An introduction to the Smarandache Square Complementary function

Felice Russo
Via A. Infante
67051 Avezzano (Aq) Italy
felice.russo@katamail.com

Abstract

In this paper the main properties of Smarandache Square Complementary function has been analysed. Several problems still unsolved are reported too.

The Smarandache square complementary function is defined as [4],[5]:

$$Ssc(n)=m$$

where m is the smallest value such that $m \cdot n$ is a perfect square.

Example: for n=8, m is equal 2 because this is the least value such that $m \cdot n$ is a perfect square.

The first 100 values of Ssc(n) function follows:

n	Ssc(n)	n	Ssc(n)	n	Ssc(n)	n	Ssc(n)
1	1	26	26	51	51	76	19
2	2	27	3	52	13	77	77
3	3	28	7	53	53	78	78
4	1	29	29	54	6	79	79
5	5	30	30	55	55	80	5
6	6	31	31	56	14	81	1
7	7	32	2	57	57	82	82
8	2	33	33	58	58	83	83
9	1	34	34	59	59	84	21
10	10	35	35	60	15	85	85
11	11	36	1	61	61	86	86
12	3	37	37	62	62	87	87
13	13	38	38	63	7	88	22
14	14	39	39	64	1	89	89
15	15	40	10	65	65	90	10
16	1	41	41	66	66	91	91
17	17	42	42	67	67	92	23
18	2	43	43	68	17	93	93
19	19	44	11	69	69	94	94
20	5	45	5	70	70	95	95
21	21	46	46	71	71	96	6
22	22	47	47	72	2	97	97
23	23	48	3	73	73	98	2
24	6	49	1	74	74	99	11
25	1	50	2	75	3	100	1

Let's start to explore some properties of this function.

Theorem 1: $Ssc(n^2)=1$ *where n=1,2,3,4...*

In fact if $k=n^2$ is a perfect square by definition the smallest integer m such that $m \cdot k$ is a perfect square is m=1.

Theorem 2: *Ssc(p)=p where p is any prime number*

In fact in this case the smallest m such that $m \cdot p$ is a perfect square can be only m=p.

Theorem 3: $Ssc(p^n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ p & \text{if } n \text{ is odd} \end{cases}$ *where p is any prime number.*

First of all let's analyse the even case. We can write:

$$p^n = p^2 \cdot p^2 \cdot \ldots\ldots \cdot p^2 = \left| p^{\frac{n}{2}} \right|^2$$

and then the smallest m such that $p^n \cdot m$ is a perfect square is 1.

Let's suppose now that n is odd. We can write:

$$p^n = p^2 \cdot p^2 \cdot \ldots\ldots \cdot p^2 \cdot p = \left| p^{\left\lfloor \frac{n}{2} \right\rfloor} \right|^2 \cdot p = p^{2 \left\lfloor \frac{n}{2} \right\rfloor} \cdot p$$

and then the smallest integer m such that $p^n \cdot m$ is a perfect square is given by m=p.

Theorem 4: $Ssc(p^a \cdot q^b \cdot s^c \cdot \ldots\ldots \cdot t^x) = p^{odd(a)} \cdot q^{odd(b)} \cdot s^{odd(c)} \cdot \ldots \cdot t^{odd(x)}$ *where p,q,s,....t are distinct primes and the odd function is defined as:*

$$odd(n) = \begin{cases} 1 & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$$

Direct consequence of theorem 3.

Theorem 5: *The Ssc(n) function is multiplicative, i.e. if (n,m)=1 then* $Ssc(n\cdot m)=Ssc(n)\cdot Ssc(m)$

Without loss of generality let's suppose that $n=p^a\cdot q^b$ and $m=s^c\cdot t^d$ where p, q, s, t are distinct primes. Then:

$$Ssc(n\cdot m)=Ssc(p^a\cdot q^b\cdot s^c\cdot t^d)=p^{odd(a)}\cdot q^{odd(b)}\cdot s^{odd(c)}\cdot t^{odd(d)}$$

according to the theorem 4.

On the contrary:

$$Ssc(n)=Ssc(p^a\cdot q^b)=p^{odd(a)}\cdot q^{odd(b)}$$

$$Ssc(m)=Ssc(s^c\cdot t^d)=s^{odd(c)}\cdot t^{odd(d)}$$

This implies that: $Ssc(n\cdot m)=Ssc(n)\cdot Ssc(m)$ qed

Theorem 6: *If* $n=p^a\cdot q^b\cdot\ldots\ldots\cdot p^s$ *then* $Ssc(n)=Ssc(p^a)\cdot Ssc(p^b)\cdot\ldots\ldots\cdot Ssc(p^s)$ *where p is any prime number.*

According to the theorem 4:

$$Ssc(n)=p^{odd(a)}\cdot p^{odd(b)}\cdot\ldots\ldots\cdot p^{odd(s)}$$

and:

$$Ssc(p^a)=p^{odd(a)}$$
$$Ssc(p^b)=p^{odd(b)}$$

and so on. Then:

$$Ssc(n)=Ssc(p^a)\cdot Ssc(p^b)\cdot\ldots\ldots\cdot Ssc(p^s) \qquad \text{qed}$$

Theorem 7: *Ssc(n)=n if n is squarefree, that is if the prime factors of n are all distinct. All prime numbers, of course are trivially squarefree [3].*

Without loss of generality let's suppose that $n=p\cdot q$ where p and q are two distinct primes.

According to the theorems 5 and 3:

$Ssc(n) = Ssc(p \cdot q) = Ssc(p) \cdot Ssc(q) = p \cdot q = n$ qed

Theorem 8: *The Ssc(n) function is not additive.*:

In fact for example: Ssc(3+4)=Ssc(7)=7<>Ssc(3)+Ssc(4)=3+1=4

Anyway we can find numbers m and n such that the function Ssc(n) is additive. In fact if:

m and n are squarefree
k=m+n is squarefree.

then Ssc(n) is additive.
In fact in this case Ssc(m+n)=Ssc(k)=k=m+n and Ssc(m)=m Ssc(n)=n according to theorem 7.

Theorem 9: $\sum_{n=1}^{\infty} \frac{1}{Ssc(n)}$ *diverges*

In fact:

$$\sum_{n=1}^{\infty} \frac{1}{Ssc(n)} > \sum_{p=2}^{\infty} \frac{1}{Ssc(p)} = \sum_{p=2}^{\infty} \frac{1}{p}$$ where p is any prime number.

So the sum of inverese of Ssc(n) function diverges due to the well known divergence of series [3]:

$$\sum_{p=2}^{\infty} \frac{1}{p}$$

Theorem 10: *Ssc(n)>0 where n=1,2,3,4 ...*

This theorem is a direct consequence of Ssc(n) function definition. In fact for any n the smallest m such that $m \cdot n$ is a perfect square cannot be equal to zero otherwise $m \cdot n$=0 and zero is not a perfect square.

Theorem 11: $\sum_{n=1}^{\infty} \frac{Ssc(n)}{n}$ *diverges*

In fact being $Ssc(n) \geq 1$ this implies that:

$$\sum_{n=1}^{\infty} \frac{Ssc(n)}{n} > \sum_{n=1}^{\infty} \frac{1}{n}$$

and as known the sum of reciprocal of integers diverges. [3]

Theorem 12: $Ssc(n) \leq n$

Direct consequence of theorem 4.

Theorem 13: *The range of Ssc(n) function is the set of squarefree numbers.*

According to the theorem 4 for any integer n the function Ssc(n) generates a squarefree number.

Theorem 14: $0 < \frac{Ssc(n)}{n} \leq 1$ *for n>=1*

Direct consequence of theorems 12 and 10.

Theorem 15: $\frac{Ssc(n)}{n}$ *is not distributed uniformly in the interval]0,1]*

If n is squarefree then Ssc(n)=n that implies $\frac{Ssc(n)}{n} = 1$

If n is not squarefree let's suppose without loss of generality that $n = p^a \cdot q^b$ where p and q are primes.

Then:

$$\frac{Ssc(n)}{n} = \frac{Ssc(p^a) \cdot Ssc(p^b)}{p^a \cdot q^b}$$

We can have 4 different cases.

1) a even and b even

$$\frac{Ssc(n)}{n}=\frac{Ssc(p^a)\cdot Ssc(p^b)}{p^a\cdot q^b}=\frac{1}{p^a\cdot q^b}\le\frac{1}{4}$$

2) a odd and b odd

$$\frac{Ssc(n)}{n}=\frac{Ssc(p^a)\cdot Ssc(p^b)}{p^a\cdot q^b}=\frac{p\cdot q}{p^a\cdot q^b}=\frac{1}{p^{a-1}\cdot q^{b-1}}\le\frac{1}{4}$$

3) a odd and b even

$$\frac{Ssc(n)}{n}=\frac{Ssc(p^a)\cdot Ssc(p^b)}{p^a\cdot q^b}=\frac{p\cdot 1}{p^a\cdot q^b}=\frac{1}{p^{a-1}\cdot q^b}\le\frac{1}{4}$$

4) a even and b odd

Analogously to the case 3 .

This prove the theorem because we don't have any point of Ssc(n) function in the interval]1/4,1[

Theorem 16: *For any arbitrary real number* $\varepsilon > 0$, *there is some number n>=1 such that:*

$$\frac{Ssc(n)}{n}<\varepsilon$$

Without loss of generality let's suppose that $q = p_1 \cdot p_2$ where p_1 and p_2 are primes such that $\frac{1}{q}<\varepsilon$ and ε is any real number grater than zero. Now take a number n such that:

$$n = p_1^{a_1}\cdot p_2^{a_2}$$

For a_1 and a_2 odd:

$$\frac{Ssc(n)}{n}=\frac{p_1 \cdot p_2}{p_1^{a_1} \cdot p_2^{a_2}}=\frac{1}{p_1^{a_1-1} \cdot p_2^{a_2-1}}<\frac{1}{p_1 \cdot p_2}<\varepsilon$$

For a_1 and a_2 even:

$$\frac{Ssc(n)}{n}=\frac{1}{p_1^{a_1} \cdot p_2^{a_2}}<\frac{1}{p_1 \cdot p_2}<\varepsilon$$

For a_1 odd and a_2 even (or viceversa):

$$\frac{Ssc(n)}{n}=\frac{p_1}{p_1^{a_1} \cdot p_2^{a_2}}=\frac{1}{p_1^{a_1-1} \cdot p_2^{a_2}}<\frac{1}{p_1 \cdot p_2}<\varepsilon$$

Theorem 17: $Ssc(p_k\#)=p_k\#$ where $p_k\#$ is the product of first k primes (primorial) [3].

The theorem is a direct consequence of theorem 7 being $p_k\#$ a squarefree number.

Theorem 18: *The equation* $\frac{Ssc(n)}{n}=1$ *has an infinite number of solutions.*

The theorem is a direct consequence of theorem 2 and the well-known fact that there is an infinite number of prime numbers [6]

Theorem 19: *The repeated iteration of the Ssc(n) function will terminate always in a fixed point (see [3] for definition of a fixed point).*

According to the theorem 13 the application of Scc function to any n will produce always a squarefree number and according to the theorem 7 the repeated application of Ssc to this squarefree number will produce always the same number.

Theorem 20: *The diophantine equation Ssc(n)=Ssc(n+1) has no solutions.*

We must distinguish three cases:

1) n and n+1 squarefree
2) n and n+1 not squareefree
3) n squarefree and n+1 no squarefree and viceversa

Case 1. According to the theorem 7 Ssc(n)=n and Ssc(n+1)=n+1 that implies that Ssc(n)<>Ssc(n+1)

Case 2. Withou loss of generality let's suppose that:

$$n = p^a \cdot q^b$$
$$n+1 = p^a \cdot q^b + 1 = s^c \cdot t^d$$

where p,q,s and t are distinct primes.

According to the theorem 4:

$$Ssc(n) = Ssc(p^a \cdot q^b) = p^{odd(a)} \cdot q^{odd(b)}$$
$$Ssc(n+1) = Ssc(s^c \cdot t^d) = s^{odd(c)} \cdot t^{odd(d)}$$

and then Ssc(n)<>Ssc(n+1)

Case 3. Without loss of generality let's suppose that $n = p \cdot q$. Then:

$$Ssc(n) = Ssc(p \cdot q) = p \cdot q$$
$$Ssc(n+1) = Ssc(p \cdot q + 1) = Ssc(s^a \cdot t^b) = s^{odd(a)} \cdot t^{odd(b)}$$

supposing that $n+1 = p \cdot q + 1 = s^a \cdot t^b$

This prove completely the theorem.

Theorem 21: $\sum_{k=1}^{N} Ssc(k) > \frac{6 \cdot N}{\pi^2}$ *for any positive integer N.*

The theorem is very easy to prove. In fact the sum of first N values of Ssc function can be separated into two parts:

$$\sum_{k_1=1}^{N} Ssc(k_1) + \sum_{k_2=1}^{N} Ssc(k_2)$$

where the first sum extend over all k_1 squarefree numbers and the second one over all k_2 not squarefree numbers.
According to the Hardy and Wright result [3], the asymptotic number Q(n) of squarefree numbers $\leq N$ is given by:

$$Q(N) \approx \frac{6 \cdot N}{\pi^2}$$

and then:

$$\sum_{k=1}^{N} Ssc(k) = \sum_{k_1=1}^{N} Ssc(k_1) + \sum_{k_2=1}^{N} Ssc(k_2) > \frac{6 \cdot N}{\pi^2}$$

because according to the theorem 7, $Ssc(k_1) = k_1$ and the sum of first N squarefree numbers is always greater or equal to the number Q(N) of squarefree numbers $\leq N$, namely:

$$\sum_{k_1=1}^{N} k_1 \geq Q(N)$$

Theorem 22: $\sum_{k=1}^{N} Ssc(k) > \frac{N^2}{2 \cdot \ln(N)}$ *for any positive integer N.*

In fact:

$$\sum_{k=1}^{N} Ssc(k) = \sum_{k'=1}^{N} Ssc(k') + \sum_{p=2}^{N} Ssc(p) > \sum_{p=2}^{N} Ssc(p)$$

because by theorem 2, Ssc(p)=p. But according to the result of Bach and Shallit [3], the sum of first N primes is asymptotically equal to:

$$\frac{N^2}{2\cdot\ln(N)}$$

and this completes the proof.

Theorem 23: *The diophantine equations* $\frac{Ssc(n+1)}{Ssc(n)}=k$ *and* $\frac{Ssc(n)}{Ssc(n+1)}=k$ *where k is any integer number have an infinite number of solutions.*

Let's suppose that n is a perfect square. In this case according to the theorem 1 we have:

$$\frac{Ssc(n+1)}{Ssc(n)}=Ssc(n+1)=k$$

On the contrary if n+1 is a perfect square then:

$$\frac{Ssc(n)}{Ssc(n+1)}=Ssc(n)=k$$

Problems.

1) Is the difference |Ssc(n+1)-Ssc(n)| bounded or unbounded?

2) Is the Ssc(n) function a Lipschitz function ?
A function is said a Lipschitz function [3] if:

$$\frac{|Ssc(m)-Ssc(k)|}{|m-k|}\geq M \quad \text{where M is any integer}$$

3) Study the function FSsc(n)=m. Here m is the number of different integers k such that Ssc(k)=n.

4) Solve the equations Ssc(n)=Ssc(n+1)+Ssc(n+2) and Ssc(n)+Ssc(n+1)=Ssc(n+2). Is the number of solutions finite or infinite?

5) Find all the values of n such that $Ssc(n) = Ssc(n+1) \cdot Ssc(n+2)$

6) Solve the equation $Ssc(n) \cdot Ssc(n+1) = Ssc(n+2)$

7) Solve the equation $Ssc(n) \cdot Ssc(n+1) = Ssc(n+2) \cdot Ssc(n+3)$

8) Find all the values of n such that $S(n)^k + Z(n)^k = Ssc(n)^k$ where S(n) is the Smarandache function [1], Z(n) the pseudo-Smarandache funtion [2] and k any integer.

9) Find the smallest k such that between Ssc(n) and Ssc(k+n), for n>1, there is at least a prime.

10) Find all the values of n such that Ssc(Z(n))-Z(Ssc(n))=0 where Z is the Pseudo Smarandache function [2].

11) Study the functions Ssc(Z(n)), Z(Ssc(n)) and Ssc(Z(n))-Z(Ssc(n)).

12) Evaluate $\lim_{k \to \infty} \frac{Ssc(k)}{\theta(k)}$ where $\theta(k) = \sum_{n \le k} \ln(Ssc(n))$

13) Are there m, n, k non-null positive integers for which $Ssc(m \cdot n) = m^k \cdot Ssc(n)$?

14) Study the convergence of the Smarandache Square compolementary harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{Ssc^a(n)}$$

where a>0 and belongs to R

15) Study the convergence of the series:

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{Ssc(x_n)}$$

where x_n is any increasing sequence such that $\lim_{n \to \infty} x_n = \infty$

16) Evaluate:

$$\lim_{n \to \infty} \frac{\sum_{k=2}^{n} \frac{\ln(Ssc(k))}{\ln(k)}}{n}$$

Is this limit convergent to some known mathematical constant?

17) Solve the functional equation:

$$Ssc(n)^r + Ssc(n)^{r-1} + \dots\dots + Ssc(n) = n$$

where r is an integer ≥ 2.

18) What about the functional equation:

$$Ssc(n)^r + Ssc(n)^{r-1} + \dots\dots + Ssc(n) = k \cdot n$$

where r and k are two integers ≥ 2.

19) Evaluate $\sum_{k=1}^{\infty} (-1)^k \cdot \frac{1}{Ssc(k)}$

20) Evaluate $\frac{\sum_n Ssc(n)^2}{\left|\sum_n Ssc(n)\right|^2}$

21) Evaluate:

$$\lim_{n\to\infty}\left|\sum_n \frac{1}{Ssc(f(n))} - \sum_n \frac{1}{f(Ssc(n))}\right|$$

for f(n) equal to the Smarandache function S(n) [1] and to the Pseudo Smarandache function Z(n) [2].

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