Q-Smarandache Fuzzy Implicative Ideal of QSmarandache BH-Algebra

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Q-Smarandache Fuzzy Implicative Ideal of Q-Smarandache BH-Algebra

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Abstract
In this paper, The notions of Q-Smarandache fuzzy implicative ideal and Q- Smarandache fuzzy sub implicative ideal of a Q-Smarandache BH-Algebra introduced, examples are given, and related properties investigated the relationships among these notions and other types of Q-Smarandache fuzzy ideal of a Q-Smarandache BH-Algebra are Studies.

Keywords: BCK-algebra, BH-algebra, BH-algebra,Q- Smarandache a filter of Smarandache BH-algebra.

1 Introduction

The concept of fuzzy set was introduced by zadeh [1]in 1966,Y.Imai and K. Iseki introduced new notion called BCK-algebra[2]In 1991, applied it to the fundamental theory of groups. O.G. Xi [3]In 2005,Y.B.Jun introduced the notion of a Smarandache BCI-algebra, Smarandache ideal of a Smarandache BCI-algebra [4] in 2009, A.B. Saeid and A.Namdar, introduced the notion of a Q-Smarandache BCH-algebra and Q-Smarandache ideal of Q-Smarandache BCH-algebra [5] in 2013, H. H. Abbass and S. J. Mohammed introduced notions of the Smarandache BH-algebra, Q-Smarandache (ideal, closed ideal, fantastic ideal, completely closed ideal) of a Q-Smarandache BH-algebra[6] In 2014, H. H. Abbass and S. A. Neamah introduced the notions the (im-

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plicative, medial) BH-algebra and sub- implicative ideal of a BH-algebra[7]. in 2015, H.H.Abbass and H.K.Gatea introduced the notion Q-Smarandache implicative ideal of a Q-Smarandache BH-Algebra[8]. in this paper we introduce the notion of Q-Smarandache fuzzy implicative ideal and Q-Smarandache fuzzy sub implicative ideal of a Q-Smarandache BH-Algebra. Note in this paper, X is Q-Smarandache BH-Algebra.

2. Preliminaries

In this section, we give some basic concepts about a BCI-algeba, a BCK-algebra,a BCH-algebra, a BH-algeba,a Q-Smarandache BH-algebra, and a Q-Smarandach ideal of a BH-algebra.

Definition 2.1. [9]. A BCI-algebra is an algebra $(X, *, 0)$, where X is a nonempty set, $*$ is a binary operation and 0 is a constant, satisfying the following axioms: for all $x, y, z \in X$:

i. $((x * y) * (x * z)) * (z * y) = 0$,

ii. $(x * (x * y)) * y = 0$,

iii. $x * x = 0$,

iv. $x * y = 0$ and $y * x = 0$ imply $x = y$.

Definition 2.2. [3]. BCK-algebra is a BCI-algebra satisfying the axiom: $0 * x = 0$ for all $x \in X$.

Definition 2.3. [10]. A BH-algebra is a nonempty set X with a constant 0 and a binary operation * satisfying the following conditions:

i. $x * x = 0, \forall x \in X$.

ii. $x * y = 0$ and $y * x = 0$ imply $x = y, \forall x, y \in X$.

iii. $x * 0 = x, \forall x \in X$.

Remark 2.4. [10].

i. Every BCK-algebra is a BCI-algebra.

ii. Every BCK-algebra is a BCH/ BH-algebra.

Remark 2.5. [11] Let X and Y be BH-algebras. A mapping $f : X \to Y$ is called a homomorphism if $f(x * y) = f(x) * f(y) \ \forall x, y \in X$. A homomorphism f is called a monomorphism (resp., epimorphism) if it is injective (resp., surjective). For any

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homomorphism f : f : X $\rightarrow$Y the set $\{x\in X: f(x)=0'\}$ called the kernel of f, denoted by Ker(f), and the set $\{f(x):x\in X\}$ is called the image of f, denoted by Im(f) Notice that f(0) = 0.

Definition 2.6. [12]. BCK-algebra (X, *, 0) is said to be Bounded BCK-algebra satisfying the identity: $x *(y *x) = x \ \forall x, y\in X$.

Definition 2.7. [13] A BH-algebra X is called BH*-algebra if $(x*y)*x = 0, \forall x, y\in X$.

Definition 2.8. [6]. A Smarandache BH-algebra is defined to be a BH-algebra X in which there exists a proper subset Q of X such that

i. $0 \in Q$ and $|Q| \geq 2$.

ii. Q is a BCK-algebra under the operation of X.

Definition 2.9. [8] A Q-Smaradache BH-algebra is said to be a Q-Smaradache implicative BH- algebra if it satisfies the condition, $(x * (x * y)) * (y * x)=y * (y * x)$ $\forall x, y\in Q$.

Definition 2.10. [8] A Q-Smarandache BH-algebra X is called a Q-Smarandache medial BH-algebra if $x * (x *y) = y, \forall x, y\in Q$.

Definition 2.11. [6]. A nonempty subset I of X is called a Q-Smarandache ideal of X, denoted by a Q-S.I of X if it satisfies:

(J_1) $0 \in I$

(J_2) $\forall y \in I$ and $x*y \in I \Rightarrow x \in I, \ \forall x \in Q$.

Definition 2.12. [8]. A Q-Smarandache ideal I of X is called a Q-Smarandache implicative ideal of X, denoted by a Q-S.I.I of X if:

$(x*(y*x))*z \in I$ and $z \in I$ imply $x\in I, \forall x, y \in Q$.

Definition 2.13. [8]. A nonempty subset I of X is called a Q-Smarandache P-ideal of X if satisfies (J_1) and :

(J_3) $(x *z) *(y * z) \in I$ and $y \in I$ imply $x \in I, \forall x, z \in Q$.

Definition 2.14. [2]. A fuzzy set A in a BH-algebra X is said to be a fuzzy subalgebra of X if it satisfies: $A(x*y) \geq \min \{A(x) , A(y) \}, \forall x ,y \in X$.

Definition 2.15. [14] A fuzzy subset A of a BH-algebra X is said to be a fuzzy ideal if and only if:

(I_1) $A(0) \geq A(x), \forall x\in X$.

(I_2) $A(x) \geq \min\{A(x*y), A(y)\}, \forall x, y\in X$.

Definition 2.16. [15]. A fuzzy subset A of a BH-algebra X is called a fuzzy implicative ideal of X, denoted by a F.I.I if it satisfies$(I_1)$and

(I_3) $A(x) \geq \min\{A((x*(y*x))*z), A(z)\}, \forall x, y, z \in X$.

Definition 2.17. [16]. A fuzzy subset A of a BH-algebra X is called a fuzzy sub implicative ideal of X, denoted by a F.S.I.I if it satisfies (I_1) and

(I_4) $A(y * (y * x)) \geq \min\{A(((x * (x * y)) * (y * x) *z), A(z)\}, \forall x, y, z \in X$.

Definition 2.18. [17]. Let A be a fuzzy set in, $\forall \alpha \in [0, 1]$, the set $A_\alpha = \{ x \in X, A(x) \geq \alpha \}$ is called a level subset of A.

Note that, A_α is a subset of X in the ordinary sense.

Definition 2.19. [17].
Let X and Y be any two sets, A be any fuzzy set in X and f: X → Y be any function. The set $f^{-1}(y) = \{x \in X \mid f(x) = y\}, \forall y \in Y$. The fuzzy set B in Y defined by B(y) $= \begin{cases} \sup\{A(x) \mid x \in f^{-1}(y)\}; \text{ if } & f^{-1}(y) \neq \emptyset \\ 0 \quad ; & \text{otherwise} \end{cases}$, $\forall y \in Y$, is called the image of A under f and is denoted by f(A).

Definition 2.18. [12].

Let X and Y be any two sets f : X → Y be any function and B be any fuzzy set in f(A). The fuzzy set A in X defined by: $A(x) = B(f(x)) \ \forall x \in X$ is called the primage of B under f and is denoted by $f^{-1}(B)$

Definition 2.21. [6]. A fuzzy subset A of X is said to be a Q-Smarandache fuzzy ideal of X, denoted by a Q-S.F.I of X*:*

(F_1) $A(0) \geq A(x)$, $\forall x \in X$

(F_2) $A(x) \geq \min\{A(x*y))\ A(y)\}$, $\forall x \in Q$, $y \in X$

3. Main results

In this section, we introduce the concepts of a Q-Smarandache fuzzy implicative ideal and Q-Smarandache fuzzy sub implicative ideal of a Q-Smarandache BH-algebra, also we study some properties of it with examples.

Definition 3.1. A fuzzy subset A of X is called a Q-Smarandache fuzzy implicative ideal of X, denoted by a Q-S.F.I.I of X if it satisfies (F_1) and,

(F_3) $A(x) \geq \min\{A(((x * (y * x)) * z)), A(z)\}$, for all $x, y \in Q$, $z \in X$.

Example 3.2. .

Consider X = {0, 1, 2} with binary operation defined by the following table:

*	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

where Q = {0, 2} is a BCK-algebra. The fuzzy subset A defined by A(0) = 0.7, A(1) = 0.5 and A(2) = 0.2 by calculation we knew that A is Q-S.F.I.I.

Proposition 3.3. Every Q-S.F.I.I is Q-S.F.I. of X.

Proof. Let A be a Q-S.F.I.I, To prove that A is Q-S.F.I. by Definition (3.1) the condition (F_1) is satisfied .Now let x, ∈Q and y ∈ X. we have A(x) ≥ min{A((x _ (x * x)) *y),A(y)},(since A is a Q-S.F.I.I) it follows that A(x) ≥ min{A((x * (0)) * y), A(y)},(since x * x = 0, ∀x, ∈Q) implies that A(x) ≥ min{A((x * y),A(y)}(since x * 0 = x, ∀x ∈Q). Hence A is Q-S.F.I of X.

Remark 3.4. A Q-S.F.I of X may not be a Q-S.F.I.I of X as in the following example.

Example 3.5. Consider X = {0, 1, 2, 3} with binary operation "*" defined by the following table:

*	0	1	2	3
0	0	0	2	3
1	1	0	2	2
2	2	1	0	1
3	3	2	3	0

where Q = {0, 1} is a BCK-algebra. The fuzzy subset A defined by A(0) = A(2) = 0.5 and A(1) = A(3) = 0.2 is Q-S.F.I of X but it is not a Q-S.F.I.I of X. Since if x = 1, y = 0, z = 2,then

A(1) < min{A((1 * (0 * 1)) * 2), A(2)}.

Theorem 3.6. Let A be a Q-S.F.I of X. Then A is a Q-S.F.I.I of X if and only if the level subset A_α is a Q-S.I.I of X, ∀ α ∈ [0, A(0)],such that A(0) = $Sup_{x \in X} A(x)$.

Proof. Let A be a Q-S.F.I.I of X. To prove A_α is a Q-S.I.I of X.[it is clear that A(0) ≥ α]. So 0 ∈ A_α. Hence A_α satisfies I1 .Now let x, y ∈ Q, z ∈ X such that ((x ∗ (y ∗ x)) ∗ z) ∈ A_α and z ∈ A_α it follows that A((x ∗ (y ∗ x)) ∗ z) ≥ α and A(z) ≥ α thus min{A((x ∗ (y ∗ x)) ∗ z), A(z)} ≥ α. But A(x) ≥ min{A((x ∗ (y ∗ x)) ∗ z), A(z)} [Since A is a Q-S.F.I.I of X. By definition 3.1(F3)] So A(x) ≥ α ⇒ x ∈ A_α Therefore, A_α is a Q-S.I.I of X.

Conversely,

Let A_α be a Q-S.I.I. of X, ∀ α ∈ [0,A(0)] and α = $Sup_{x \in X} A(x)$. To prove that A is a Q-S.F.I.I of X. 0 ∈ A_α .[Since A_α is a Q-S.I.I. of X].

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imply A(0 ≥ α we get A(0) ≥ A(x). Let x, y ∈ Q, z ∈ X such that min{A((x ∗ (y ∗ x)) ∗ z), A(z)} = α then A((x ∗ (y ∗ x)) ∗ z) ≥ α and A(z) ≥ α

it follows that ((x ∗ (y ∗ x)) ∗ z) ∈ A_α and z ∈ A_α thus x ∈ A_α[Since A_α be an Q-S.I.I of X] imply A(x) ≥ α we get A(x) ≥ min{A(((x ∗ (y ∗ x)) ∗ z)),A(z)}.

Therefore, A is a **Q-S.F.I.I** of X.

Corollary 3.6.1. A fuzzy subset A is a Q-S.F.I.I of X if and only if the set X_A is an Q-S.I.I of X, where X_A = { x∈X | A(x) = A(0) }

Proof. Let A be a Q-S.F.I.I of X. To prove XA is a Q-S.I.I of X.

i .If x = 0 then A(0) = A(0) ⟹ 0 ∈X_A

ii: Let x, y ∈Q, z∈X such that (x*(y*x)) *z∈X_A and z∈X_A .

follows that A((x*(y*x))*z) = A(0) and A(z) = A(0) . we have A(x) ≥ min {A((x*(y*x)) *z) , A(z) } = min {A(0) ,A(0)}[Since A is a **Q-S.F.I.I** of X] it follows that

A(x) ≥A(0) Hence A(x) = A(0) [Since A is a Q-S.F.I.I of X, A(x) ≥A(0)] we get x∈X_A. Therefore , X_A is a **Q-S.I.I** of X

Conversely,

Let X_A be a **Q-S.I.I** of X . To prove A is a **Q-S.F.I.I** of X .

Since X_A = $A_{A(0)}$

Therefore, A is a. Q-S.F.I.I of X [By Theorem 3.6].

Proposition 3.7. Let A be a fuzzy subset of X defined by

$$A\,(x) = \begin{cases} \alpha_1\,; & x \in X_A \\ \alpha_2\,; & \text{otherwies}\,, \end{cases}$$ where α1, α_2 ∈ [0, 1] such that $\alpha_1 > \alpha_2$

Then A is a Q-S.F.I.I of X if and only if XA is an Q-S.I.I of X.

Proof. Let A be a Q-S.F.I.I of X. To prove X_A is an Q-S.I.I of X.

i. A(0) = α_1 ⟹ 0 ∈ X_A[Since A(0) ≥ A(x); ∀x ∈ X. By definition 3.1(F1)].

ii Let x, y ∈ Q, z ∈ X_A such that (x * (y * x)) * z ∈ X_A and z ∈ X_A.

we obtain A((x * (y * x)) * z) = A(0) = α_1 and A(z) = A(0) = α_1 it follows that A(x) ≥ min{A((x * (y * x)) * z),A(z)} = α_1 [Since A is a Q-S.F.I.I of X,

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by definition3.1(F_1)], Thus $A(x) = \alpha_1$ we get $x \in X_A$. Hence X_A is a Q-S.I.I of X.

Conversely, Let X_A be an Q-S.I.I of X. To prove A is a Q-S.F.I.I of X.

i . Since $0 \in X_A$, then $A(0) = \alpha_1 \Rightarrow A(0) = \alpha_1 \geq A(x)$. we get $A(0) \geq A(x)$, $\forall x \in X$.

ii . Let $x, y \in Q$, $z \in X$. Then we have four cases:

Case 1: If $(x * (y * x)) * z \in X_A$ and $z \in X_A$. it follows that $x \in X_A$.[Since XA is an Q-S.I.I of X]. we get $A((x * (y * x)) * z) = A(z) = A(x) = \alpha1$. Hence $A(x) \geq \min\{A((x * (y * x)) * z), A(z)\}$.

Case 2: If $(x * (y * x)) * z \in X_A$ and $z \notin X_A$ it follows that $A((x * (y * x)) * z) = \alpha_1$ and $A(z) = \alpha_2$. we get $\min\{A((x * (y * x)) * z), A(z)\} = \alpha_2$. Hence $A(x) \geq \min\{A((x * (y * x)) * z), A(z)\}$.

Case 3: If $(x * (y * x)) * z \notin X_A$ and $z \in X_A$ it follows that $\{A((x * (y * x)) * z) = \alpha2$ and $A(z) = \alpha1$. we get $\min\{A((x * (y * x)) * z), A(z)\} = \alpha_2$. Hence $A(x) \geq \min\{A((x * (y * x)) * z), A(z)\}$.

Case 4: If $(x * (y * x)) * z \notin X_A$ and $z \notin X_A$ it follows that $A(x * (y * x)) * z) = A(z) = \alpha_2$. we get $\min\{A((x * (y * x)) * z), A(z)\} = \alpha_2$. Hence $A(x) \geq \min\{A((x * (y * x)) * z), A(z)\}$. Therefore, A is a Q-S.F.I.I of X.

Remark 3.8. Let A be a fuzzy subset of X and $w \in X$. The set $\{x \in X | A(w) \leq A(x)\}$ is denoted by $\uparrow A(w)$.

Proposition 3.9. Let A be a **Q-S.F.I** of X and $w \in X$. If A satisfies the condition $\forall x, y \in Q$ $A(x) \geq A(x * (y * x)$ (b_2). Then $\uparrow A(w)$ is a **Q-S.I.I** of X.

Proof. Let A be a Q-S.F.I of X. Then $A(0) \geq A(x)$, $\forall x \in X$ [By Definition 2.21(F_1)].

it follows that $A(0) \geq A(w)$ [Since $w \in X$] we get $0 \in \uparrow A(w)$

Now, Let $x, y \in Q$, $z \in X$ such that $(((x*(y*x)) *z)) \in \uparrow A(w)$ and $z \in \uparrow A(w)$

thus $A(w) \leq A((x*(y*x)) *z)$ and $A(w) \leq A(z)$ implies that

$A(w) \leq \min\{A(((x * (y * x)) * z)), A(z)\} \leq A(x * (y * x))$[Since A is a **Q-S.F.I** of X] But $A(x*(y*x)) \leq A(x)$. [By (b_2)]. we get $A(w) \leq A(x)$. Hence $x \in \uparrow A(w)$ Therefore, A(w) is a Q-S.I.I of X.

Proposition 3.10. Let $w \in X$. If A is a Q-S.F.I.I of X, then $\uparrow A(w)$ is a Q-S.I.I of X.

Proof. Let A be a Q-S.F.I of X. Then $A(0) \geq A(x)$, $\forall x \in X$ it follows that $A(0) \geq A(w)$[Since $w \in X$]. Hence $0 \in \uparrow A(w)$. Let $x, y \in Q$, $z \in X$ such that $(x*(x*y))*z \in \uparrow A(w)$ and $z \in \uparrow A(w)$ Then $A(w) \leq A((x*(y*x))*z)$ and $A(w) \leq A(z)$ it follows

$A(w) \leq \min\{A((x*(y*x))*z), A(z)\}$ But $\min A((x*(y*x))*z), A(z) \leq A(x)$ [By Definition 2.21(F_2)] we get $A(w) \leq A(x)$. Hence $x \in \uparrow A(w)$. Therefore, $\uparrow A(w)$ is a **Q-S.I.I** of X.

Proposition 3.11.

Let $\{A_i / I \in r\}$ be a family of **Q-S.F.I.I** of X . Then $\bigcap_{\alpha \in \lambda} A_i$ is a **Q-S.F.I.I** of X.

Let $\{Ai / i \in r\}$ be a family of **Q-S.F.I.I** of X

i. Let $x \in X$. Then

$$\bigcap_{i \in \Gamma} A_i(0) = \inf\{A_i(0) | i \in r\}) \geq \inf\{A_i(x) | i \in r\} = \bigcap_{i \in \Gamma} A_i(x)$$

(ii). Let $x, y \in Q$, $z \in X$. Then , we have

$$\bigcap_{i \in \Gamma} A_i(x) = \inf\{A_i(x) | i \in r\} \geq \inf\{\min\{A_i((x * (y * x)) * z), A_i(z) | i \in \Gamma\}\}$$

$$= \min\{\inf\{A_i((x*(y*x))*z), A_i(z) | i \in r\}$$

$$= \min\{\inf\{A_i((x*(y*x))*z)) | i \in r\}, \inf\{A_i(z)\}\}$$

$$= \min\{\bigcap_{i \in \Gamma} A_i(x) \quad {}_i((x*(y*x))*z) \mid i \in \Gamma\}, \bigcap_{i \in \Gamma} A_i(z) \mid \alpha \in \Gamma\}\}$$

$$\Rightarrow \bigcap_{i \in \Gamma} A_i(x) \geq \min\{\{\bigcap_{i \in \Gamma} A_i(x*(y*x))*z) |\}, \{\bigcap_{\alpha \in \lambda} A_i(z)\}\}$$

Therefore, $\bigcap_{i \in \Gamma} A_i(x)$ is a **Q-S.F.I.I** of X.

Remark 3.12. The union of a Q-S.F.I.I of X may not be a Q-S.F.I.I of X as in The following example.

Example 3.13. Consider X = {0, 1, 2, 3, 4, 5} with binary operation ″*″ defined by the following table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1

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2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Where Q={0,2} is a BCK-algebra . The fuzzy subset A,B defined by

A(0) = A(1) = 0.9, A(2) = A(3) = A(4) = A(5) = 0.4 and

B(0) = B(5) = 0.9, B(1) = B(2) = B(3) = B(4) = 0.4 are two Q-S.F.I.I, but

$A\cup B(0)= A\cup B(1)= A\cup B(5)= 0.9$ and

$A\cup B(2)= A\cup B(3)= A\cup B(4)= 0.4$

is not a Q-S.F.I.I of X, Since

$(A\cup B)(2) = 0.4 < \min \{(A\cup B)((2*(0*2))*5), (A\cup B)(5) \}$

Proposition 3.14.

Let $\{A_i / i\in\Gamma\}$ be a chain of **Q-S.F.I.I** of X .Then $\bigcup_{i\in\Gamma} A_i (x)$ is a **Q-S.F.I.I** of X.

Proof .

Let $\{A_i | i\in\Gamma\}$ be a chain of Q-S.F.I.I of X

i: Let $x\in X$. Then

$$\bigcup_{i\in\Gamma} A_i (0) = \sup \{ A_i(0) | i\in \Gamma \}) \geq \sup \{ A_i(x) | i\in \Gamma \} = \bigcup_{i\in\Gamma} A_i (x)$$

[Since A_i is a **Q-S.F.I.I** of X **,** $i\in\Gamma$, by Definition 3.1(i)]

$$\Rightarrow \bigcup_{i\in\Gamma} A_i (0) \geq \bigcup_{i\in\Gamma} A_i (x)$$

ii: Let $x, y \in Q$, $z\in X$. Then , we have

$$\bigcup_{i\in\Gamma} A_i (x) = \sup \{ A_i(x) | i\in \Gamma \}) \geq \sup \{\min\{ A_i(x*(y*x)*z), A_i (z) | i\in \Gamma \}\}$$

[Since Ai is a Q-S.F.I.I of X , $i\in\lambda$ by Definition 3.1(i)]

$\Rightarrow = \min\{\sup\{A_i (x*(y*x)*z), A_i(z) | i \in \Gamma\}\}$[since A_i is a chain, $i \in \Gamma$]

$\Rightarrow = \min\{\sup\{A_i(x*(y*x)*z) | i \in \Gamma\}, \sup\{ A_i(z)) | i \in \Gamma \}$

$\Rightarrow \quad = \min \{ \bigcup_{i\in\Gamma} A_i (x*(y*x)*z) \mid i\in \Gamma \},\{ \bigcup_{i\in\Gamma} A_i(z) \mid i\in \Gamma \}\}$

$$\Rightarrow \bigcup_{i\in\Gamma} A_i(x) \geq \min \{\bigcup_{i\in\Gamma} A_i(x*(y*x)*z) \mid i\in \Gamma \},\{ \bigcup_{i\in\Gamma} A_i(z) \mid i\in \Gamma \}\}$$

Therefore, $\bigcup_{i\in\Gamma} A_i(x)$ is a Q-S.F.I.I of X.

Theorem 3.15. Let A be a Q-S.F.I of X . Then A is a **Q-S.F.I.I** of X if and only if A satisfies the following inequality : ∀x , y∈Q A(x) ≥A(x*(y*x)) (b_2) .

Proof. Let A be a Q-S.F.I.I of X and x, y ∈ Q then

A(x) ≥ min{A((x *(y*x)) *0), A(0) } it follows that ≥ min {A(x*(x*y)) , A(0) }[since x*(y*x)*0 = x*(y*x)] Therefore the condition (b1) is satisfied.
Conversely,

Let A be a Q-S.F.I of X . Then (F1) satisfied.

Now, let x, y∈Q , then A(x*(y*x))≥ min {A((x*(x*y)) *z) , A(z) } [Since A is a Q - S.F.I of X. By (2.21)(F2)] we have A(x) ≥min {A((x*(y*x)) *z) , A(z) }. Hence, A is a Q-S.F.I.I of X.

Definition 3.16. A fuzzy subset A of X is called a Q-Smarandache fuzzy P-ideal of X, denoted by a Q-S.F.P.I of X if satisfies (F1) and :

(F_4) A(x) ≥ min {A((x*z)*(y*z)) , A(y)}, for all x, z∈ Q, y∈X.

Example 3.17. Consider X = {0, 1, 2, 3} with binary operation ″*″ defined by the following table:

*	0	1	2	3
0	0	0	2	0
1	1	0	1	2
2	2	2	0	1
3	3	2	2	0

where Q = {0, 1} is a BCK-algebra. The fuzzy subset A defined by

A(0) = A(1) = A(2) = 0.8 and A(3) = 0.2 is a Q-S.F.P.I of X.

Theorem 3.18. Every Q-S.F.P.I is a Q-S.F.I of X.

Proof. Let A be a Q-S.F.P.I of X. Then(F1) satisfied.

Now, let x, z ∈ Q and y ∈ X, z = 0 in (F_4) we get:

A(x) ≥ min{A((x ∗ 0) ∗ (y ∗ 0)),A(y)}[Since X is a Q-Smrandache BH-algebra x ∗ 0 = x]. A(x) ≥ min{A(x ∗ y), A(y)}

Therefore, A is **Q-S.F.I** of X.

Theorem 3.19. Every Q-S.F.P.I is a Q-S.F.I.I of X.

Proof. Let A be a Q-S.F.P.I of X. Then.(F1) satisfied[By definition 3.16(F1)]. And Let a, c, x, y ∈ Q and d ∈ X .Then

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$A(a) \geq \min\{A((a * c) * (d * c)), A(d)\}$ [By (F_4)]. Put $a = x$, $d = 0$, $c = y * x$, we get

$A(x) \geq \min\{A((x * (y * x)) * (0 * (y * x))), A(0)\}$

$= \min\{A((x * (y * x)) * 0), A(0)\}$ [Since Q is BCK $0 * x = 0$]

$= \min\{A(x * (y * x)), A(0)\}$ [Since Q is BCK ; $x * 0 = x$]

$= A(x * (y * x))$ [Since $A(0) \geq A(x)$, $\forall x \in X$]

Therefore, A is a **Q-S.F.I.I** of X [by Theorem 3.15]

Remark 3.20. In the following example, we see that the converse of Theorem (3.21) may not be true in general.

Example 3.21. Consider X ={0,1, 2} with binary operation ″*″ defined by table where Q = {0,2} is a BCK-algebra. The fuzzy subset A defined by

A(0) = 0.7, A(1) = 0.5 and A(2) = 0.2

Then A is Q-S.F.I.I of X, but A is not a Q-S.F.P.I of X, since if x = 2,y = 1,z= 2,

then $A(2)=0.2 \leq \min\{A((2*2)*(1*2)), A(1)\}=0.5$

Theorem 3.22. Let A be a Q-S.F.I , such that Q is a bounded BCK- algebra . Then A is a Q-S.F.I.I of X.

Proof. It's clear that $A(0) \geq A(X)$. $\forall x \in X$

Now, let $x, y \in Q$ and $z \in X$, Then
$A(x*(y*x)) \geq \min\{A((x*(y*x))*z), A(z)\}$, [Since A is a Q-S.F.I of X, by 2.21(F_2)] implies that $A(x) \geq \min\{A((x*(y*x))*z, A(z)\}$[Since Q is bounded BCK- algebra, by 2.6] Therefore, A is a Q-S.F.I.I of X

Definition 3.23. A fuzzy subset A of X is called a Q-Smarandache fuzzy sub-implicative ideal of X, denoted by (a Q-S.F.S.I.I) of X if it satisfies: (F1) and

(F_5) $A(y * (y * x)) \geq \min\{A(((x * (x * y)) * (y * x)) * z), A(z)\}$ for all $x, y \in Q$, $z \in X$

<u>Example 3.24.</u>

Consider X = {0, 1, 2, 3} with binary operation ″∗″ defined by the following table:

*	0	1	2	3
0	0	0	0	3
1	1	0	1	3
2	2	2	0	3
3	3	3	3	0

Where Q={0,2} is a BCK-algebra . The fuzzy subset A is defined by

A(0) = A(1) = 0.9 and A(2) = A(3) = 0.3 it easy to check that A is Q-S.F.S .I.I of X

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Proposition 3.25. Every Q-S.F.S.I.I is Q-S.F.I. of X.

Proof. Let A be a Q-S.F.S.I.I. Then (F1) it is satisfied. Now let $x \in Q$ and $y \in X$.

$A(x) = A(x * 0) = A(x * (x * x)) \geq \min\{A(((x * (x * x)) * (x * x)) * y), A(y)\}$

[Since A is a **Q-S.F.S.I.I** of X, by Definition 3.23 (F_5)]

$= \min\{A(((x * 0) * 0) * y), A(y)\}$ [Since , $x * x = 0$]

$= \min\{A((x * 0) * y), A(y)\}$ [Since , $x * 0 = x$]

$= \min\{A(x * y), A(y)\}$ [Since , $x * 0 = x$]Thus $A(x) \geq \min\{A(x * y), A(y)\}$

Therefore, A is a Q-S.F.I of X **.**

Proposition 3.26. Let A be a Q-S.F.I of X. Then A is a Q-S.F.S.I.I of X if and only if A satisfies the following inequality:$\forall x, y \in Q$ $A(y * (y * x)) \geq A((x * (x * y)) * (y * x))$ (b3) .

Proof. Let A be a Q-S.F.S.I.I. and $x, y \in Q$, Then

$A(y * (y * x)) \geq \min\{A(((x * (x * y)) * (y * x)) * 0), A(0)\} = \min\{A((x * (x * y)) * (y * x)), A(0)\}$ [Since Q is BCK; $x * 0 = x$] it follows that $= A((x * (x * y)) * (y * x))$ [Since A is a Q-S.F.I of X ,$A(0) \geq A(x)$].Then the condition (b4) is satisfied.

Conversely,

Let A be a Q-S.F.I. Then (F_1) satisfied , Let $x, y \in Q$. Then

$A((x * (x * y)) * (y * x)) \geq \min\{A(((x * (x * y)) * (y * x)) * z), A(z)\}$ [Since A is a Q-S.F.I of X by Definition 2.21] By (b3) we have $A(y * (y * x)) \geq A((x * (x * y)) * (y * x))$ implies that $A(y * (y * x)) \geq \min\{A(((x * (x * y)) * (y * x)) * z), A(z)\}$

Therefore, A is Q-S.F.S.I.I of X.

Theorem 3.27. Let X be a Q-Smarandache implicative BH-algebra. Then every Q-S.F.I of X is a Q-S.F.S.I.I of X.

Proof. Let A be a Q-S.F.I of X. Then (F1) satisfied[By (2.21)] and let $x, y \in Q$. Then $A(y * (y * x)) \geq \min\{A((y * (y * x)) * z), A(z)\}$ (since A is a Q-S.F.I.I) we get $\geq \min\{A(((x * (x * y)) * (y * x)) * z)), A(z$ [since $x * x = 0, \forall x \in Q$] namely $A(y * (y * x)) \geq \min\{A(((x * (x * y)) * (y * x)) * z), A(z)\}$. (since $x * 0 = x, \forall x \in Q$])

Therefore, A is a Q-S.F.S.I.I of X.

Corollary 3.27.2. Let X be a Q-Smarandache implicative BH-algebra and A be Q-S.F.I.I of X . Then A is a Q-S.F.S.I.I of X .

Proof. Directly from proposition 3.3 and Theorem 3.27

Proposition 3.28. Let X be a Q-Smarandache medial BH-algebra and A be a Q-S.F.I of X .Then A is a Q-S.F.S.I.I of X .

Proof. Let A be a Q-S.F.I of X. Then (F1) satisfied [By 2.21] and let x, y∈ Q, and z ∈ X. Then A((x ∗ (x ∗ y)) ∗ (y ∗ x)) ≥ min{A(((x ∗ (x ∗ y)) ∗ (y ∗ x)) ∗ z),A(z)}. we get A((y ∗ (y ∗ x)) ≥ min{A(((x ∗ (x ∗ y)) ∗ (y ∗ x)) ∗ z),A(z)} [Since X is a Q-Smarandache medial BH-algebra]. Hence A is a Q-S.F.S.I.I of X .

Corollary 3.28.3. Let X be an Q-Smarandache medial BH-algebra and A be a Q.S.F.I.I of X .Then A is a Q.S.F.S.I.I of X .

Proof. Directly from proposition 3.3 and proposition 3.28.

Theorem 3.29. Let X be an Q-Smarandache medial BH-algebra and A be Q.S.F.S.I.I satisfies the condition ∀ x, y ∈ Q, A((x ∗(x ∗ y)) ∗ (y ∗ x)) ≥ A(x ∗ (y ∗ x)) (b_4). Then A is Q.S.F.I.I.

Proof. Let A be a Q-S.F.S.I.I of X . Then（F1） is satisfied

Now let x, y ∈ Q and z∈ X. Then By (b4) we have A((x ∗(x ∗ y))∗(y ∗ x)) ≥ A(x∗(y ∗ x)). Thus, A(y*(y*x)) ≥ min{A(((x*(x*y))*(y*x))*z), A(z)})}[Since A is a Q-S.F.S.I.I of X] if z= 0, then A(y ∗ (y ∗ x)) ≥ min{A(((x ∗ (x ∗ y)) ∗ (y ∗ x)) ∗ 0), A(0)} we obtain A(y ∗ (y ∗ x)) ≥ min{A((x ∗ (x ∗ y)) ∗ (y ∗ x)), A(0)} [Since Q is a BCK-algebra, x * 0 = x]. It follows that A(y ∗ (y ∗ x)) ≥ A((x ∗ (x ∗ y)) ∗ (y ∗ x)) By (b_4), We have A((x *(x * y)) * (y * x)) ≥ A(x * (y *x)).Thus A(y ∗ (y ∗ x)) ≥ A(x ∗ (y ∗ x)), But A(x) = A(x * (y * x))[Since X is a medial, y * (y * x) = x].So, A(x) ≥ A(y * (y * x)) Hence, A is a Q-S.F.I.I of X [By 3.15(b_2)]

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