Scientia Magna
Vol. 4 (2008), No. 1, 23-25

On an equation involving the Smarandache reciprocal function and its positive integer solutions

Zhibin Ren

Department of Mathematics, Weinan Teachers University ,
Weinan Shaanxi 714000,P.R.China

E-mail: zbr@wntc.edu.cn

Abstract For any positive integer n, the Smarandache reciprocal function $S_c(n)$ is defined as $S_c(n) = \max\{m : \ y \mid n! \text{ for all } 1 \leq y \leq m, \text{ and } \ m+1 \dagger n!\}$. That is, $S_c(n)$ is the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$. The main purpose of this paper is using the elementary method and the Vinogradov's important work to prove the following conclusion: For any positive integer $k \geq 3$, there exist infinite group positive integers $(m_1, \ m_2, \ \cdots, \ m_k)$ such that the equation

$$S_c(m_1 + m_2 + \cdots + m_k) = S_c(m_1) + S_c(m_2) + \cdots + S_c(m_k).$$

This solved a problem posed by Zhang Wenpeng during the Fourth International Conference on Number Theory and the Smarandache Problems.

Keywords The Smarandache reciprocal function, equation, positive integer solutions.

§1. Introduction and result

For any positive integer n, the Smarandache reciprocal function $S_c(n)$ is defined as the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$. That is, $S_c(n) = \max\{m : y \mid n! \text{ for all } 1 \leq y \leq m, \text{ and } \ m+1 \dagger n!\}$. For example, the first few values of $S_c(n)$ are:

$$S_c(1) = 1, \ S_c(2) = 2, \ S_c(3) = 3, \ S_c(4) = 4, \ S_c(5) = 6, \ S_c(6) = 6, \ S_c(7) = 10,$$
$$S_c(8) = 10, \ S_c(9) = 10, \ S_c(10) = 10, \ S_c(11) = 12, \ S_c(12) = 12, \ S_c(13) = 16,$$
$$S_c(14) = 16, \ S_c(15) = 16, \ S_c(16) = 16, \ S_c(17) = 18, \ S_c(18) = 18, \ \cdots\cdots.$$

This function was first introduced by A.Murthy in reference [2], where he studied the elementary properties of $S_c(n)$, and proved the following conclusion:

If $S_c(n) = x$ and $n \neq 3$, then $x + 1$ is the smallest prime greater than n.

During the Fourth International Conference on Number Theory and the Smarandache Problems, Professor Zhang Wenpeng asked us to study such a problem: For any positive integer k, whether there exist infinite group positive integers $(m_1, m_2, \cdots, m_k)$ such that the equation

$$S_c(m_1 + m_2 + \cdots + m_k) = S_c(m_1) + S_c(m_2) + \cdots + S_c(m_k).$$

I think that this problem is interesting, because it has some close relations with the Goldbach problem. The main purpose of this paper is using the elementary method and the Vinogradov's important work to study this problem, and solved it completely. That is, we shall prove the following conclusion:

Theorem. For any positive integer $k \geq 3$, there exist infinite group positive integers $(m_1, m_2, \cdots, m_k)$ such that the equation

$$S_c(m_1 + m_2 + \cdots + m_k) = S_c(m_1) + S_c(m_2) + \cdots + S_c(m_k).$$

It is clear that if $k = 1$, then our Theorem is trivial. Whether there exist infinite group positive integers (m_1, m_2) such that $S_c(m_1+m_2) = S_c(m_1)+S_c(m_2)$? This is an open problem.

If the Goldbach's conjecture is true (i.e., every even number $2N \geq 6$ can be written as $2N = p_1 + p_2$, a sum of two odd primes), then there exist infinite group positive integers $(m_1,\ m_2)$ such that the equation $S_c(m_1 + m_2) = S_c(m_1) + S_c(m_2)$.

§2. Proof of the theorem

In this section, we shall prove our Theorem directly. First from the Vinogradov's important work Three Primes Theorem (See Theorem 6.14 of reference [5]) we know that for any odd number $2N + 1$ large enough, there must exist three odd primes p_1, p_2 and p_3 such that the equation:

$$2N + 1 = p_1 + p_2 + p_3. \tag{1}$$

For any positive integer $k \geq 3$ and prime p (large enough), by using the mathematical inductive method and the Vinogradov's work (1) we can deduce that $p + k - 1$ can be written as a sum of k odd primes:

$$p + k - 1 = p_1 + p_2 + \cdots + p_k. \tag{2}$$

In fact if $k = 3$, then for any prime p large enough, $p + 2$ is an odd number, so from (1) we know that $p + 2 = p_1 + p_2 + p_3$. So (2) is true. If $k = 4$, then we take $p_1 = 3$, so from (1) we also have

$$p + 3 = 3 + p_2 + p_3 + p_4 = p_1 + p_2 + p_3 + p_4.$$

So (2) is true if $k = 4$. If $k \geq 5$, we take p be such a prime so as to odd number $p+k-1-3\cdot(k-3)$ large enough, from (1) we know that there must exist three odd primes p_{k-2}, p_{k-1} and p_k such that the equation:

$$p + k - 1 - 3 \cdot (k - 3) = p_{k-2} + p_{k-1} + p_k$$

or

$$p + k - 1 = \underbrace{3 + 3 + \cdots + 3}_{k-3} + p_{k-2} + p_{k-1} + p_k = p_1 + p_2 + \cdots + p_k,$$

where $p_1 = p_2 = \cdots = p_{k-3} = 3$. So (2) is true for all $k \geq 3$.

Now we use (2) to complete the proof of our Theorem. For any positive integer $k \geq 3$, we take prime p large enough, then from (2) we have the identity

$$p - 1 = p_1 - 1 + p_2 - 1 + p_3 - 1 + \cdots\cdots + p_k - 1. \tag{3}$$

Note that $S_c(p_i - 1) = p_i - 1$ for all prime p_i, taking $m = p - 1$, $m_i = p_i - 1$, $i = 1,\ 2,\ \cdots,\ k$, from (3) we may immediately deduce the identity

$$\begin{aligned} p - 1 &= S_c(p-1) = S_c(m) = S_c(m_1 + m_2 + \cdots + m_k) \\ &= p_1 - 1 + p_2 - 1 + p_3 - 1 + \cdots\cdots + p_k - 1 \\ &= S_c(m_1) + S_c(m_2) + S_c(m_3) + \cdots + S_c(m_k). \end{aligned}$$

That is,

$$S_c(m_1 + m_2 + \cdots + m_k) = S_c(m_1) + S_c(m_2) + \cdots + S_c(m_k).$$

Since there are infinite prime p, so there exist infinite group positive integers $(m_1,\ m_2,\ \cdots,\ m_k)$ such that the equation

$$S_c(m_1 + m_2 + \cdots + m_k) = S_c(m_1) + S_c(m_2) + \cdots + S_c(m_k).$$

This completes the proof of Theorem.

References

[1] F.Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] A.Murthy, Smarandache reciprocal function and an elementary inequality, Smarandache Notions Journal, **11**(2000), No. 1-2-3, 312-315.

[3] Zhang Wenpeng, The elementary number theory (in Chinese), Shaanxi Normal University Press, Xi'an, 2007.

[4] Tom M. Apostol. Introduction to Analytic Number Theory. New York, Springer-Verlag, 1976.

[5] Pan Chengdong and Pan Chengbiao, Goldbach Conjecture, Science Press, Beijing, 1992.

[6] I.Balacenoiu and V.Seleacu, History of the Smarandache function, Smarandache Notions Journal, **10**(1999), No. 1-2-3, 192-201.