

Smarandache Quasigroups

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Abstract In this paper, we have introduced Smarandache quasigroups which are Smarandache non-associative structures. W.B.Kandasamy [2] has studied Smarandache groupoids and Smarandache semigroups etc. Substructure of Smarandache quasigroups are also studied.

Keywords Quasigroup; Smarandache Quasigroup.

1. Introduction

W.B.Kandasamy has already defined and studied Smarandache groupoids, Smarandache semigroups etc. A quasigroup is a groupoid whose composition table is LATIN SQUARE. We define Smarandache quasigroup as a quasigroup which contains a group properly.

2. Preliminaries

Definition 2.1. A groupoid S such that for all $a, b \in S$ there exist unique $x, y \in S$ such that $ax = b$ and $ya = b$ is called a quasigroup.

Thus a quasigroup does not have an identity element and it is also non-associative.

Example 2.1. Here is a quasigroup that is not a loop.

| | | | | | |
|---|---|---|---|---|---|
| * | 1 | 2 | 3 | 4 | 5 |
| 1 | 3 | 1 | 4 | 2 | 5 |
| 2 | 5 | 2 | 3 | 1 | 4 |
| 3 | 1 | 4 | 2 | 5 | 3 |
| 4 | 4 | 5 | 1 | 3 | 2 |
| 5 | 2 | 3 | 5 | 4 | 1 |

We note that the definition of quasigroup Q forces it to have a property that every element of Q appears exactly once in every row and column of its operation tables. Such a table is called a LATIN SQUARE. Thus, quasigroup is precisely a groupoid whose multiplication table is a LATIN SQUARE.

Definition 2.2. If a quasigroup $(Q, *)$ contains a group $(G, *)$ properly then the quasigroup is said to be Smarandache quasigroup.

A Smarandache quasigroup is also denoted by S-quasigroup.

Example 2.2. Let Q be a quasigroup defined by the following table;

| | | | | | |
|-------|-------|-------|-------|-------|-------|
| * | a_0 | a_1 | a_2 | a_3 | a_4 |
| a_0 | a_0 | a_1 | a_3 | a_4 | a_2 |
| a_1 | a_1 | a_0 | a_2 | a_3 | a_4 |
| a_2 | a_3 | a_4 | a_1 | a_2 | a_0 |
| a_3 | a_4 | a_2 | a_0 | a_1 | a_3 |
| a_4 | a_2 | a_3 | a_4 | a_0 | a_1 |

Clearly, $A = \{a_0, a_1\}$ is a group *w.r.t.* $*$ which is a proper subset of Q . Therefore Q is a Smarandache quasigroup.

Definition 2.3. A quasigroup Q is idempotent if every element x in Q satisfies $x * x = x$.

Theorem 2.1. If a quasigroup contains a Smarandache quasigroup then the quasigroup is a Smarandache quasigroup.

Proof. Follows from definition of Smarandache quasigroup.

Example 2.3. $(Q, *)$ defined by the following table is a quasigroup.

| | | | | |
|---|---|---|---|---|
| * | 1 | 2 | 3 | 4 |
| 1 | 1 | 3 | 4 | 2 |
| 2 | 4 | 2 | 1 | 3 |
| 3 | 2 | 4 | 3 | 1 |
| 4 | 3 | 1 | 2 | 4 |

$(Q, *)$ is an idempotent quasigroup.

Definition 2.4. An element x in a quasigroup Q is called idempotent if $x.x = x$.

Consider a quasigroup;

| | | | | | |
|---|---|---|---|---|---|
| * | 1 | 2 | 3 | 4 | 5 |
| 1 | 3 | 1 | 4 | 2 | 5 |
| 2 | 5 | 2 | 3 | 1 | 4 |
| 3 | 1 | 4 | 2 | 5 | 3 |
| 4 | 4 | 5 | 1 | 3 | 2 |
| 5 | 2 | 3 | 5 | 4 | 1 |

Here 2 is an idempotent element.

Example 2.4. The smallest quasigroup which is neither a group nor a loop is a quasigroup of order 3 as given by the following table;

| | | | |
|-------|-------|-------|-------|
| * | q_1 | q_2 | q_3 |
| q_1 | q_1 | q_2 | q_3 |
| q_2 | q_3 | q_1 | q_2 |
| q_3 | q_2 | q_3 | q_1 |

3. A new class of Quasigroups

V.B.Kandasamy [2] has defined a new class of groupoids as follows;

Definition 3.1. Let $Z_n = \{0, 1, 2, \dots, n-1\}$, $n \geq 3$. For $a, b \in Z_n$ define a binary operation $*$ on Z_n as: $a * b = ta + ub \pmod{n}$ where t, u are two distinct element in $Z_n \setminus \{0\}$ and $(t, u) = 1$. Here $+$ is the usual addition of two integers and ta means the product of two integers t and a . We denote this groupoid by $Z_n(t, u)$.

Theorem 3.1. Let $Z_n(t, u)$ be a groupoid. If $n = t + u$ where both t and u are primes then $Z_n(t, u)$ is a quasigroup.

Proof. When t and u are primes every row and column in the composition table will have distinct n element. As a result $Z_n(t, u)$ is a quasigroup.

Corollary 3.1. If $Z_p(t, u)$ is a groupoid and $t + u = p$, $(t, u) = 1$ then $Z_p(t, u)$ is a quasigroup.

Proof. Follows from the theorem.

Example 3.1. Consider $Z_5 = \{0, 1, 2, 3, 4\}$. Let $t = 2$ and $u = 3$. Then $5 = 2 + 3$, $(2, 3) = 1$ and the composition table is:

| | | | | | |
|---|---|---|---|---|---|
| * | 0 | 1 | 2 | 3 | 4 |
| 0 | 0 | 3 | 1 | 4 | 2 |
| 1 | 2 | 0 | 3 | 1 | 4 |
| 2 | 4 | 2 | 0 | 3 | 1 |
| 3 | 1 | 4 | 2 | 0 | 3 |
| 4 | 3 | 1 | 4 | 2 | 0 |

Thus $Z_5(2, 3)$ is a quasigroup.

Definition 3.2. Let $Z_n = \{0, 1, 2, \dots, n-1\}$, $n \geq 3$, $n < \infty$. Define $*$ on Z_n as $a * b = ta + ub \pmod{n}$ where t and $u \in Z_n \setminus \{0\}$ and $t = u$. For a fixed integer n and varying t and u we get a class of quasigroups of order n .

Example 3.2. Consider $Z_5 = \{0, 1, 2, 3, 4\}$. Then $Z_5(3, 3)$ is a quasigroup as given by the following table:

| | | | | | |
|---|---|---|---|---|---|
| * | 0 | 1 | 2 | 3 | 4 |
| 0 | 0 | 3 | 1 | 4 | 2 |
| 1 | 3 | 1 | 4 | 2 | 0 |
| 2 | 1 | 4 | 2 | 0 | 3 |
| 3 | 4 | 2 | 0 | 3 | 1 |
| 4 | 2 | 0 | 3 | 1 | 4 |

Definition 3.3. Let $Z_n = \{0, 1, 2, \dots, n-1\}$, $n \geq 3$, $n < \infty$. Define $*$ on Z_n as $a * b = ta + ub \pmod{n}$ where t and $u \in Z_n \setminus \{0\}$ and $t = 1$ and $u = n-1$. For a fixed integer n and varying t and u we get a class of quasigroups of order n .

Example 3.3. Consider $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Then $Z_8(1, 7)$ is a quasigroup as given by the following table:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 0 | 0 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
| 1 | 1 | 0 | 7 | 6 | 5 | 4 | 3 | 2 |
| 2 | 2 | 1 | 0 | 7 | 6 | 5 | 4 | 3 |
| 3 | 3 | 2 | 1 | 0 | 7 | 6 | 5 | 4 |
| 4 | 4 | 3 | 2 | 1 | 0 | 7 | 6 | 5 |
| 5 | 5 | 4 | 3 | 2 | 1 | 0 | 7 | 6 |
| 6 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 7 |
| 7 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

Definition 3.4. Let $Z_n = \{0, 1, 2, \dots, n-1\}$, $n \geq 3$, $n < \infty$. Define $*$ on Z_n as $a * b = ta + ub \pmod{n}$ where t and $u \in Z_n \setminus \{0\}$ and $(t, u) = 1$, $t + u = n$ and $|t - u|$ is a minimum. For a fixed integer n and varying t and u we get a class of quasigroups of order n .

Example 3.4. Consider $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Then $Z_8(3, 5)$ is a quasigroup as given by the following table:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 0 | 0 | 5 | 2 | 7 | 4 | 1 | 6 | 3 |
| 1 | 3 | 0 | 5 | 2 | 7 | 4 | 1 | 6 |
| 2 | 6 | 3 | 0 | 5 | 2 | 7 | 4 | 1 |
| 3 | 1 | 6 | 3 | 0 | 5 | 2 | 7 | 4 |
| 4 | 4 | 1 | 6 | 3 | 0 | 5 | 2 | 7 |
| 5 | 7 | 4 | 1 | 6 | 3 | 0 | 5 | 2 |
| 6 | 2 | 7 | 4 | 1 | 6 | 3 | 0 | 5 |
| 7 | 5 | 2 | 7 | 4 | 1 | 6 | 3 | 0 |

Definition 3.5. Let $(Q, *)$ be a quasigroup. A proper subset V of Q is called a subquasigroup of Q if V itself is a quasigroup under $*$.

Definition 3.6. Let Q be a quasigroup. A subquasigroup V of Q is said to be normal subquasigroup of Q if:

1. $aV = Va$
2. $(Vx)y = V(xy)$
3. $y(xV) = (yx)V$

for all $a, x, y \in Q$.

Example 3.5. Let Q be a quasigroup defined by the following table:

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 4 | 3 | 2 | 6 | 5 | 8 | 7 |
| 2 | 2 | 1 | 4 | 3 | 5 | 6 | 7 | 8 |
| 3 | 3 | 2 | 1 | 4 | 7 | 8 | 6 | 5 |
| 4 | 4 | 3 | 2 | 1 | 8 | 7 | 5 | 6 |
| 5 | 6 | 5 | 7 | 8 | 1 | 2 | 3 | 4 |
| 6 | 5 | 6 | 8 | 7 | 2 | 3 | 4 | 1 |
| 7 | 8 | 7 | 6 | 5 | 3 | 4 | 1 | 2 |
| 8 | 7 | 8 | 5 | 6 | 4 | 1 | 2 | 3 |

Here $V = \{1, 2, 3, 4\}$ is a normal subquasigroup of Q .

Definition 3.7. A subquasigroup is said to be simple if it has no proper nontrivial normal subgroup.

4. Substructures of Smarandache Quasigroups

Definition 4.1. Let $(Q, *)$ be a Smarandache quasigroup. A nonempty subset H of Q is said to be a Smarandache subquasigroup if H contains a proper subset K such that k is a group under $*$.

Example 4.1. Let $Q = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the quasigroup defined by the following table:

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 6 | 5 | 8 | 7 |
| 2 | 2 | 1 | 4 | 3 | 5 | 6 | 7 | 8 |
| 3 | 3 | 4 | 1 | 2 | 7 | 8 | 6 | 5 |
| 4 | 4 | 3 | 2 | 1 | 8 | 7 | 5 | 6 |
| 5 | 6 | 5 | 7 | 8 | 1 | 2 | 3 | 4 |
| 6 | 5 | 6 | 8 | 7 | 2 | 3 | 4 | 1 |
| 7 | 8 | 7 | 6 | 5 | 3 | 4 | 1 | 2 |
| 8 | 7 | 8 | 5 | 6 | 4 | 1 | 2 | 3 |

Consider $S = \{1, 2, 3, 4\}$ then S is a subquasigroup which contains a group $G = \{1, 2\}$. Therefore S is a Smarandache subquasigroup.

Example 4.2. There do exist Smarandache quasigroup which do not possess any Smarandache subquasigroup. Consider the quasigroup Q defined by the following table:

| * | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 3 | 1 | 4 | 2 | 5 |
| 2 | 5 | 2 | 3 | 1 | 4 |
| 3 | 1 | 4 | 2 | 5 | 3 |
| 4 | 4 | 5 | 1 | 3 | 2 |
| 5 | 2 | 3 | 5 | 4 | 1 |

Clearly, Q is Smarandache quasigroup as it contains a group $G = \{2\}$. But there is no subquasigroup, not to talk of Smarandache subquasigroup.

Definition 4.2. Let Q be a S -quasigroup. If $A \subset Q$ is a proper subset of Q and A is a subgroup which can not be contained in any proper subquasigroup of Q we say A is the largest subgroup of Q .

Example 4.3. Let $Q = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the quasigroup defined by the following table:

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 6 | 5 | 8 | 7 |
| 2 | 2 | 1 | 4 | 3 | 5 | 6 | 7 | 8 |
| 3 | 3 | 4 | 1 | 2 | 7 | 8 | 6 | 5 |
| 4 | 4 | 3 | 2 | 1 | 8 | 7 | 5 | 6 |
| 5 | 6 | 5 | 7 | 8 | 1 | 2 | 3 | 4 |
| 6 | 5 | 6 | 8 | 7 | 2 | 3 | 4 | 1 |
| 7 | 8 | 7 | 6 | 5 | 3 | 4 | 1 | 2 |
| 8 | 7 | 8 | 5 | 6 | 4 | 1 | 2 | 3 |

Clearly, $A = \{1, 2, 3, 4\}$ is the largest subgroup of Q .

Definition 4.3. Let Q be a S -quasigroup. If A is a proper subset of Q which is subquasigroup of Q and A contains the largest group of Q then we say A to be the Smarandache hyper subquasigroup of Q .

Example 4.4. Let Q be a quasigroup defined by the following table:

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 3 | 6 | 5 | 8 | 7 |
| 2 | 2 | 1 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 3 | 4 | 1 | 2 | 7 | 8 | 6 | 5 |
| 4 | 4 | 3 | 2 | 1 | 8 | 7 | 5 | 6 |
| 5 | 6 | 5 | 7 | 8 | 1 | 2 | 3 | 4 |
| 6 | 5 | 6 | 8 | 7 | 2 | 3 | 4 | 1 |
| 7 | 8 | 7 | 6 | 5 | 3 | 4 | 1 | 2 |
| 8 | 7 | 8 | 5 | 6 | 4 | 1 | 2 | 3 |

Here $A = \{1, 2, 3, 4\}$ is the subquasigroup of Q which contains the largest group $\{1, 2\}$ of Q . A is a Smarandache hyper subquasigroup of Q .

Definition 4.4. Let Q be a finite S -quasigroup. If the order of every subgroup of Q divides the order of the S -quasigroup Q then we say Q is a Smarandache Lagrange quasigroup.

Example 4.5. In the above example 4.4, Q is a S -quasigroup whose only subgroups are $\{1\}$ and $\{1, 2\}$. Clearly, order of these subgroups divide the order of the quasigroup Q . Thus Q is the Smarandache Lagrange quasigroup.

Definition 4.5. Let Q be a finite S -quasigroup. p is the prime such that p divides the order of Q . If there exist a subgroup A of Q of order p or p^l , ($l > 1$) we say Q has a Smarandache p -Sylow subgroup.

Example 4.6. Let $Q = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the quasigroup defined by the following table:

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 6 | 5 | 8 | 7 |
| 2 | 2 | 1 | 4 | 3 | 5 | 6 | 7 | 8 |
| 3 | 3 | 4 | 1 | 2 | 7 | 8 | 6 | 5 |
| 4 | 4 | 3 | 2 | 1 | 8 | 7 | 5 | 6 |
| 5 | 6 | 5 | 7 | 8 | 1 | 2 | 3 | 4 |
| 6 | 5 | 6 | 8 | 7 | 2 | 3 | 4 | 1 |
| 7 | 8 | 7 | 6 | 5 | 3 | 4 | 1 | 2 |
| 8 | 7 | 8 | 5 | 6 | 4 | 1 | 2 | 3 |

Consider $A = \{1, 2, 3, 4\}$ then A is a subgroup of Q whose order 2^2 divides order of Q . Therefore Q has a Smarandache 2-Sylow subgroup.

Definition 4.6. Let Q be a finite S -quasigroup. An element $a \in A$, $a \subset Q$ (A a proper subset of Q and A is the subgroup under the operation of Q) is said to be a Smarandache Cauchy element of Q if $a^r = 1$, ($r > 1$) and 1 is the unit element of A and r divides the order of Q otherwise a is not a Smarandache Cauchy element of Q .

Definition 4.7. Let Q be a finite S -quasigroup if every element in every subgroup of Q is a Smarandache Cauchy element of Q then we say that Q is a Smarandache Cauchy quasigroup.

Example 4.6. In the above example 4.6 there are three subgroup of Q . They are $\{1\}$, $\{1, 2\}$ and $\{1, 2, 3, 4\}$. Each element in each subgroup is a Smarandache Cauchy element as $1^2 = 2^2 = 3^2 = 4^2 = 1$ in each respective subgroup. Thus Q is a Smarandache Cauchy group.

References

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