Scientia Magna
Vol. 5 (2009), No. 4, 31-35

Semigroup of continuous functions and Smarandache semigroups [1]

F. Ayatalloh Zadeh Shirazi † and A. Hosseini ‡

† Faculty of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Enghelab Ave., Tehran, Iran

‡ Department of Mathematics, Faculty of Science, University of Guilan, Manzarieh Ave., Rasht, Iran, P.O. Box: 1914

E-mail: fatemah@khayam.ut.ac.ir a_hosseini@guilan.ac.ir

Abstract The main aim of the following text is to study the semigroup of continuous functions from a topological space to itself under the operation of composition of maps, in the point of view of Smarandache semigroup's approach.

Keywords Continuous map, Smarandache semigroup, topological space.

§1. Introduction

As it has been mentioned [3, page 29], a Smarandache semigroup S is a semigroup which is not a group and has a proper subset A with at least two elements such that A under the operation of S is a group. One may want to restrict the elements of S under particular properties, e.g., one may ask about $\overline{G}$ in X^X where X is a compact Hausdorff topological space and G is a semigroup of continuous functions on X, this case has been studied in [1]. In this paper our main interest is on semigroup of continuous functions on X, i.e., $C(X, X)$.

Let X be a topological space. By $C(X, X)$ we mean the set of all continuous maps like $f : X \to X$, which is clearly a semigroup under the composition of maps. In the next section which is the main section of this paper, we want to study the conditions under which $C(X, X)$ is a Smarandache semigroup.

Remark. If $f : X \to Y$ is a map and $D \subseteq X$, by $f|_D : D \to Y$ we mean the restriction of f to D. Moreover $id_X : X \to X$ denotes the identity function on X, i.e., $id_X(x) = x (x \in X)$.

§2. $C(X, X)$ and Smarandache semigroup's concept

In this section we want to be as close as possible to the cases in which $C(X, X)$ is a Smarandache semigroup (under the composition of maps). From now on suppose X is a topological space with at least two elements and consider $C(X, X)$ under the composition of maps operation (so $C(X, X)$ is a semi-group).

Lemma 2.1. $C(X, X)$ is not a group.

[1] With thanks to Prof. A. Mamourian for his helpful comments, ideas and encouragement.

Proof. Let x_1, x_2 be two distinct elements of X and $f_i(x) = x_i$, then f_1, f_2 are to idempotents of $C(X, X)$ but $f_1, f_2 = id_X$.

Corollary 2.2. $C(X, X)$ contains a group with at least two elements if and only if it is a Smarandache semigroup.

Proof. Use Lemma 2.1.

Theorem 2.3. If X satisfies one of the following conditions, then $C(X, X)$ is a Smarandache semigroup:

(1) There exists a homeomorphism $f : X \to X$ such that $f = id_X$;

(2) There exists nonempty disjoint topological spaces Y and Z such that $X = Y \cup Z$ and X is topological disjoint union of Y and Z, i.e., $\{U \cup V : U$ is an open subset of Y and V is an open subset of $Z\}$ is the topology of X.

Proof. (1) $f_n : n \in Z$ is a subgroup of $C(X, X)$ with at least two elements ($id_X(= f^0)$ and $f(= f^1)$), now use Corollary 2.2.

(2) Let $x_1 \in Y$ and $x_2 \in Z$. Define $g : X \to X$ with $g(x) = x_2$ for $x \in Y$ and $g(x) = x_1$ for $x \in Z$. $g, g \circ g$ is a subset of $C(X, X)$ with two elements and it is a group under the composition of maps. Corollary 2.2 completes the proof.

Counterexample 2.4.

(1) Consider $Y := \mathbb{N}$ with topology $\{\{1, \cdots, n\} : n \in \mathbb{N}\} \cup \{\emptyset, \mathbb{N}\}$ and $Z := \{0\}$ with topology $\{Z, \emptyset\}$. Then $X = Y \cup Z$ as topological disjoint union of Z and Y satisfies item (2) in Theorem 2.3, so $C(X, X)$ is a Smarandache semigroup, however X does not satisfy item (1) in Theorem 2.3.

(2) Consider $X := F$ with topology $\{\{-n, -n+1, \cdots, n\} : n \in \mathbb{N}\} \cup \{X, \emptyset\}$ satisfies item (1) in Theorem 2.3 ($-id_X : X \to X$ is a homeomorphism and $-id_X \neq id_X$) so $C(X, X)$ is a Smarandache semigroup, however X does not satisfy item (2) in Theorem 2.3 since for any two nonempty open subsets U, V of X, $\{0\} \subseteq U \cap V$ and $U \cap V \neq \emptyset$.

(3) Consider $X := \{1, 2\}$ with topology $\{\{1\}, X, \emptyset\}$, then $C(X, X)$ is the set consisting of three elements: id_X, constant function 1, and constant function 2, so it is clear that all of the elements of $C(X, X)$ are idempotents, thus $C(X, X)$ does contain any group with at least two elements and by Corollary 2.2 it is not a Smarandache semigroup.

Corollary 2.5. $C(X \times X, X \times X)$ is a Smarandache semigroup.

Proof. $f : X \times X \to X \times X$ with $f(x, y) = (y, x)((x, y) \in X \times X)$ is a homeomorphism and $f \neq id_{X \times X}$, so by Theorem 2.3, $C(X \times X, X \times X)$ is a Smarandache semigroup.

Note 2.6. Comparing Corollary 2.5 and item (3) in Counterexample 2.4, leads us to the fact that there are cases in which $C(X, X)$ is not a Smarandache semigroup but $C(X \times X, X \times X)$ is a Smarandache semigroup.

Proposition 2.7. Let Y be a topological space. If $C(X, X)$ is a Smarandache semigroup, then $C(X \times Y, X \times Y)$ is a Smarandache semigroup too.

Proof. Since $C(X, X)$ is a Smarandache semigroup, thus there exists a group $G \subseteq C(X, X)$ with more than two elements. For $K := \{(f, id_Y) : f \in G\}$ (where $(f, id_Y) : X \times Y \to X \times Y$ is $(f, id_Y)(x, y) = (f(x), y)$) is a group and subset of $C(X \times Y, X \times Y)$ with more than one element, so by Corollary 2.2, $C(X \times Y, X \times Y)$ is a Smarandache semigroup.

Note 2.8. If X is discrete, then it is topological disjoint union of two spaces, so $C(X, X)$

is a Smarandache semigroup by Theorem 2.3.

Theorem 2.9. If (X, X) is a Smarandache semigroup and $G \subseteq C(X, X)$ with at least two elements is a group, then there exists $W \subseteq X$ with at least two elements such that for all $f \in G$:

(1) $f|_W : Y \to W$ is a homeomorphism.

(2) $f|_W = id_W$ if and only if f is the identity of G.

Proof. Suppose k is the identity of G and $f \in G$. For all $x \in X, k(f(x)) = f(k(x)) = f(x)$, therefore $f(x) \in \{z \in X : k(z) = z\} =: W$. Therefore $f|_W : W \to W$ is a continuous function. It is clear that $k|_W = id_W$. Moreover there exists $g \in G$ such that $f \circ g = g \circ f = k$, also $g|_W : W \to W$ is continuous. Using $f|_W \circ g|_W = g|_W \circ f|_W = k|_W = id_W$ leads us to the fact that $f|_W : W \to W$ is a homeomorphism.

Since $f(X) \subseteq W$, thus $W \neq \emptyset$. We claim that W has at least two elements, otherwise $W = a$, f is constant function a (note that $f(X) \subseteq W = \{a\}$), and it is idempotent, since $f \in G$ is arbitrary, thus all of the elements of G are idempotent, which is a contradiction since G is a group with more than one element and in a group there exists just one idempotent element.

Moreover suppose $f \in G$ is such that $f|_W = id_W$. There exists $g \in G$ such that $f \circ g = k$. In addition we know $g(X) \subseteq W$, thus for all $x \in X$ we have $k(x) = f(g(x)) = f|_W(g(x)) = id_W(g(x)) = g(x)$ which shows $g = k$, thus $k = f \circ g = f \circ k = f$.

Considering Theorem 2.3 and Theorem 2.9 make us to ask:

Problem 2.10. If $C(X, X)$ is a Smarandache semigroup, does at least one of the conditions:

(1) There exists a homeomorphism $f : X \to X$ such that $f = id_X$;

(2) There exists nonempty disjoint topological spaces Y and Z such that $X = Y \cup Z$ and X is topological disjoint union of Y and Z. hold?

§3. A short glance to other topological properties else continuity

In this section we deal with examples of semigroups of resp. close, open, clopen, and proper maps from topological space X to itself, with the operation of composition of maps.

Remark. In topological space X:

(1) $f : X \to X$is called open if for any open subset U of $X, f(U)$ is open;

(2) $f : X \to X$ is called close if for any close subset C of $X, f(C)$ is close;

(3) $f : X \to X$ is called clopen if it is both close and open;

(4) $f : X \to X$ is called proper if for any compact subset K of $X, f^{-1}(K)$ is compact.

Remark. If X has at least two elements, then X^X, the set of all functions $f : X \to X$, is a Smarandache semigroup.

Lemma 3.1. If X has at least two elements, then $S = \{f \in X^X : \forall x \in X(f^{-1}(x) \text{ is finite})\}$ is a Smarandache semigroup.

Proof. Let a, b be two distinct elements of X and X has at least three elements. Define $f : X \to X$ with $f(x) = a$ for $x = a, b$ and $f(x) = x$ for $x \in X - \{a, b\}$. f and id_X are two different idempotent elements of S, so S is not a group. Define $g : X \to X$ with $g(a) = b$,

$g(b) = a$ and $g(x) = x$for $x \in X - \{a, b\}$, $g, g \circ g$ is a subset of S with two elements and it is a group, thus S is a Smarandache semigroup.

Example 3.2. Let S be the semigroup of all closed maps like $f : X \to X$, under composition of maps:

(i) If X with at least two elements is discrete, then $S = X^X$ is a Smarandache semigroup;

(ii) If $X = \mathbb{Z}$ with topology $\{X, \emptyset\} \cup \{X - \{-n, \cdots, -1, 0, 1, \cdots, n\} : n \in \mathbb{N} \cup \{0\}\}$, then X is not discrete and S is a Smarandache semigroup; since:
$-id_X$ and constant function 0 are two different idempotent elements of S, so S is not a group,
$-\{id_X, -id_X\}$ is a subgroup of S with two elements;

(iii) If $X = \{1, 2\}$ with topology $\{X, \{1\}, \emptyset\}$, then S has two elements and it is not a Smarandache semigroup.

Example 3.3. Let S be the semigroup of all open maps like $f : X \to X$, under composition of maps:

(i) If X with at least two elements is discrete, then $S = X^X$ is a Smarandache semigroup;

(ii) If $X = \mathbb{Z}$ with topology $\{X, \emptyset\} \cup \{\{-n, \cdots, -1, 0, 1, \cdots, n\} : n \in \mathbb{N} \cup \{0\}\}$, then X is not discrete and S is a Smarandache semigroup (use a similar method described in Example 3.2);

(iii) If $X = \{1, 2\}$ with topology $\{X, \{1\}, \emptyset\}$, then S has two elements and it is not a Smarandache semigroup.

Example 3.4. Let S be the semigroup of all clopen maps like $f : X \to X$, under composition of maps:

(i) If X with at least two elements is discrete, then $S = X^X$ is a Smarandache semigroup;

(ii) If $X = \mathbb{Z} \cup \pi\mathbb{Z}$ with topological basis $\{\{-n, \cdots, -1, 0, 1, \cdots, n\} \cup A : n \in \mathbb{N} \cup \{0\}, A \subseteq \pi\mathbb{Z}\}$, then X is not discrete and S is a Smarandache semigroup (use a similar method described in Example 3.2);

(iii) If $X = \{1, 2\}$ with topology $\{X, \{1\}, \emptyset\}$, then S has one element and is not a Smarandache semigroup.

Example 3.5. Let S be the set of all proper maps like $f : X \to X$, under composition of maps (use Lemma 3.1):

(i) If X with at least two elements is discrete, then $S = \left\{f \in X^X : \forall x \in X(f^{-1}(x) \text{ is finite})\right\}$ is a Smarandache semigroup;

(ii) If $X = \mathbb{Z} \cup \pi\mathbb{Z}$ with topological basis $\{\{-n, \cdots, -1, 0, 1, \cdots, n\} \cup A : n \in \mathbb{N} \cup \{0\}, A \subseteq \pi\mathbb{Z}\}$, then S is a Smarandache semigroup, and X is not discrete (in this case we have $S = \left\{f \in X^X : \forall x \in X(f^{-1}(x) \text{ is finite})\right\}$ too).

§4. More examples

In this section suppose G (resp. $\overline{G}$) is a domain (resp. closed domain) in $\mathbb{C}$, also suppose it is bounded with nonempty interior. We have the following examples which deal with product operation (not composition of maps) on complex valued maps with a paticular property.

(1) $C(\overline{G})$: the set of continuous complex valued functions on closed domain $\overline{G}$ [4, page 3]. $C(\overline{G})$ under product operation is a Smarandache semigroup.

(2) $C^m(G)$: the set of continuous complex valued functions on domain G, with continuous partial derivations up to the m-th order [4, page 3]. $C^m(G)$ under product operation is a Smarandache semigroup.

(3) $C_\alpha(\overline{G})$: the set of all bounded functions $f : \overline{G} \to \mathbb{C}$ for closed domain $\overline{G}$ such that there exists $0 < H < +\infty$ with

$$\forall z_1, z_2 \in \overline{G}(|f(z_1) - f(z_2)| \leq H|z_1 - z_2|^\alpha)$$

for $0 < \alpha \leq 1$ [4, page 7]. $C_\alpha(\overline{G})$ under product operation is a Smarandache semigroup.

(4) $C^m_\alpha(\overline{G}) = \left\{ f \in C^m(\overline{G}) : \forall k \in \{0, \cdots, m\} \frac{\partial^m f}{\partial x^{m-k} - \partial y^k} \in C_\alpha(\overline{G}) \right\}$ for $0 < \alpha \leq 1$ and a closed domain $\overline{G}$ [4, page 7]. $C^m_\alpha(\overline{G})$ under product operation is a Smarandache semigroup.

(5) $A(G)$: the set of all analytic functions on G. $A(G)$ under product operation a Smarandache semigroup.

(6) $\ell^\infty(X)$: the set of all bounded functions like $f : X \to C$ [2, Chapter 6] where X has more than one element. $\ell^\infty(X)$ under product operation is a Smarandache semigroup.

(7) $\ell^p(X)$: the set of all functions like $f : X \to C$ where X at least two elements, with $\sum_{x \in X} |f(x)|^p < +\infty$, for fixed $p \in [1, +\infty)$ [2, Chapter 6] where X has more than one element. $\ell^p(X)$ under product operation is a Smarandache group.

Proof. For items $(1), \cdots, (6)$: let S denotes the related semigroup. For $h(x) = 0$, $f(x) = 1$ we have $h, f \in S$ and it is clear that under product operation S is not a group, moreover $f, -f$ is a proper subset of S with two elements which is group under product operation, so S is a Smarandache semigroup.

(7): Suppose x_1, x_2 be two distinct elements of X, for $i = 1, 2$ let $f_i(x_i) = 1$ and $f_i(x) = 0$ for $x \in X - \{x_i\}$, then $0 = f_1 f_2, f_1, f_2 \in \ell^p(X)$ and $\ell^p(X)$ under product operation is not group, but it contains $\{f_1, -f_1\}$ which is a group, so $\ell^p(X)$ under product operation is a Smarandache semigroup.

References

[1] Ayatollah Zadeh Shirazi, F., Hosseini, A., Some faces of Smarandache semigroups' concept in transformation semigroups' approach, Scientia Magna, **3**(2007), 1-4.

[2] Folland, G. B., Real analysis, Modern techniques and their applications, Second edition, Pure and Applied Mathematics (New York), A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1999.

[3] Vasantha Kandasamy, W. B., Smarandache semigroups, American Research Press, Rehoboth, 2002.

[4] Vekua, I. N., Generalized analytic functions, (translated from Russian), International Series of Monographs on Pure and Applied Mathematics, Pergamon Press, **25**(1962).