leq .5$.

$$\Rightarrow 1 + (S_\mu)_{P_1} + (S_\mu)_{P_2} \geq 2, 1 + (S_\sigma)_{P_1} + (S_\sigma)_{P_2} \geq 2, 1 + (S_\gamma)_{P_1} + (S_\gamma)_{P_2} \geq 2.$$

$$f_\mu = \frac{1}{1 + (S_\mu)_{P_1} + (S_\mu)_{P_2}} \leq .5 \, f_\sigma = \frac{1}{1 + (S_\sigma)_{P_1} + (S_\sigma)_{P_2}} \leq .5 \, f_\gamma = \frac{1}{1 + (S_\gamma)_{P_1} + (S_\gamma)_{P_2}} \geq .5.$$

This becomes a contradiction to the fact PN graph is a $\mathscr{S}PNPG$. Thus the number of points of intersections between $\mathscr{S}$ edges cannot be two. If the count of point of intersections of PN edges increases, the $PNPV$ decreases. When the count of the point of intersection of $\mathscr{S}$ edges is 1, then the $PNPV$ $f_\mu \geq 0.5, f_\sigma \geq 0.5, f_\gamma \geq 0.5$. A $\mathscr{S}PNPG$ is a $PNPG$ without any crossing between edges. Thus, the largest number of points of intersections among the $\mathscr{S}$ edges in G is 1 parameter. The region bounded by PN edges is a face of a PN graph. Every PN Face (PNF) in its boundary is characterized by PN edges. If every edge in the boundary of a PNF have $\mu_o, \sigma_o, \gamma_o$ values $(1, 1, 1)$ and $(0, 0, 0)$, then it is a crisp face. When one among those edges is removed or has $\mu_o, \sigma_o, \gamma_o$ values $(0, 0, 0)$ and $(1, 1, 1)$ correspondingly, the PNF does not exist. The existence of a PNF depends on the minimal strength of PN edges in its boundary. A PNF and its $\mu_o, \sigma_o, \gamma_o$ Values of PNG are expressed below.

Definition 3.3. Let G be a $PNPG$ and $O = \{(ds, \mu_o(ds)_k, \sigma_o(ds)_k, \gamma_o(ds)_k, k = 1 \text{ to } m/ds \in V \times V\}$. A PNF of G is a region and is bounded by the set of PN lines $E' \subset E$, of a pictorial demonstration of G. The M, I and NM of PNF are:

$$\min\left\{\frac{\mu_o(ds)_k}{\min\{\mu_C(d), \mu_C(s)\}}, k = 1, 2, \ldots \frac{m}{ds} \in E'\right\},$$

$$\min\left\{\frac{\sigma_o(ds)_k}{\min\{\sigma_C(d), \sigma_C(s)\}}, k = 1, 2, \ldots \frac{m}{ds} \in E'\right\},$$

$$\max\left\{\frac{\gamma_o(ds)_k}{\max\{\gamma_C(d), \gamma_C(s)\}}, k = 1, 2, \ldots \frac{m}{ds} \in E'\right\}.$$

Definition 3.4. A PNF is $(\mathscr{S})$ PNF if the value of M, I is larger than 0.5, NM is below 0.5, and weak otherwise. The infinite region in every $PNPG$ is termed as an outer PNF, and other faces are called inner $PNFs$.

Example 3.2. The $PNPG$ as in Fig. 3, has the following faces: PNF F_1 is bounded by the edges $(ab, .4, .4, .1)$, $(bc, .5, .5, .1)$, $(ac, .4, .4, .1)$. Outer PNF F_2 surrounded by edges $(ac, .4, .4, .1)$, $(ad, .4, .4, .1)$, $(bd, .5, .5, .1)$, $(bc, .5, .5, .1)$. PNF F_3 is bounded by lines $(ab, .4, .4, .1)$, $(bd, .5, .5, .1)$, $(ad, .4, .4, .1)$.

Clearly, the M, I and NM value of a PNF F_1 is $(.8, .8, .5)$. Thus, F_1 is a $\mathscr{S}PNF$.

Definition 3.5. Let G be a $PNPG$ and let $O = \{(ds, \mu_o(ds)_k, \sigma_o(ds)_k, \gamma_o(ds)_k, k = 1 - m/ds \in V \times V\}$. Let $F_1, F_2, \ldots, F_k$ be the $\mathscr{S}PNFs$ of G. The PN Dual Graph ($PNDG$) of G is a $PNPG$ $G' = (V', C', O')$ with $V = \{x_k, k = 1 - k\}$, and the vertex x_k of G' is for F_k of G. The M, I, NM values

of vertices are $C' = (\mu'_C, \sigma'_C, \gamma'_C) : V' \rightarrow [0, 1]^3$ such that

$$\mu'_C(x_k) = \max \left\{ \mu'_o(ua)_k, \, k = 1 \text{ to } \frac{p}{ua} \text{ is in the boundary of } \mathscr{S}PNF \, F_k \right\},$$

$$\sigma'_C(x_k) = \max \left\{ \sigma'_o(ua)_k, \, k = 1 \text{ to } \frac{p}{ua} \text{ is in the boundary of } \mathscr{S}PNF \, F_k \right\},$$

$$\gamma'_C(x_k) = \min \left\{ \gamma'_o(ua)_k, \, k = 1 \text{ to } \frac{p}{ua} \text{ is in the boundary of } \mathscr{S}PNF \, F_k \right\}.$$
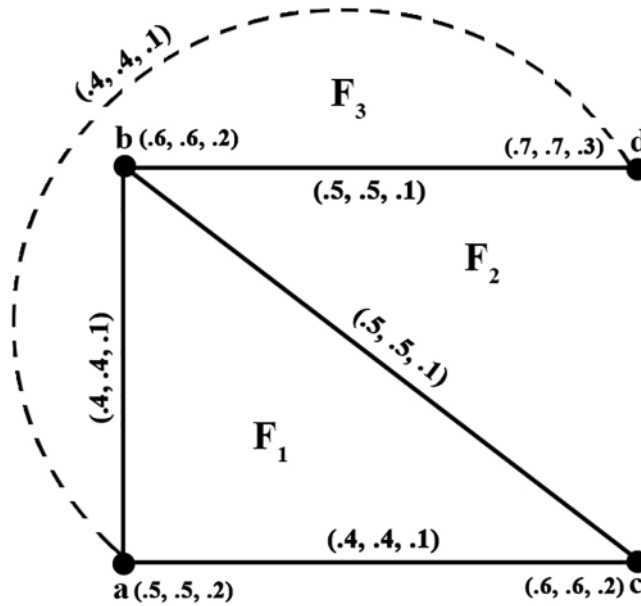

Figure 3: Faces in pythagorean neutrosophic planar graph

Two common faces F_k and F_b of G might exist between one common line. There may be more than 1 edge among 2 vertices x_k and x_b in *PNDG* G'. $\mu'_o (x_k x_b)$ represent the M value of the lth edge among x_k and x_b and $\gamma'_o(x_k x_b)$ represent the NM value of the lth edge amidst x_k and x_b. M, I and NM values of *PN* edges of the *PNDG* are presented by $\mu'_o(x_k x_b)_l = \mu'_o(ua)_b$, $\sigma'_o(x_k x_b)_l = \sigma'_o(ua)_b$, $\gamma'_o(x_k x_b)_l = \gamma'_o(ua)_b$, with $(ua)_b$ is an edge in the boundary between 2 $\mathscr{S}PNF$ F_k and F_b and $l = 1$ to S, where $\mathscr{S}$ is the count of lines among x_k and x_b. *PNDG* of *PNPG* does not hold point of intersection of edges for a some representation, so it is *PNPG* with *PNPV* (1, 1, 1). The *PNF* of *PNDG* can be similarly expressed as in *PNPG*.

Theorem 3.3. Let G be a *PNPG* whose count of vertices, total of *PN* edges, count of $\mathscr{S}PNF$ are symbolized by m, p, n correspondingly. G' be the *PNDG* of G, then count of vertices, edges, *PNF* of G' equals m, p, n correspondingly.

Theorem 3.4. Let $G = (V, C, O)$ be a *PNPG* without weak lines and the *PNPG* of G be $G' = (V', C', O')$. The M, I and NM values of *PN* lines of G' equals values of G.

Definition 3.6. Let $G = (C, O)$ be a *PNPG* where $O = \left\{ (ds, \mu_o(ds)_k, \sigma_o(ds)_k, \gamma_o(ds)_k, b) = 1 \right.$ to $\left. \frac{n}{ab} \in V \times V \right\}$. Let $F_1, F_2, \ldots, F_k$ be $\mathscr{S}PNFs$ of G. Then *PNDG* of G is a *PNPG* $G' = (C', O')$, where $V' = \{ r_b, \, b = 1 \text{ to } K \}$ and the vertex r_b of G' is taken for F_b of G.

The M, I, NM by mapping $G' = (V', C', O') : V' \rightarrow [0, 1]^3$ such that

$$\mu'_C(r_b) = max \left\{ \mu'_O(ua)_b, \, b = 1 \, to \, \frac{m}{ab} \, is \, in \, the \, neighbourhood \, of \, \mathcal{SPNF} \, F_b \right\},$$

$$\sigma'_C(r_b) = max \left\{ \sigma'_O(ua)_b, \, b = 1 \, to \, \frac{m}{ab} \, is \, in \, the \, neighbourhood \, of \, \mathcal{SPNF} \, F_b \right\},$$

$$\gamma'_C(r_b) = min \left\{ \gamma'_O(ua)_b, \, b = 1 \, to \, \frac{m}{ab} \, is \, in \, the \, neighbourhood \, of \, \mathcal{SPNF} \, F_b \right\}.$$

Between F_k and F_b of G, at least one common edge may occur. Among two vertices, there may exist beyond a single edge $r_k r_b$ in PNDG G'. M, I and NM values of PN edges of $PNDG$ are $\mu'_C(r_k r_b)_S = \mu^S_O(ua)_k$, $\sigma'_C(r_k r_b)_S = \sigma^S_O(ua)_k$, $\gamma'_C(r_k r_b)_S = \gamma^S_O(ua)_k$ where $(ab)^S$ is in the surrounding among $\mathcal{SPN}$ faces F_k and F_b and $S = 1 - l$, is the count of common edges in the neighborhood of F_k and F_b. The PNDG G' of PNDG G has no crossing among lines for some definite geometric representation, PNPG of PNPV $(1, 1, 1)$.

Example 3.3. Take a PNG $G = (V, C, O)$ as displayed in Fig. 4 with $V = \{a_1, a_2, a_3, a_4, a_5\}$. Let C and O be PN vertex set and PN edge set.

$C =< (a_1, .7, .5, .3), (a_1, .69, .55, .4), (a_1, .35, .45, .5), (a_4, .76, .8, .3) > .$

$O =< (a_1 a_2, .6, .48, .2), (a_2 a_3, .3, .4, .43), (a_3 a_4, .3, .4, .45), (a_4 a_1, .65, .4, .25), (a_1 a_3, .3, .4, .4) > .$

PNF F_1 is enclosed by the edges $(a_1 a_3, .3, .4, .4), (a_3 a_4, .3, .4, .45), (a_4 a_1, .65, .4, .25)$.

PNF F_2 is enclosed by the edges $(a_1 a_2, .6, .48, .2), (a_2 a_3, .3, .4, .43), (a_1 a_3, .3, .4, .4)$.

PNF F_3 is enclosed by the edges $(a_1 a_2, .6, .48, .2), (a_2 a_3, .3, .4, .43), (a_3 a_4, .3, .4, .45), (a_4 a_1, .65, .4, .25)$.
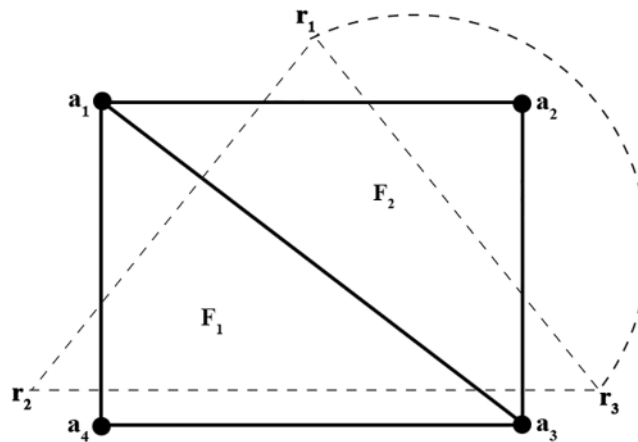

Figure 4: Pythagorean neutrosophic dual graph

We symbolize the vertices of PN Dual Graph $(PNDG)$ by a dot and edges by dashed lines. We take a vertex for each face of $PNDG$ with $V' = \{r_1, r_2, r_3, r_4\}$.

$\mu_{C'}(r_1) = max \, \{.3, .3, .65\} = .65, \sigma_{C'}(r_1) = max \, \{.4, .4, .4\} = .4, \gamma_{C'}(r_1) = min \, \{.4, .45, .25\} = .25,$

$\mu_{C'}(r_2) = max \, \{.6, .3, .3\} = .6, \sigma_{C'}(r_2) = max \, \{.48, .4, .4\} = .48, \gamma_{C'}(r_2) = min \, \{.2, .43, .4\} = .2,$

$\mu_{C'}(r_3) = max \ \{.6, .3, .3, .65\} = .65, \ \sigma_{C'}(r_3) = max \ \{.48, .4, .4, .4\} = .48, \ \gamma_{C'}(r_3)$

$= min \ \{.2, .43, .45, .25\} = .2$

The vertex set V' has the vertices $< r_1, (.65, .4, .25) >, < r_2, (.6, .48, .2) >, < r_3, (.65, .48, .2) >$.

There is one common edge a_1a_3 amidst F_1, F_2 in G. Thus, there exists a single line among vertices r_1 and r_2 in $PNDG$ of G. The edges for the $PNDG$ are constructed as in Fig. 4.

Definition 3.7. An isomorphism of two $PNPGs$ G_1 and G_2, $y \ : G_1 \rightarrow G_2$ is a bijective mapping $y \ : V_1 \rightarrow V_2$ that holds the following

1. $\mu_{C_1}(r) = \mu_{C_2}(y(r)), \ \sigma_{C_1}(r) = \sigma_{C_2}(y(r)), \ \gamma_{C_1}(r) = \gamma_{C_2}(y(r)).$
2. $\mu_{O_1}(rs) = \mu_{O_2}(y(r) \ y(s)), \ \sigma_{O_1}(rs) = \sigma_{O_2}(y(r) \ y(s)), \ \gamma_{O_1}(rs) = \gamma_{O_2}(y(r) \ y(s)), \ \forall \ r \in V_1, \ \forall \ rs \in E_1.$

Example 3.4. Consider two $PNPGs$ $G_1 = (C_1, O_1)$ and $G_2 = (C_2, O_2)$ as in Fig. 5 such that

$C_1 = \{< a_1, .8, .7, .2 >, < a_2, .7, .6, .4 >, \ < a_3, .6, .5, .5 >, < a_4, .5, .4, .4 >\},$

$O_1 = \{< a_1a_2, .6, .5, .3 >, < a_2a_3, .5, .4, .4 >, \ < a_3a_4, .4, .3, .3 >, < a_4a_1, .3, .2, .3 >\},$

$C_2 = \{< r_1, .5, .4, .4 >, < r_3, .7, .6, .4 >, \ < r_2, .6, .5, .5 >, < r_4, .8, .7, .2 >\},$

$O_2 = \{< r_1r_2, .4, .3, .3 >, < r_2r_3, .5, .4, .4 >, \ < r_3r_4, .6, .5, .3 >, < r_4r_1, .3, .2, .3 >\}.$
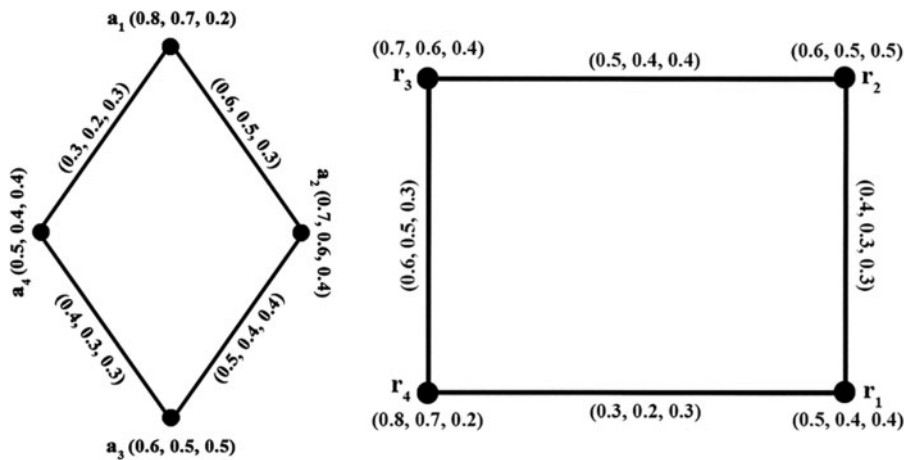

Figure 5: Pythagorean neutrosophic planar graph

$y \ : \ V_1 \rightarrow \ V_2$ given by $y(a_1) = r_4, \ y(a_2) = r_3, \ y(a_3) = r_2, \ y(a_4) = r_1$ satisfies

$\mu_{C_1}(r_k) = \mu_{C_2}(y(r_k)), \ \sigma_{C_1}(r_k) = \sigma_{C_2}(y(r_k)), \ \gamma_{C_1}(r_k) = \gamma_{C_2}(y(r_k)), \ \mu_{O_1}(r_kr_b) = \mu_{O_2}(y(r_k)y(r_b)), \sigma_{O_1}(r_kr_b) = \sigma_{O_2}(y(r_k)y(r_b)), \ \gamma_{O_1}(r_kr_b) = \gamma_{O_2}(y(r_k)y(r_b))$ for all $r_k \in V_1, \ r_kr_b \in E_1$, where $i, j = 1$ to 4. Thus G_1 is isomorphic to G_2.

The M, I and NM of the edges of $PNDG$ are $\mu_{o'}(r_1r_2) = \mu_o(a_1a_3) = .3, \ \sigma_{o'}(r_1r_2) = \sigma_o(a_1a_3) = .4,$ $\gamma_{o'}(r_1r_2) = \gamma_o(a_1a_3) = .4, \ \mu_{o'}(r_2r_3) = \mu_o(a_1a_2) = .6, \ \sigma_{o'}(r_2r_3) = \sigma_o(a_1a_2) = .48, \ \gamma_{o'}(r_2r_3) = \gamma_o(a_1a_2) = .2, \ \mu_{o'}(r_3r_1) = \mu_o(a_3a_4) = .65, \ \sigma_{o'}(r_3r_1) = \sigma_o(a_3a_4) = .4, \ \gamma_{o'}(r_3r_1) = \gamma_o(a_3a_4) = .45, \ \mu_{o'}(r_1r_3) = \mu_o(a_4a_1) = .65, \ \sigma_{o'}(r_1r_3) = \sigma_o(a_4a_1) = .4, \ \gamma_{o'}(r_1r_3) = \gamma_o(a_4a_1) = .25, \ \mu_{o'}(r_2r_3) = \mu_o(a_2a_3) = .3, \ \sigma_{o'}(r_2r_3) = \sigma_o(a_2a_3) = .4, \ \gamma_{o'}(r_2r_3) = \gamma_o(a_2a_3) = .43.$

Thus, the *PNDG* edge set is,

$O' = <(r_1r_2, .3, .4, .4), (r_2r_3, .6, .48, .2), (r_3r_1, .65, .4, .45), (r_1r_3, .65, .4, .25), (r_2r_3, .3, .4, .43)>$. Thus G_1 is a PNDG of G_2.

Definition 3.8. A weak isomorphism of two *PNPGs* G_1 and G_2, $y : G_1 \rightarrow G_2$ is a bijective mapping $y : V_1 \rightarrow V_2$ that holds the following:

1. y is a homomorphism.
2. $\mu_{C_1}(r) = \mu_{C_2}(y(r)), \sigma_{C_1}(r) = \sigma_{C_2}(y(r)), \gamma_{C_1}(r) = \gamma_{C_2}(y(r)) \,\forall\, r \in V_1$.

Example 3.5. Consider two *PNPG*, $G_1 = (C_1, O_1)$ and $G_2 = (C_2, O_2)$ as in Fig. 6 such that

$C_1 = \{< a_1, .9, .5, .3 >, < a_2, .8, .6, .2 >, < a_3, .7, .4, .4 >, < a_4, .6, .3, .4 >, < a_5, .5, .4, .3 >\}$,

$O_1 = \{< a_1a_2, .7, .4, .2 >, < a_2a_3, .6, .4, .3 >, \; < a_3a_4, .4, .3, .3 >, < a_4a_1, .5, .3, .2 >,$

$\qquad < a_4a_5, .5, .3, .2 >, < a_5a_1, .4, .3, .1 >, < a_2a_5, .4, .3, .15 >\}$,

$C_2 = \{< r_1, .7, .4, .4 >, < r_2, .6, .3, .4 >, \; < r_3, .8, .6, .2 >, < r_4, .5, .4, .3 >, < r_5, .9, .5, .3 >\}$,

$O_2 = \{< r_1r_2, .4, .2, .1 >, < r_2r_3, .5, .3, .2 >, \; < r_3r_5, .5, .2, .1 >, < r_3r_4, .3, .2, .1 >, < r_2r_4, .4, .2, .1 >$
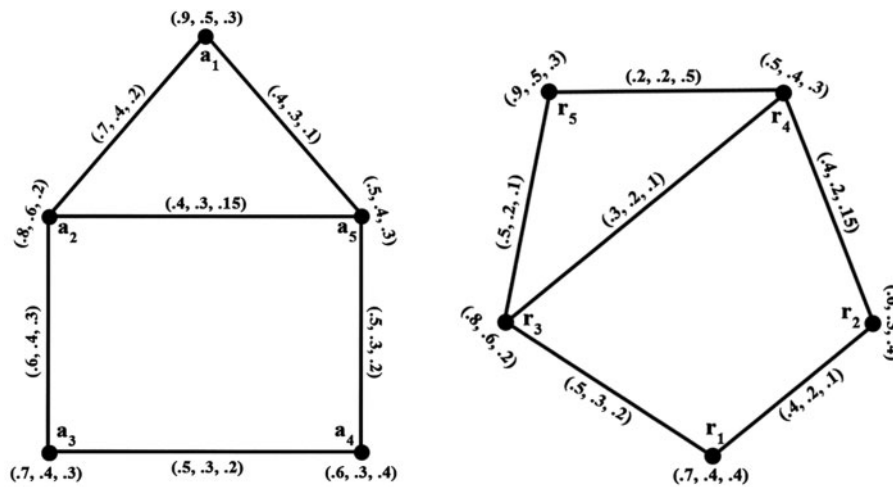
$\qquad < r_4r_5, .2, .2, .5 >\}$.

Figure 6: Pythagorean neutrosophic planar graphs

A mapping y: $V_1 \rightarrow V_2$ given by $y(a_1) = r_5, y(a_2) = r_3, y(a_3) = r_1, y(a_4) = r_2, y(a_5) = r_4$ satisfies $\mu_{C_1}(r_k) = \mu_2(y(r_k)), \sigma_{C_1}(r_k) = \sigma_2(y(r_k)), \gamma_{C_1}(r_k) = \gamma_2(y(r_k))$ for all $r_k \in V_1$, where $k, b = 1, 2, 3, 4, 5$.

But $\mu_{C_1}(r_kr_b) \neq \mu_{C_2}(y(r_k) \, y(r_b)), \sigma_{C_1}(r_kr_b) \neq \sigma_{C_2}(y(r_k) \, y(r_b)), \gamma_{C_1}(r_kr_b) \neq \gamma_{C_2}(y(r_k) \, y(r_b))$.

Thus G_1 is a weak isomorphic to G_2.

Definition 3.9. A co-weak isomorphism of two *PNPGs* G_1 and G_2, y: $G_1 \rightarrow G_2$ is a bijective mapping $y : V_1 \rightarrow V_2$ that holds

1. y is a homomorphism.
2. $\mu_{C_1}(rs) = \mu_{C_2}(y(r) \, y(s)), \sigma_{C_1}(rs) = \sigma_{C_2}(y(r) \, y(s)), \gamma_{C_1}(rs) = \gamma_{C_2}(y(r) \, y(s)), \forall\, rs \in E_1$.

Example 3.6. Take *PNPG*, $G_1 = (C_1, O_1)$ and $G_2 = (C_2, O_2)$ as in Fig. 7 such that

$C_1 = \{< a_1, .8, .7, .6 >, < a_2, .7, .6, .5 >, < a_3, .9, .5, .4 >, < a_4, .8, .6, .2 >, < a_5, .7, .5, .3 >\}$

$O_1 = \{< a_1a_2, .6, .5, .4 >, < a_2a_3, .5, .4, .35 >, < a_1a_4, .65, .5, .3 >, < a_2a_5, .55, .4, .25 >,$
$\quad < a_4a_5, .6, .4, .1 >, < a_2a_5, .6, .4, .35 >\}$

$C_2 = \{< p_1, .75, .65, .5 >, < p_2, .6, .5, .45 >, < p_3, .8, .4, .3 >, < p_4, .7, .5, .1 >, < p_5, .65, .45, .25 >\}$

$O_2 = \{< p_1p_2, .6, .5, .4 >, < p_2p_3, .5, .4, .35 >, < p_1 p_4, .65, .5, .3 >, < p_2 p_5, .55, .4, .25 >,$
$\quad < p_2 p_4, .5, .4, .2 >, < p_3 p_5, .6, .4, .2 >, < p_4 p_5, .6, .4, .1 >, < p_2 p_5, .6, .4, .35 >\}$.
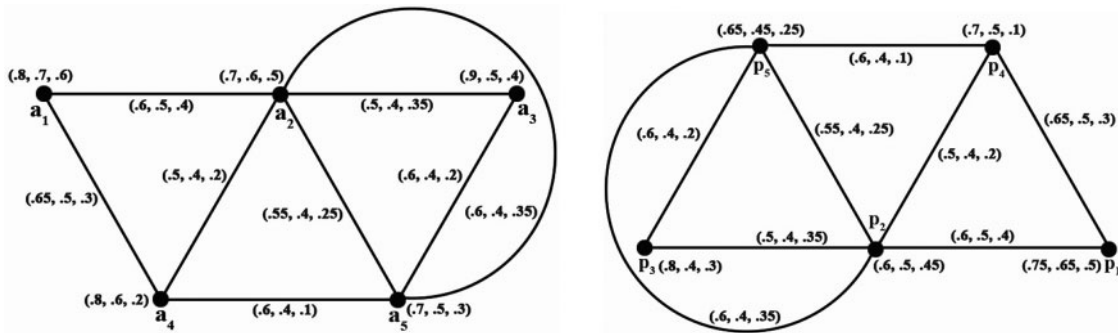

Figure 7: Pythagorean neutrosophic planar graphs

A mapping $y : V_1 \rightarrow V_2$ illustrated by $y(a_1) = p_1$, $y(a_2) = p_2$, $y(a_3) = p_3$, $y(a_4) = p_4$, $y(a_5) = p_5$ satisfies $\mu_{O_1}(r_kr_b) = \mu_{O_2}(y(r_k)y(r_b))$, $\sigma_{O_1}(r_kr_b) = \sigma_{O_2}(y(r_k)y(r_b))$, $\gamma_{O_1}(r_kr_b) = \gamma_{O_2}(y(r_k)y(r_b))$.

forall $r_kr_b \in E_1$, where $k, b = 1, 2, 3, 4, 5$ but $\mu_{C_1}(r_k) \neq \mu_{C_2}(y(r_k))$, $\sigma_{C_1}(r_k) \neq \sigma_{C_2}(y(r_k))$, $\gamma_{C_1}(r_k) \neq \gamma_{C_2}(y(r_k))$. Thus G_1 is a weak isomorphic to G_2.

4 Application in Decision Making Problem

4.1 Algorithm

The following algorithm is our proposed technique for multi-criteria decision making.

Step 1: Input the alternatives $B = (B_1, B_2, \ldots, B_n)$ and set of criteria's $C = (C_1, C_2, \ldots, C_m)$ and create the PNF relation $\left(M^{(k)} = m_{lp}^{(k)}\right)_{nxn}$ according to each criteria.

Step 2: Aggregate all $m_{lp}^{(k)} = \left(\mu_{lp}^{(k)}, \beta_{lp}^{(k)}, \sigma_{lp}^{(k)}\right)$ $(l, p = 1, 2, \ldots, n)$ regarding criteria C_j and derive $M^{(k)} = (m_{lp})_{nxn}$ where m_{lp} is the value assigned for alternate m_l over m_p according to criteria C_j by PNF averaging (PNFA) operator. $p_i^{(k)} = PNFA\left(m_{i1}^{(k)}, m_{i2}^{(k)}, \ldots, m_{in}^{(k)}\right)$, (k= 1, 2, \ldots, m)

$$= \left(\sqrt{1 - \left(\prod_{j=1}^{n}\left(1 - \mu_{ij}^2\right)\right)^{1/n}}, \sqrt{1 - \left(\prod_{j=1}^{n}\left(1 - \beta_{ij}^2\right)\right)^{1/n}}, \left(\prod_{j=1}^{n}\left(1 - \sigma_{ij}^2\right)\right)^{1/n}\right), i = 1, 2, \ldots, m$$

Step 3: Calculate the aggregated value of each criteria C_m and compute the aggregated matrix.

Step 4: Use the score function,

$$S\left(B_i^{(j)}\right) = \frac{1 + \mu + \beta - \sigma}{3}, (i = 1 \text{ to } n, j = 1 \text{ to } m).$$

calculate the score matrix for the problem.

Step 5: Calculate the choice matrix by the function

$$S(B_i) = \sum_{j=1}^{n} \frac{B_{ij}}{n}, \ (i = 1 \ to \ m).$$

Step 6: Now arrange the alternatives in an order and choose the maximum as the optimal decision.

4.2 Numerical Approach

In this competitive world, time is the most precious asset for everyone. In the given 24 h a day, saving time and using it for multiple duties, and chores is an important quality. Even though time management is in our hands, travelling from one point destination for the people who doesn't drive is hard in recent times, cabs and travelling applications is one of the trending and useful facility in our cities. By the existing trends and techniques, the fastest and money-saving possibility is vital in day-to-day life. Consider the following scenario: a person wants to travel from a point to his destination and is given a set of mobile booking applications to choose a ride. Let there be these 5 cabs booking applications namely B_i ($i = 1 \ to \ 5$) that are effective nowadays. The decision-makers provide their priors by comparing these applications concerning criteria's C_j ($j = 1, 2, 3, 4, 5$).

C_1 = *Availability*, C_2 = *Travelling speed*, C_3 = *Safety*, C_4 = *Cost*, C_5 = *User friendly*

Step 1: The alternatives are $B = (B_1, B_2, B_3, B_4, B_5)$ and the criteria's are $C = (C_1, C_2, C_3, C_4, C_5)$. The Pythagorean neutrosophic fuzzy relation according to criteria $C_i = (i = 1, 2, \ldots, 5)$ is given in structure as in Fig. 8 and values are detailed in Tables 1 to 5.

Step 2 and 3: Using PNFA operator, the aggregated values are calculated as follows:

$C_1 : B_1^{(1)} = (.5269, .3984, .0423)$, $B_2^{(1)} = (.4687, .3543, .08662)$, $B_3^{(1)} = (.521, .325, .0321)$,

$\quad B_4^{(1)} = (.7275, .3791, .04975)$, $B_5^{(1)} = (.69123, .61604, .0321)$.

$C_2 : B_1^{(2)} = (.5967, .2851, .0234)$, $B_2^{(2)} = (.7223, .26435, .0184)$, $B_3^{(2)} = (.48703, .3332, .04975)$,

$\quad B_4^{(2)} = (.2121, .2851, .0321)$, $B_5^{(2)} = (.6953, .5701, .0321)$.

$C_3 : B_1^{(3)} = (.6088, .3248, .0321)$, $B_2^{(3)} = (.51595, .4729, .0558)$, $B_3^{(3)} = (.7763, .3248, .0243)$,

$\quad B_4^{(3)} = (.5847, .3922, .0443)$, $B_5^{(3)} = (.6214, .3543, .0321)$.

$C_4 : B_1^{(4)} = (.6247, .3407, .0321)$, $B_2^{(4)} = (.5523, .3332, .0321)$, $B_3^{(4)} = (.5376, .3734, .0377)$,

$\quad B_4^{(4)} = (.70114, .3017, .0321)$, $B_5^{(4)} = (.6955, .3248, .0377)$.

$C_5 : B_1^{(5)} = (.5316, .347, .0497)$, $B_2^{(5)} = (.7029, .3161, .0243)$, $B_3^{(5)} = (.5523, .3922, .0423)$,

$\quad B_4^{(5)} = (.5931, .3108, .0423)$, $B_5^{(5)} = (.6807, .2766, .0243)$.
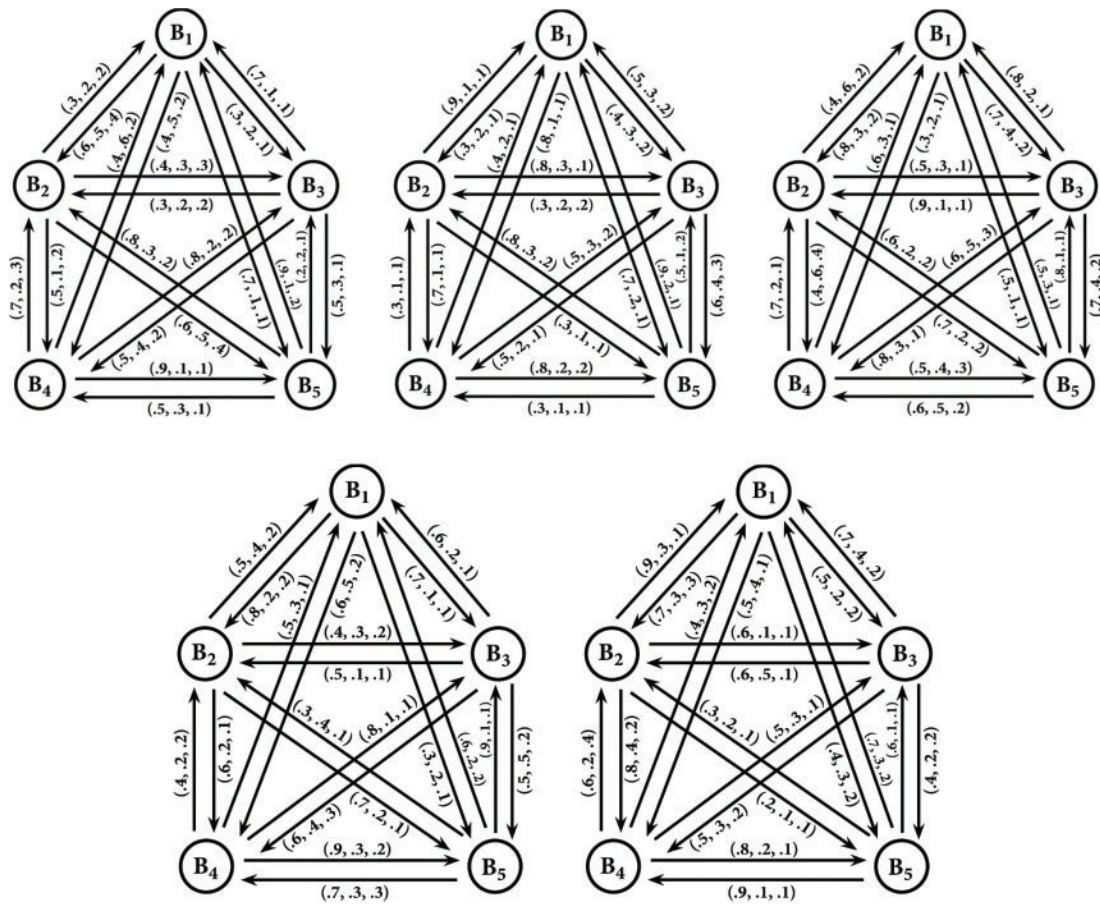
Figure 8: Pythagorean neutrosophic fuzzy directed graph for $M^{(k)}$ ($k = 1, 2, 3, 4, 5$)

Table 1: Pythagorean neutrosophic fuzzy relation of criteria C_1

M^1	B_1	B_2	B_3	B_4	B_5
B_1	(.46, .46, .46)	(.6, .5, .4)	(.3, .2, .1)	(.4, .5, .2)	(.7, .1, .1)
B_2	(.3, .2, .2)	(.46, .46, .46)	(.4, .3, .3)	(.5, .1, .2)	(.6, .5, .4)
B_3	(.7, .1, .1)	(.3, .2, .2)	(.46, .46, .46)	(.5, .4, .2)	(.5, .3, .1)
B_4	(.4, .6, .2)	(.7, .2, .3)	(.8, .2, .2)	(.46, .46, .46)	(.9, .1, .1)
B_5	(.9, .1, .2)	(.8, .3, .2)	(.2, .2, .1)	(.5, .3, .1)	(.46, .46, .46)

Table 2: Pythagorean neutrosophic fuzzy relation of criteria C_2

M^2	B_1	B_2	B_3	B_4	B_5
B_1	(.46, .46, .46)	(.3, .2, .1)	(.4, .3, .2)	(.8, .1, .1)	(.7, .2, .1)

(Continued)

Table 2: Continued

M²	B₁	B₂	B₃	B₄	B₅
B₂	(.9, .1, .1)	(.46, .46, .46)	(.8, .3, .1)	(.7, .1, .1)	(.3, .1, .1)
B₃	(.5, .3, .2)	(.3, .2, .2)	(.46, .46, .46)	(.5, .2, .1)	(.6, .4, .3)
B₄	(.4, .2, .1)	(.3, .1, .1)	(.5, .3, .2)	(.46, .46, .46)	(.8, .2, .2)
B₅	(.9, .2, .1)	(.8, .3, .2)	(.5, .1, .2)	(.3, .1, .1)	(.46, .46, .46)

Table 3: Pythagorean neutrosophic fuzzy relation of criteria C_3

M³	B₁	B₂	B₃	B₄	B₅
B₁	(.46, .46, .46)	(.8, .3, .2)	(.7, .4, .2)	(.3, .2, .1)	(.5, .1, .1)
B₂	(.4, .6, .2)	(.46, .46, .46)	(.5, .3, .1)	(.4, .6, .4)	(.7, .2, .2)
B₃	(.8, .2, .1)	(.9, .1, .1)	(.46, .46, .46)	(.8, .3, .1)	(.7, .4, .2)
B₄	(.6, .3, .1)	(.7, .2, .1)	(.6, .5, .3)	(.46, .46, .46)	(.5, .4, .3)
B₅	(.5, .3, .1)	(.6, .2, .2)	(.8, .1, .1)	(.6, .5, .2)	(.46, .46, .46)

Table 4: Pythagorean neutrosophic fuzzy relation of criteria C_4

M⁴	B₁	B₂	B₃	B₄	B₅
B₁	(.46, .46, .46)	(.8, .2, .2)	(.7, .1, .1)	(.6, .5, .2)	(.3, .2, .1)
B₂	(.5, .4, .2)	(.46, .46, .46)	(.4, .3, .2)	(.6, .2, .1)	(.7, .2, .1)
B₃	(.6, .2, .1)	(.5, .1, .1)	(.46, .46, .46)	(.6, .4, .3)	(.5, .5, .2)
B₄	(.5, .3, .1)	(.4, .2, .2)	(.8, .1, .1)	(.46, .46, .46)	(.9, .3, .2)
B₅	(.6, .2, .2)	(.3, .4, .1)	(.9, .1, .1)	(.7, .3, .3)	(.46, .46, .46)

Table 5: Pythagorean neutrosophic fuzzy relation of criteria C_5

M⁵	B₁	B₂	B₃	B₄	B₅
B₁	(.46, .46, .46)	(.7, .3, .3)	(.5, .2, .2)	(.5, .4, .1)	(.4, .3, .2)
B₂	(.9, .3, .1)	(.46, .46, .46)	(.6, .1, .1)	(.8, .4, .2)	(.2, .1, .1)
B₃	(.7, .4, .2)	(.6, .5, .1)	(.46, .46, .46)	(.5, .3, .2)	(.4, .2, .2)
B₄	(.4, .3, .2)	(.6, .2, .4)	(.5, .3, .1)	(.46, .46, .46)	(.8, .2, .1)
B₅	(.7, .3, .2)	(.3, .2, .1)	(.6, .1, .1)	(.9, .1, .1)	(.46, .46, .46)

Step 4: By using the score function, the score matrix is calculated in Table 6.

Step 5: Deriving the choice values of alternatives using the function, we get

$S(B_1) = .62702, S(B_2) = .6324, S(B_3) = .629, S(B_4) = .6192, S(B_5) = .691.$

Table 6: Score matrix of alternatives to concerning criteria

Criteria\value	B_1	B_2	B_3	B_4	B_5
C_1	.628	.579	.605	.686	.758
C_2	.6195	.656	.488	.488	.744
C_3	.634	.644	.644	.644	.648
C_4	.644	.618	.657	.657	.661
C_5	.6096	.665	.621	.621	.644

Step 6: The order of ranking obtained for the problem is

$B_5 > B_2 > B_3 > B_1 > B_4$.

Thus, the alternate with maximum value B_5 is chosen to be the optimal decision.

4.3 Comparative Analysis

The proposed model is compared with the decision-making method in [39] and is verified that the same ranking is obtained. Table 7 provides a comparison of both algorithms, showing the optimal alternative and results. Both algorithms provide the same optimum decision, as can be seen in the comparison table.

Table 7: Comparison analysis

Method	Ranking	Optimal alternative
Decision-making method in [39]	$B_5 > B_2 > B_3 > B_1 > B_4$	B_5
Our proposed method	$B_5 > B_2 > B_3 > B_1 > B_4$	B_5

5 Discussion

Graph theory is known for its vast applications in various fields most vitally in designing networking problems. In particular, numerous graph theoretical concepts have been introduced to model vagueness in networking problems. PN graphs, an extension of FGs, and a fusion of Pythagorean and Neutrosophic graphs have better flexibility to be applied in real-world problems. The article has initiated the concept of PN Multi Graph and PN Planar Graph employing the concept of PNGs. The concept of PN Dual Graph, isomorphism, weak and co-weak isomorphism has been explored for PN Planar Graphs and their results have been examined. An algorithm has been proposed using Pythagorean Neutrosophic fuzzy graphs with a numerical example for a real-life problem. The limitation of the set and study is that it is limited when it is compared with the newly proposed sets, but the advancements pave for the new concept of planar graphs in Pythagorean neutrosophic environment. The advantage of this proposed study is this set is more fuzzifying than the previous studies because of the set and their properties. This research can be extended further to investigate Interval- valued PNGs, bipolar PNGs, and their implementation in real-life situations.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.

[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, pp. 87–96, 1986.

[3] F. Smarandache, "A unifying field in logics," in *Neutrosophy: Neutrosophic Probability, Set and Logic*, Rehoboth: American Research Press, 1999.

[4] R. R. Yager, "Pythagorean fuzzy subsets," in *Proc. of the Joint IFSA World Congress and NAFIPS Annual Meeting*, Edmonton, AB, Canada, pp. 57–61, 2013.

[5] A. Kaufmann, "Introduction a la Theorie des Sour-Ensembles Flous; Masson etCie 1," *Compléments et Nouvelles Applications*, Masson, Paris, vol. 4, 1977.

[6] L. A. Zadeh, "Similarity relations and fuzzy orderings," *Information Sciences*, vol. 3, pp. 177–200, 2020.

[7] A. Rosenfeld, "Fuzzy graphs," In: L. A. Zadeh, K. S. Fu and M. Shimura (Eds.), *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, pp. 77–95, Cambridge, MA, USA: Academic Press, 1975.

[8] P. Bhattacharya, "Some remarks on fuzzy graphs," *Pattern Recognition Letters*, vol. 5, pp. 297–302, 1987.

[9] J. N. Mordeson and C. S. Peng, "Operations on fuzzy graphs," *Information Sciences*, vol. 79, pp. 159–170, 1994.

[10] A. Shannon and K. T. Atanassov, "A first step to a theory of intuitionistic fuzzy graphs," In: D. Lakov, (Ed.), *Proc. of the Fuzzy Based Expert Systems*, Sofia, Bulgaria, pp. 59–61, 1994.

[11] A. Shannon and K. T. Atanassov, "Intuitionistic fuzzy graphs from $\alpha-,\beta-,(\alpha,\beta)$-levels," *Notes Intuitionistic Fuzzy Sets*, vol. 1, pp. 32–35, 1995.

[12] R. Parvathi, M. G. Karunambigai and K. T. Atanassov, "Operations on intuitionistic fuzzy graphs," in *Proc. of the IEEE Int. Conf. on Fuzzy Systems*, Jeju Island, Korea, pp. 1396–1401, 2009.

[13] M. Akram and W. A. Dudek, "Intuitionistic fuzzy hypergraphs with applications," *Information Sciences*, vol. 218, pp. 182–193, 2013.

[14] M. Akram and B. Davvaz, "Strong intuitionistic fuzzy graphs," *Filomat*, vol. 26, pp. 177–196, 2012.

[15] S. Naz, S. Ashraf and M. Akram, "A novel approach to decision-making with pythagorean fuzzy information," *Mathematics*, vol. 6, pp. 1–28, 2018.

[16] M. Akram and S. Naz, "Energy of pythagorean fuzzy graphs with applications," *Mathematics*, vol. 6, pp. 136, 2018.

[17] M. Akram, A. Habib, F. Ilyas and J. M. Dar, "Specific types of pythagorean fuzzy graphs and application to decision-making," *Mathematical and Computational Applications*, vol. 23, pp. 42, 2018.

[18] M. Akram, J. M. Dar and S. Naz, "Certain graphs under pythagorean fuzzy environment," *Complex & Intelligent Systems*, vol. 5, pp. 127–144, 2019.

[19] N. Abdul-Jabbar, J. H. Naoom and E. H. Ouda, "Fuzzy dual graphs," *Al-Nahrain Journal of Science*, vol. 12, pp. 168–171, 2009.

[20] R. R. Yager, "On the theory of bags," *International Journal of General Systems*, vol. 13, pp. 23–37, 1986.

[21] A. Pal, S. Samanta and M. Pal, "Concept of fuzzy planar graphs," in *Proc. of the Science and Information Conf.*, London, UK, vol. 14, pp. 557–563, 2013.

[22] A. Pal, S. Samanta and M. Pal, "New concept of fuzzy planar graphs," *International Journal of Advanced Research in Artificial Intelligence*, vol. 3, pp. 52–59, 2014.

[23] N. Alshehri and M. Akram, "Intuitionistic fuzzy planar graphs," *Discrete Dynamics in Nature and Society*, vol. 2014, pp. 1–9, 2014.

[24] M. Akram, S. Samanta and M. Pal, "Application of bipolar fuzzy sets in planar graphs," *International Journal of Applied and Computational Mathematics*, vol. 3, pp. 773–785, 2017.

[25] R. Dhavaseelan, R. Vikramaprasad and V. Krishnaraj, "Certain types of neutrosophic graphs," *International Journal of Mathematical Sciences and Applications*, vol. 5, pp. 333–339, 2015.

[26] S. Broumi, M. Talea, A. Bakali and F. Smarandache, "Single-valued neutrosophic graphs," *Journal of New Theory*, vol. 10, pp. 86–101, 2016.

[27] M. Akram and S. Shahzadi, "Neutrosophic soft graphs with application," *Journal of Intelligent & Fuzzy Systems*, vol. 32, pp. 841–858, 2017.

[28] M. Akram and S. Shahzadi, "Representation of graphs using intuitionistic neutrosophic soft sets," *Journal of Mathematical Analysis*, vol. 7, pp. 1–23, 2016.

[29] M. Akram, S. Shahzadi and A. B. Saeid, "Single-valued neutrosophic hypergraphs," *TWMS Journal of Applied and Engineering Mathematics*, vol. 8, pp. 122–135, 2018.

[30] G. Muhiuddin, S. Hameed, A. Rasheed and U. Ahmad, "Cubic planar graph and its application to road network," *Mathematical Problems in Engineering*, vol. 2022, pp. 1–12, 2022.

[31] T. Mahapatra, S. Sahoo, G. Ghorai and M. Pal, "Interval valued m-polar fuzzy planar graph and its application," *Artificial Intelligence Review*, vol. 54, no. 3, pp. 1649–1675, 2021.

[32] R. Mahapatra, S. Samanta and M. Pal, "Generalized neutrosophic planar graphs and its application," *Journal of Applied Mathematics and Computing*, vol. 65, no. 1, pp. 693–712, 2021.

[33] G. Sur and Ö. Erkan, "Surface quality optimization of CFRP plates drilled with standard and step drill bits using TAGUCHI, TOPSIS and AHP method," *Engineering Computations*, vol. 38, no. 5, pp. 2163–2187, 2021.

[34] Ö. Erkan, D. Mustafa, I. Birahan and N. T. Ibrahim, "Selection of optimal machining conditions for the composite materials by using taguchi and GONNs," *Measurement*, vol. 48, pp. 306–313, 2014.

[35] D. Ajay and P. Chellamani, "Pythagorean neutrosophic fuzzy graphs," *International Journal of Neutrosophic Science*, vol. 11, pp. 108–114, 2020.

[36] R. Jansi, K. Mohana and F. Smarandache, "Correlation measure for pythagorean neutrosophic fuzzy sets with T and F as dependent neutrosophic components," *Neutrosophic Sets and Systems*, vol. 30, pp. 1–16, 2019.

[37] D. Ajay, P. Chellamani, G. Rajchakit, N. Boonsatit and P. Hammachukiattikul, "Regularity of pythagorean neutrosophic graphs with an illustration in MCDM," *AIMS Mathematics*, vol. 7, no. 5, pp. 9424–9442, 2022.

[38] D. Ajay, S. John Borg and P. Chellamani, "Domination in pythagorean neutrosophic graphs with an application in fuzzy intelligent decision making," in *Int. Conf. on Intelligent and Fuzzy Systems*, Cham,Turkey, Springer, pp. 667–675, 2022.

[39] P. Chellamani and D. Ajay, "Pythagorean neutrosophic dombi fuzzy graphs with an application to MCDM," *Neutrosophic Sets and Systems*, vol. 47, pp. 411–431, 2021.

[40] D. Ajay and P. Chellamani, "Operations on pythagorean neutrosophic graphs," *AIP Conference Proceedings*, vol. 2516, no. 1, pp. 200028, 2022.