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Smarandachely Roman Edge s-Dominating Function

Karam Ebadi

Department of Studies in Mathematics of University of Mysore

Manasagangotri, Mysore-570006, Karnataka, India.

L. Pushpalatha

(Department of Mathematics, Yuvaraja's College, Mysore, India)

Email: Karam_Ebadi@yahoo.com, pushpakrishna@yahoo.com

Abstract: For an integer $n \geq 2$, let $I \subset \{0, 1, 2, \cdots, n\}$. A *Smarandachely Roman s-dominating function* for an integer s, $2 \leq s \leq n$ on a graph G = (V,E) is a function $f : V \to \{0, 1, 2, \cdots, n\}$ satisfying the condition that $|f(u) - f(v)| \geq s$ for each edge $uv \in E$ with $f(u)$ or $f(v) \in I$. Similarly, a *Smarandachely Roman edge s-dominating function* for an integer s, $2 \leq s \leq n$ on a graph $G = (V, E)$ is a function $f : E \to \{0, 1, 2, \cdots, n\}$ satisfying the condition that $|f(e) - f(h)| \geq s$ for adjacent edges $e, h \in E$ with $f(e)$ or $f(h) \in I$. Particularly, if we choose $n = s = 2$ and $I = \{0\}$, such a Smarandachely Roman s-dominating function or Smarandachely Roman edge s-dominating function is called *Roman dominating function* or *Roman edge dominating function*. The Roman edge domination number $\gamma_{re}(G)$ of G is the minimum of $f(E) = \sum_{e \in E} f(e)$ over such functions. In this paper, we find lower and upper bounds for Roman edge domination numbers in terms of the diameter and girth of G.

Key Words: Smarandachely Roman s-dominating function, Smarandachely Roman edge s-dominating function, diameter, girth.

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§1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. As usual $|V| = n$ and $|E| = q$ denote the number of vertices and edges of the graph G, respectively. The open neighborhood $N(v)$ of the vertex v is the set $\{u \in V(G)|\ uv \in E(G)\}$ and its closed neighborhood $N[v] = N(v) \cup \{v\}$. Similarly, the open neighborhood of a set $S \subseteq V$ is the set $N[S] = \bigcup_{v \in S} N(v)$, and its closed neighborhood is $N(S) = N(S) \cup S$. The minimum and maximum vertex degrees in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

The degree of an edge $e = uv$ of G is defined by $deg\ e = deg\ u + deg\ v - 2$ and $\delta'(G)$ ($\Delta'(G)$) is the minimum (maximum) degree among the edges of G (the degree of a edge is the

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number of edges adjacent to it). A vertex of degree one is called a pendant vertex or a leaf and its neighbor is called a support vertex.

A set $D \subseteq V$ is said to be a dominating set of G, if every vertex in $V - D$ is adjacent to some vertex in D. The minimum cardinality of such a set is called the domination number of G and is denoted by $\gamma(G)$. For a complete review on the topic of domination and its related parameters, see [5].

Mitchell and Hedetniemi in [6] introduced the notion of edge domination as follows. A set F of edges in a graph G is an edge dominating set if every edge in $E - F$ is adjacent to at least one edge in F. The minimum numbers of edges in such a set is called the edge domination number of G and is denoted by $\gamma_e(G)$. This concept is also studied in [1].

For an integer $n \geq 2$, let $I \subset \{0, 1, 2, \cdots, n\}$. A *Smarandachely Roman s-dominating function* for an integer s, $2 \leq s \leq n$ on a graph G = (V,E) is a function $f : V \to \{0, 1, 2, \cdots, n\}$ satisfying the condition that $|f(u) - f(v)| \geq s$ for each edge $uv \in E$ with $f(u)$ or $f(v) \in I$. Similarly, a *Smarandachely Roman edge s-dominating function* for an integer s, $2 \leq s \leq n$ on a graph $G = (V, E)$ is a function $f : E \to \{0, 1, 2, \cdots, n\}$ satisfying the condition that $|f(e) - f(h)| \geq s$ for adjacent edges $e, h \in E$ with $f(e)$ or $f(h) \in I$. Particularly, if we choose $n = s = 2$ and $I = \{0\}$, such a Smarandachely Roman s-dominating function or Smarandachely Roman edge s-dominating function is called *Roman dominating function* or *Roman edge dominating function*.

The concept of Roman dominating function (RDF) was introduced by E. J. Cockayne, P. A. Dreyer, S. M. Hedetniemi and S. T. Hedetniemi in [3]. (See also [2,4,7]). A Roman dominating function on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of a Roman dominating function on G.

A *Roman edge dominating function* (REDF) on a graph $G = (V, E)$ is a function $f : E \to \{0, 1, 2\}$ satisfying the condition that every edge e for which $f(e) = 0$ is adjacent to at least one edge h for which $f(h) = 2$. The weight of a Roman edge dominating function is the value $f(E) = \sum_{e \in E} f(e)$. The Roman edge domination number of a graph G, denoted by $\gamma_{re}(G)$, equals the minimum weight of a Roman edge dominating function on G. This concept is also studied in Soner et al. in [8]. A $\gamma - set$, $\gamma_r - set$ and γ_{re}-set, can be defined as a minimum dominating set (MDS), a minimum Roman dominating set (MRDS) and a minimum Roman edge dominating set (MREDS), respectively.

The purpose of this paper is to establish sharp lower and upper bounds for Roman edge domination numbers in terms of the diameter and the girth of G.

Soner et al. in [8] proved that:

Theorem A *For a graph G of order p,*

$$\gamma_e(G) \leq \gamma_{re}(G) \leq 2\gamma_e(G).$$

Theorem B *For cycles C_p with $p \geq 3$ vertices,*

$$\gamma_{re}(C_p) = \lceil 2p/3 \rceil.$$

Here we observe the following properties.

Property 1 *For any connected graph G with $p \geq 3$ vertices,*

$$\gamma_{re}(G) = \gamma_r(L(G)).$$

Property 2 *a) If an edge e has degree one and h is adjacent to e, then every such h must be in every REDS of G.*

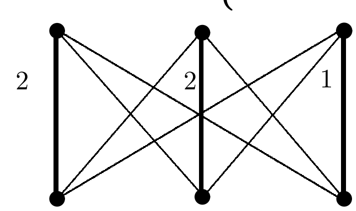
b) For the path graph P_k with $k \geq 2$ vertices,

$$\gamma_{re}(P_k) = \lfloor 2k/3 \rfloor.$$

c) For the complete bipartite graph $K_{m,n}$ with $m \leq n$ vertices,

$$\gamma_{re}(K_{m,n}) = \begin{cases} \text{2m-1} & \text{if } m = n, \\ \text{2m} & \text{otherwise.} \end{cases}$$

ex:



$$\gamma_{re}(K_{3,3}) = 5$$

d) $\gamma_{re}(G \cup H) = \gamma_{re}(G) + \gamma_{re}(H)$.

In the following theorem, we establish the result relating to maximum edge degree of G.

Theorem 1 *Let $f = (E_0, E_1, E_2)$ be any $\gamma_{re} - function$ and G has no isolated edges, then*

$$2q/(\Delta'(G) + 1) - |E_1| \leq \gamma_{re}(G) \leq q - \Delta'(G) + 1.$$

Furthermore, equality hold for P_3, P_4, and C_3.

Proof Let $f = (E_0, E_1, E_2)$ be any $\gamma_{re} - function$. Since E_2 dominates the set E_0, so $S = (E_1 \cup E_2)$ is a edge dominating set of G. Then

$$2|S|\Delta'(G) \geq 2\sum_{e \in S} deg(e) = 2\sum_{e \in S} |N(e)| \geq 2|\bigcup_{e \in S} N(e)| \geq 2|E - S| \geq 2q - 2|S|.$$

Thus

$$2q/(\Delta'(G) + 1) \leq 2|S| = 2(|E_1| + |E_2|) = |E_1| + \gamma_{re}(G).$$

Converse, let $deg \ \ e = \Delta'(G)$, if for every edge $x \in N(e)$ is adjacent to an edge h which is not adjacent to e. Then clearly, $E(G) - N(e) \cup h$ is an REDS. Thus $\gamma_{re}(G) \leq q - \Delta'(G) + 1$ follows.
□

Corollary 1 *Let $f = (E_0, E_1, E_2)$ be any $\gamma_{re} - function$ and G has no isolated edges. If $|E_1| = 0$, then*

$$2q/(\Delta'(G)+1) \leq \gamma_{re}(G) \leq q - \Delta'(G) + 1.$$

In this section sharp lower and upper bounds for $\gamma_{re}(G)$ in terms of diam(G) are presented. Recall that the eccentricity of vertex v is $ecc(v) = max\{d(u,v) : u \in V, \ u \neq v\}$ and the diameter of G is $diam(G) = max\{ecc(v) : v \in V\}$. Throughout this section we assume that G is a nontrivial graph of order $n \geq 2$.

Theorem 2 *If a graph G has diameter two, then $\gamma_{re}(G) \leq 2\delta'$. Further, the equality holds if $G = P_3$.*

Proof Since G has diameter two, $N(e)$ dominates $E(G)$ for all edge $e \in E(G)$. Now, let $e \in E(G)$ and $deg \ \ e = \delta'$. Define $f : E(G) \longrightarrow \{0,1,2\}$ by $f(e_i) = 2$ for $e_i \in N(e)$ and $f(e_i) = 0$ otherwise. Obviously f is a Roman edge dominating function of G. Thus $\gamma_{re}(G) \leq 2\delta'$. For P_3, $\gamma_{re}(P_3) = 2 = 2 \times 1$. □

Theorem 3 *For any connected graph G on n vertices,*

$$\lceil (diam(G)+1)/2 \rceil \leq \gamma_{re}(G)$$

With equality for P_n, $(2 \leq n \leq 5)$.

Proof The statement is obviously true for K_2. Let G be a connected graph with vertices $n \geq 3$. Suppose that $P = e_1e_2...e_{diam(G)}$ is a longest diametral path in G. By Theorem B, $\gamma_{re}(P) = \lceil 2diam(G)/3 \rceil$, and $\lceil (diam(G)+1)/2 \rceil < \lceil 2(diam(G)+1)/3 \rceil$, then $\lceil (diam(G) + 1)/2 \leq \lceil 2diam(G)/3 \rceil \leq \gamma_{re}(P)$, let $f = (E_0, E_1, E_2)$ be a $\gamma_{re}(P) - function$. Define $g : E(G) \longrightarrow \{0,1,2\}$ by $g(e) = f(e)$ for $e \in E(P)$ and $g(h_i) \leq 1$ for $h_i \in E(G) - E(P)$, then $w(g) = w(f) + \sum_{h_i \in E(G)-E(P)} h_i$. Obviously g is a REDF for G and hence

$$\lceil (diam(G)+1)/2 \rceil \leq \gamma_{re}(G).$$ □

Theorem 4 *For any connected graph G on n vertices,*

$$\gamma_{re}(G) \leq q - \lfloor (diam(G)-1)/3 \rfloor.$$

Furthermore, this bound is sharp for C_n and P_n.

Proof Let $P = e_1e_2...e_{diam(G)}$ be a diametral path in G. Moreover, let $f = (E_0, E_1, E_2)$ be a $\gamma_{re}(P) - function$. By Property 2(b), the weight of f is $\lceil 2diam(G)/3 \rceil$. Define $g : E(G) \longrightarrow \{0,1,2\}$ by $g(e) = f(e)$ for $e \in E(P)$ and $g(e) = 1$ for $e \in E(G) - E(P)$. Obviously g is a REDF for G. Hence,

$$\gamma_{re}(G) \leq w(f) + (q - diam(G)) \leq q - \lfloor (diam(G)-1)/3 \rfloor.$$ □

Theorem 5([8]) *For any connected graph G on n vertices,*

$$\gamma_{re}(G) \leq n - 1$$

and equality holds if G is isomorphic to W_5, P_3, C_4, C_5, K_n and $K_{m,m}$.

Theorem 6 *For any connected graph G on n vertices,*

$$\gamma_{re}(G) \leq n - \lceil diam(G)/3 \rceil.$$

Furthermore, this bound is sharp for P_n. And equality hold for $K_{m,m}$, P_{3k}, $(k > 0)$, K_n, W_5, C_4 and C_5.

Proof The technic proof is same with that of Theorem 3. □

In this section we present bounds on Roman edge domination number of a graph G containing cycle, in terms of its grith. Recall that the grith of G (denoted by $g(G)$) is that length of a smallest cycle in G. Throughout this section, we assume that G is a nontrivial graph with $n \geq 3$ vertices and contains a cycle. The following result is very crucial for this section.

Theorem 7 *For a graph G of order n with $g(G) \geq 3$ we have $\gamma_{re}(G) \geq \lceil 2g(G)/3 \rceil$.*

Proof First note that if G is the n-cycle then $\gamma_{re}(G) = \lceil 2n/3 \rceil$ by Theorem B. Now, let C be a cycle of length $g(G)$ in G. If $g(G) = 3$ or 4, then we need at least 1 or 2 edges, to dominate the edges of C and the statement follows by Theorem A. Let $g(G) \geq 5$. Then an edge not in $E(G)$, can be adjacent to at most one edge of C for otherwise we obtain a cycle of length less than $g(G)$ which is a contradiction. Now the result follows by Theorem A. □

Theorem 8 *For any connected graph with n vertices, $\delta'(G) \geq 2$ and $g(G) \geq 3$. Then $\gamma_{re}(G) \geq n - \lfloor g(G)/3 \rfloor$. Furthermore, the bound is sharp for $K_{m,m}$, C_n, K_n and W_n.*

Proof Let G be a such graph with n-vertices, if we prove the $\gamma_{re}(C_n) \geq n - \lfloor g(C_n)/3 \rfloor$. Then this proof satisfying the any graph of order n. Since $g(C_n) \geq g(G)$ then $n - g(C_n) \leq n - g(G)$. By Theorem B, $\gamma_{re}(C_n) = \lceil 2n/3 \rceil = \lceil 2g(C_n)/3 \rceil = n - \lceil n/3 \rceil \leq n - \lfloor n/3 \rfloor \leq n - \lfloor g(G)/3 \rfloor$. □

Theorem 9 *For a simple connected graph G with n-vertices and $\delta' \leq 2$, if $g(G) \geq 5$, then $\gamma_{re}(G) \geq 2\delta'$. The bound is sharp for C_5 and C_6.*

Proof Let G be such a graph and C be a cycle with $g(G)$ edges. If $n = 5$, then G is a $5-cycle$ and $\gamma_{re}(G) = 4 = 2\delta'$. For $n \geq 6$, since $\delta' \leq 2$, then $\gamma_{re}(G) \geq \lceil 2g(G)/3 \rceil \geq 2\delta'$ by Theorem 7. □

Theorem 10 *Let T be any tree and let $e = uv$ be an edge of maximum degree Δ'. If $1 < diam(G) \leq 5$ and $degw \leq 2$ for every vertex $w \neq u, v$, then $\gamma_{re}(G) = q - \Delta' + 1$.*

Proof Let T be a tree with $diam(T) \leq 4$ and $degw \leq 2$ for every vertex $w \neq u, v$, where $e = uv$ is an edge of maximum degree in T. If $diam(T) = 2$ or 3, then $\gamma_{re}(G) = q - \Delta' + 1 = 2$. If $diam(T) = 4$ or 5, then each non-pendent edge of T is adjacent to a pendent edge of T and hence the set $E_1 \cup E_2$ of all non-pendent edges of T forms a minimum edge dominating set and $\gamma_{re}(G) = |E_1| + 2|E_2| = q - \Delta' + 1$. □

Theorem 11([8]) *Let G be a tree or a unicyclic graph, then $\gamma_{re}(G) \leq \gamma_r(G)$.*

Theorem 12 *Let T is an $n-vertex$ tree, with $n \geq 2$, then $\gamma_{re}(T) \leq 2n/3$. The bound is sharp for P_n.*

Proof We use induction on n. The statement is obviously true for K_2. If $diamT = 2$ or 3, then T has a dominating edge, and $\gamma_{re}(T) \leq 2 \leq 2n/3$.

Hence we may assume that $diamT \geq 4$. For a subtree T' with n' vertices, where $n' \geq 2$, the induction hypothesis yields an REDF f' of T' with weight at most $2n'/3$. We find a subtree T' such that adding a bit more weight to f' will yield a small enough REDF f for T.

Let P be a longest path in T chosen to maximize the degree of its next-to-last vertex v, and let u be the non-leaf neighbor of v and let $h = uv$.

Case 1. Let $deg_T(v) > 2$. Obtain T' by deleting v and its leaf neighbors. Since $diamT \geq 4$, we have $n' \geq 2$. Define f on $E(T)$ by $f(e) = f'(e)$ except for $f(h) = 2$ and $f(e) = 0$ for each edge e adjacent to h. Not that f is an RDF for T and that $w(f) = w(f')+2 \leq 2(n-3)/3+2 \leq 2n/3$.

Case 2. Let $deg_T(v) = deg_T(u) = 2$. Obtain T' by deleting v and u and the leaf neighbor z of v. Since $diamT \geq 4$, we have $n' \geq 2$. If $n' = 2$, then T is P_5 and has an REDF of weight 3. Otherwise, the induction hypothesis applies. Define f on $E(T)$ by letting $f(e) = f'(e)$ except for $f(h) = 2$ and $f(e) = 0$ for each edge e adjacent to h. Again f is an REDF, and the computation $w(f) < 2n/3$ is the same as in Case 1.

Case 3. Let $deg_T(u) > 2$ and every penultimate neighbor of u has degree 2. Obtain T' by deleting v and its leaf neighbors and u. Define f on $E(T)$ by $f(e) = f'(e)$ except for $f(h) = 2$ and $f(e) = 0$ for each edge e adjacent to h. Not that f is an RDF for T and that $w(f) = w(f') + 2 \leq 2(n-3)/3 + 2 \leq 2n/3$. If some neighbor of u is a leaf. Obtain T' by deleting v and its leaf neighbors and u and its leaf neighbors. Define f on $E(T)$ by $f(e) = f'(e)$ except for $f(h) = 2$ and $f(e) = 0$ for each edge e adjacent to h. Not that f is an RDF for T and that $w(f) = w(f') + 2 \leq 2(n-3)/3 + 2 \leq 2n/3$. From the all cases above $w(f) = w(f') + 2 \leq 2(n-3)/3 + 2 \leq 2n/3$. This completes the proof. □

Corollary 2 *Let T is an $q-edge$ tree, with $q \geq 1$, then $\gamma_{re}(T) \leq 2(q+1)/3$.*

Theorem 13 *Let $f = (E_0, E_1, E_2)$ be any $\gamma_{re}(T)-function$ of a connected graph T of $q \geq 2$. Then*

(1) $1 \leq |E_2| \leq (q+1)/3$;

(2) $0 \leq |E_1| \leq 2q/3 - 4/3$;

(3) $(q+1)/3 \leq |E_0| \leq q-1$.

Proof By Theorem 12, $|E_1| + 2|E_2| \leq 2(q+1)/3$.

(1) If $E_2 = \varnothing$, then $E_1 = q$ and $E_0 = \varnothing$. The REDF $(0, q, 0)$ is not minimum since $|E_1| + 2|E_2| > 2(q+1)/3$. Hence $|E_2| \geq 1$. On the other hand, $|E_2| \leq (q+1)/3 - |E_1|/2 \leq (q+1)/3$.

(2) Since $|E_2| \geq 1$, then $|E_1| \leq 2(q+1)/3 - 2|E_2| \leq 2(q+1)/3 - 2 = 2q/3 - 4/3$.

(3) The upper bound comes from $|E_0| \leq q - |E_2| \leq q-1$. For the lower bound, adding on both side $2|E_0| + 2|E_1| + 2|E_2| = 2q$, $-|E_1| - 2|E_2| \geq -2(q+1)/3$ and $-|E_1| \geq -2(q+1)/3 + 2$

gives $2|E_0| \geq (2q+2)/3$. Therefor, $|E_0| \geq (q+1)/3$. $\square$

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