Scientia Magna
Vol. 4(2008), No. 1, 113-116

On a Smarandache multiplicative function and its parity[1]

Wenjing Xiong

Department of Mathematics and Physics, Shaanxi Institute of Education
Xi'an, 710061, Shaanxi, P.R.China

Abstract For any positive integer n, we define the Smarandache multiplicative function $U(n)$ as follows: $V(1) = 1$. If $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ denotes the factorization of n into prime powers, then $U(n) = \max\{\alpha_1 \cdot p_1,\ \alpha_2 \cdot p_2,\ \cdots,\ \alpha_s \cdot p_s\}$. The main purpose of this paper is using the elementary and analytic methods to study the parity of $U(n)$, and give an interesting asymptotic formula for it.

Keywords Smarandache multiplicative function, parity, asymptotic formula.

§1. Introduction and results

For any positive integer n, the famous Smarandache multiplicative function $U(n)$ is defined as $U(1) = 1$. If $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ denotes the factorization of n into prime powers, then

$$U(n) = \max\{\alpha_1 \cdot p_1,\ \alpha_2 \cdot p_2,\ \cdots,\ \alpha_s \cdot p_s\}.$$

For example, the first few value of $U(n)$ are: $U(1) = 1$, $U(2) = 2$, $U(3) = 3$, $U(4) = 4$, $U(5) = 5$, $U(6) = 3$, $U(7) = 7$, $U(8) = 6$, $U(9) = 6$, $U(10) = 5$, $U(11) = 11$, $U(12) = 4$, $U(13) = 13$, $U(14) = 7$, $U(15) = 5$, $\cdots$. About the arithmetical properties of $U(n)$, some authors had studied it, and obtained some interesting results, see references [3] and [4]. For example, Xu Zhefeng [3] proved that for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \le x} (U(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and $P(n)$ denotes the largest prime divisor of n.

In an unpublished paper, Pan Xiaowei proved that the equation

$$\sum_{d|n} U(d) = n$$

has only two positive integer solutions $n = 1$ and 28, where $\sum_{d|n}$ denotes the summation over all positive divisors of n.

[1]This work partly supported by the N.S.F.C.(10671155).

Now we let $OU(n)$ denotes the number of all integers $1 \le k \le n$ such that $U(n)$ is odd. $EU(n)$ denotes the number of all integers $1 \le k \le n$ such that $U(n)$ is even. An interesting problem is to determine the limit:

$$\lim_{n \longrightarrow \infty} \frac{EU(n)}{OU(n)}. \tag{1}$$

About this problem, it seems that none had studied it yet, at least we have not seen such a paper before. The problem is interesting, because it can help us to know more information about the parity of $U(n)$.

The main purpose of this paper is using the elementary and analytic methods to study this problem, and give an interesting asymptotic formula for $\frac{EU(n)}{OU(n)}$. That is, we shall prove the following conclusion:

Theorem. For any positive integer $n > 1$, we have the asymptotic formula

$$\frac{EU(n)}{OU(n)} = O\left(\frac{1}{\ln n}\right).$$

From this Theorem we may immediately deduce the following:

Corollary. For any positive integer n, we have the limit

$$\lim_{n \longrightarrow \infty} \frac{EU(n)}{OU(n)} = 0.$$

§2. Proof of the theorem

In this section, we shall prove our Theorem directly. First we estimate the upper bound of $EU(n)$. In fact for any integer $k > 1$, let $k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ denotes the factorization of k into prime powers, then from the definition and properties of $U(k)$ we have $U(k) = U\left(p_i^{\alpha_i}\right) = \alpha_i \cdot p_i$. If $\alpha_i = 1$, then $U(k) = p_i$ be an odd number, except $k = 2$. Let $M = \ln n$, then we have

$$EU(n) = \sum_{\substack{k \le n \\ 2 | U(k)}} 1 \le 1 + \sum_{\substack{k \le n \\ U(k) = \alpha_i p_i,\ \alpha_i \ge 2}} 1 \le 1 + \sum_{U(k) \le M} 1 + \sum_{\substack{kp^\alpha \le n \\ \alpha p > M,\ \alpha \ge 2}} 1. \tag{2}$$

Now we estimate the each term in (2) respectively. We have

$$\begin{aligned}
\sum_{\substack{kp^\alpha \le n \\ \alpha p > M,\ \alpha \ge 2}} 1 &\le \sum_{\substack{kp^2 \le n \\ 2p > M}} 1 + \sum_{\substack{kp^\alpha \le n \\ \alpha p > M,\ \alpha \ge 3}} 1 \le \sum_{\frac{M}{2} < p \le \sqrt{n}} \sum_{k \le \frac{n}{p^2}} 1 + \sum_{\substack{p^\alpha \le n \\ \alpha p > M,\ \alpha \ge 3}} \sum_{k \le \frac{n}{p^\alpha}} 1 \\
&\ll \sum_{\frac{M}{2} < p \le \sqrt{n}} \frac{n}{p^2} + \sum_{\substack{p^\alpha \le n \\ \alpha p > M,\ \alpha \ge 3}} \frac{n}{p^\alpha} \ll \frac{n}{\ln n} + \sum_{\substack{p \le \sqrt{n} \\ \alpha p > M,\ \alpha \ge p}} \frac{n}{p^\alpha} + \sum_{\substack{p \le \sqrt{n} \\ \alpha p > M,\ 3 \le \alpha < p}} \frac{n}{p^\alpha} \\
&\ll \frac{n}{\ln n} + \sum_{\substack{p \le \sqrt{n} \\ \alpha > \sqrt{M}}} \frac{n}{p^\alpha} + \sum_{\substack{p \le \sqrt{n} \\ p > \sqrt{M},\ \alpha \ge 3}} \frac{n}{p^\alpha} \\
&\ll \frac{n}{\ln n} + \frac{n}{2^{\sqrt{M} - 1}} + \frac{n}{M} \ll \frac{n}{\ln n}.
\end{aligned} \tag{3}$$

In order to estimate another term in (2), we must use a new method. For any prime $p \leq M$, let $\alpha(p) = \left[\frac{M}{p}\right]$, where $[x]$ denotes the largest integer less than or equal to x. Let $m = \prod_{p\leq M} p^{\alpha(p)}$. It is clear that for any positive integer k with $U(k) \leq M$, we have $k|m$. And for any positive divisor k of m, we also have $U(k) \leq M$. So from these properties we have

$$\begin{aligned}\sum_{U(k)\leq M} 1 &\leq \sum_{d|u} 1 = \prod_{p\leq M} (1+\alpha(p)) = \prod_{p\leq M} \left(1 + \left[\frac{M}{p}\right]\right) \\ &= \exp\left(\sum_{p\leq M} \ln\left(1+\left[\frac{M}{p}\right]\right)\right), \qquad (4)\end{aligned}$$

where $\exp(y) = e^y$.

From the Prime Theorem (see reference [5], Theorem 3.10)

$$\pi(M) = \sum_{p\leq M} 1 = \frac{M}{\ln M} + O\left(\frac{M}{\ln^2 M}\right)$$

and

$$\sum_{p\leq M} \ln p = M + O\left(\frac{M}{\ln M}\right)$$

we have

$$\begin{aligned}&\sum_{p\leq M} \ln\left(1+\left[\frac{M}{p}\right]\right) \leq \sum_{p\leq M} \ln\left(1+\frac{M}{p}\right) \\ = &\sum_{p\leq M} \left[\ln(p+M) - \ln p\right] \\ \leq &\ \pi(M)\cdot \ln(2M) - \sum_{p\leq M} \ln p \\ = &\ \frac{M\cdot \ln(2M)}{\ln M} - M + O\left(\frac{M}{\ln M}\right) = O\left(\frac{M}{\ln M}\right). \qquad (5)\end{aligned}$$

Note that $M = \ln n$, from (4) and (5) we may get the estimate:

$$\sum_{U(k)\leq M} 1 \ll \exp\left(\frac{c\cdot \ln n}{\ln\ln n}\right), \qquad (6)$$

where c is a positive constant.

It is clear that $\exp\left(\frac{c\cdot\ln n}{\ln\ln n}\right) \ll \frac{n}{\ln n}$, so combining (2), (3) and (6) we may immediately deduce the estimate:

$$EU(n) = \sum_{\substack{k\leq n\\ 2|U(k)}} 1 = O\left(\frac{n}{\ln n}\right).$$

Note that $OU(n)+EU(n) = n$, from the above estimate we can deduce the asymptotic formula:

$$OU(n) = n - EU(n) = n + O\left(\frac{n}{\ln n}\right).$$

Therefore,

$$\frac{EU(n)}{OU(n)} = \frac{O\left(\frac{n}{\ln n}\right)}{n + O\left(\frac{n}{\ln n}\right)} = O\left(\frac{1}{\ln n}\right).$$

This completes the proof of Theorem.

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