International J.Math. Combin. Vol.1 (2009), 18-26

Shortest Co-cycle Bases of Graphs

Han Ren

(Department of Mathematics,East China Normal University Shanghai 200062,P.R.China)

E-mail: hren@math.ecnu.edu.cn

Jing Ren

(Economics and Management School of Wuhan University Wuhan 430072,P.R.China)

Abstract In this paper we investigate the structure of the shortest co-cycle base(or SCB in short) of connected graphs, which are related with *map geometries*, i.e., *Smarandache* 2-*dimensional manifolds*. By using a Hall type theorem for base transformation, we show that the shortest co-cycle bases have the same structure (there is a 1-1 correspondence between two shortest co-cycle bases such that the corresponding elements have the same length). As an application in surface topology, we show that in an embedded graph on a surface any nonseparating cycle can't be generated by separating cycles. Based on this result, we show that in a 2-connected graph embedded in a surface, there is a set of surface nonseparating cycles which can span the cycle space. In particular, there is a shortest base consisting surface nonseparating cycle and all such bases have the same structure. This extends a Tutte's result [4].

Key Words: Shortest co-cycle base, nonseparating cycle, map geometries, Smarandache 2-dimensional manifolds.

MR(2000): 05C30.

§1. Introduction

In this paper, graphs are finite, undirected, connected. Used terminology is standard and may be found in [1] − [2]. Let A and B be nonempty(possibly overlapping) subsets of $V(G)$. The set $[A,B]$ is a subset of $E(G)$, namely,

$$[A,B] = \{(a,b) \in E(G) | a \in A, b \in B\}.$$

Then the edge set between S and $\overline{S}$ is a *co-cycle*(or a *cut*), denoted by $[S,\overline{S}]$, where S is a nonempty subset of $V(G)$ and $\overline{S} = V(G) - S$. Particularly, for any vertex u, $[u] = \{(u,v) | v \in V(G)\}$ is called a *vertical co-cycle*(or a *vertical cut*). Let X and Y be a pair of sets of edges of G. Then the following operations on co-cycles defined as

$$X \oplus Y = X \cup Y - X \cap Y,$$

[1]Received October 30, 2008. Accepted January 10, 2009.
[2]Supported by NNSF of China under the granted number 10671073

will form a linear vector space $\mathcal{C}^*$, called *co-cycle space* of G. It's well known that the dimension of co-cycle space of a graph G is $n-1$, where n is the number of vertices of G.

The length of a co-cycle c, denoted by $\ell(c)$, is the number of edges in c. The length of a base $\mathcal{B}$, denoted by $\ell(\mathcal{B})$, is the sum of the lengths of its co-cycles. A shortest base is that having the least number of edges.

Let $A, B \subseteq E(G)$. Then we may define an inner product denoted by (A, B) as

$$(A, B) = \sum_{e \in A \cap B} | e |, \qquad |e| = 1.$$

Since any cycle C has even number edges in any co-cycle, i.e., for any cycle C and a co-cycle $[S, \overline{S}]$

$$(C, [S, \overline{S}]) = 0$$

we have that C is *orthogonal* to $[S, \overline{S}]$, i.e.,

Theorem 1 *Let $\mathcal{C}$ and $\mathcal{C}^*$ be, respectively, the cycle space and co-cycle space of a graph G. Then $\mathcal{C}^*$ is just the orthogonal space of $\mathcal{C}$, i.e., $\mathcal{C}^{\perp} = \mathcal{C}^*$, which implies that*

$$\dim \mathcal{C} + \dim \mathcal{C}^* = |E(G)|.$$

There are many results on cycle space theory. But not many results have ever been seen in co-cycle spaces theory. Here in this paper we investigate the shortest co-cycle bases in a co-cycle space. We first set up a Hall Type theorem for base transformation and then give a sufficient and necessary condition for a co-cycle base to be of shortest. This implies that there exists a 1-1 correspondence between any two shortest co-cycle bases and the corresponding elements have the same length. As applications, we consider embedded graphs in a surface. By the definition of geometric dual multigraph, we show that a nonseparating cycle can't be generated by a collection of separating cycles. So there is a set of surface nonseparating cycles which can span the cycle space. In particular, there is a shortest base consisting surface nonseparating cycle and all such bases have the same structure. This extends a Tutte's result [4].

§2. Main results

Here in this section we will set up our main results. But first we have to do some preliminary works. Let $A = (A_1, A_2, \cdots, A_n)$ be a set of finite sets. A *distinct representatives(SDR)* is a set of $\{a_1, a_2, \cdots, a_n\}$ of n elements such that $a_i \in A_i$ for $i = 1, 2, \cdots, n$. The following result is the famous condition of Hall for the existence of SDR.

Hall's Theorem([3]) *A family $(A_1, \cdots, A_n)$ of finite sets has a system of distinct representatives(SDR) if and only if the following condition holds:*

$$\left| \bigcup_{\alpha \in J} A_\alpha \right| \geq |J|, \qquad \forall J \subseteq \{1, \cdots, n\}.$$

Let G be a connected graph with a co-cycle base $\mathcal{B}$ and c a co-cycle. We use $\mathrm{Int}(c,\mathcal{B})$ to represent the co-cycles in $\mathcal{B}$ which span c.

Another Hall Type Theorem *Let G be a connected graph with $\mathcal{B}_1$ and $\mathcal{B}_2$ as two co-cycle bases. Then the system of sets $A = \{\mathrm{Int}(c,\mathcal{B}_1) \mid c \in \mathcal{B}_2\}$, has a SDR.*

Proof What we need is to show that the system must satisfy the Hall's condition:

$$\forall J \subseteq \mathcal{B}_2 \Rightarrow \left|\bigcup_{c\in J} \mathrm{Int}(c,\mathcal{B}_1)\right| \geq \left|J\right|.$$

Suppose the contrary. Then $\exists J \subseteq \mathcal{B}_2$ such that $\left|\bigcup_{c\in J} \mathrm{Int}(c,\mathcal{B}_1)\right| < \left|J\right|$. Now the set of linear independent elements $\{c \mid c \in J\}$ is spanned by at most $\left|J\right| - 1$ vectors in $\mathcal{B}_1$, a contradiction as desired. □

Theorem 2 *Let $\mathcal{B}$ be a co-cycle base of G. Then $\mathcal{B}$ is shortest if and only if for any co-cycle c,*

$$\forall \alpha \in \mathrm{Int}(c,\mathcal{B}) \Rightarrow \ell(c) \geq \ell(\alpha).$$

Remark This result shows that in a shortest co-cycle base, a co-cycle can't be generated by shorter vectors.

Proof Let $\mathcal{B}$ be a co-cycle base of G. Suppose that there is a co-cycle c such that $\exists \alpha \in \mathrm{Int}(c), \ell(c) < \ell(\alpha)$, then $\mathcal{B} - c + \alpha$ is also a co-cycle base of G, which is a shorter co-cycle base, a contradiction as desired.

Suppose that $\mathcal{B} = \{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$ is a co-cycle base of G such that for any co-cycle $c, \ell(c) \geq \ell(\alpha), \forall \alpha \in \mathrm{Int}(c)$, but $\mathcal{B}$ is not a shortest co-cycle base. Let $\mathcal{B}^* = \{\beta_1, \beta_2, \cdots, \beta_{n-1}\}$ be a shortest co-cycle base. By Hall Type Theorem, $A = (\mathrm{Int}(\beta_1,\mathcal{B}), \mathrm{Int}(\beta_2,\mathcal{B}), \cdots, \mathrm{Int}(\beta_{n-1},\mathcal{B}))$ has an SDR $(\alpha_1', \alpha_2' \cdots, \alpha_{n-1}')$ such that $\alpha_i' \in \mathrm{Int}(\beta_i,\mathcal{B}), \ell(\beta_i) \geq \ell(\alpha_i')$. Hence $\ell(\mathcal{B}^*) = \sum\limits_{i=1}^{n-1} \ell(\beta_i) \geq \sum\limits_{i=1}^{n-1} \ell(\alpha_i') = \ell(\mathcal{B})$, a contradiction with the definition of $\mathcal{B}$. □

The following results say that some information about short co-cycles is contained in a shorter co-cycle base.

Theorem 3 *If $\{c_1, c_2, \cdots, c_k\}$ is a set of linearly independent shortest co-cycles of connected graph G, then there must be a shortest co-cycle base containing $\{c_1, c_2, \cdots, c_k\}$.*

Proof Let $\mathcal{B}$ be the shortest co-cycle base such that the number of co-cycles in $\mathcal{B} \cap \{c_1, c_2, \cdots, c_k\}$ is maximum. Suppose that $\exists c_i \notin \mathcal{B}, 1 \leq i \leq k$. Then $\mathrm{Int}(c_i,\mathcal{B})\backslash\{c_1, \cdots, c_k\}$ is not empty, otherwise $\{c_1, c_2, \cdots, c_k\}$ is linear dependent. So there is a co-cycle $\alpha \in \mathrm{Int}(c_i,\mathcal{B})\backslash\{c_1, \cdots, c_k\}$ such that $\ell(c_i) \geq \ell(\alpha)$. Then $\ell(c_i) = \ell(\alpha)$, since c_i is the shortest co-cycle. Hence $\mathcal{B}^* = \mathcal{B} - \alpha + c_i$ is a shortest co-cycle base containing more co-cycles in $\{c_1, c_2, \cdots, c_k\}$ than $\mathcal{B}$. A contradiction with the definition of $\mathcal{B}$. □

Corollary 4 *If c is a shortest co-cycle, then c is in some shortest co-cycle base.*

Theorem 5 *Let $\mathcal{B}, \mathcal{B}^*$ be two different shortest co-cycle bases of connected graph G, then exists a one-to-one mapping $\varphi : \mathcal{B} \to \mathcal{B}^*$ such that $\ell(\varphi(\alpha)) = \ell(\alpha)$ for all $\alpha \in \mathcal{B}$.*

Proof Let $\mathcal{B} = \{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}, \mathcal{B}^* = \{\beta_1, \beta_2, \cdots, \beta_{n-1}\}$. By Hall Type Theorem, $A = (\mathrm{Int}(\alpha_1, \mathcal{B}^*), \mathrm{Int}(\alpha_2, \mathcal{B}^*), \cdots, \mathrm{Int}(\alpha_{n-1}, \mathcal{B}^*))$ has a SDR $(\beta_{\sigma(1)}, \beta_{\sigma(2)}, \cdots, \beta_{\sigma(n-1)})$, where σ is a permutation of $\{1, 2, \cdots, n-1\}$. Since $\mathcal{B}^*$ is a SCB, by Theorem 2, we have $\ell(\alpha_i) \geq \ell(\beta_{\sigma(i)}), \forall i = 1, \ldots, n-1$. On the other hand, $\mathcal{B}$ and $\mathcal{B}^*$ are both shortest, i.e. $\ell(\mathcal{B}) = \ell(\mathcal{B}^*)$. So $\ell(\alpha_i) = \ell(\beta_{\sigma(i)}), \forall i = 1, \ldots, n-1$. Let $\varphi(\alpha_i) = \beta_{\sigma(i)}, \forall i = 1, \ldots n-1$. Then φ is a one-to-one mapping such that $\ell(\varphi(\alpha)) = \ell(\alpha)$ for all $\alpha \in \mathcal{B}$. □

Since a co-cycle can't be generated by longer ones in a shortest co-cycle base, we have

Corollary 6 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of shortest co-cycle bases in a graph G. Then their parts of shortest co-cycles are linearly equivalent.*

Example 1 The length of the SCB of complete graph K_n is $(n-1)^2$.

Example 2 The length of the SCB of complete graph $K_{a,b}$ $(a \leq b)$ is $2ab - b$.

Example 3 The length of the SCB of a tree with n vertex T_n is $n - 1$.

Example 4 The length of the SCB of a Halin graph with n vertex is $3(n-1)$.

Proof of Examples By theorem 1, for any vertex v, the vertical co-cycle $[v]$ is the shortest co-cycle of K_n. Clearly the set of $n-1$ vertical co-cycles is a SCB. So there're n SCBs with length $(n-1)^2$.

The proof for examples $2, 3$ and 4 is similar. □

§3. Application to surface topology

In this section we shall apply the results obtained in Section 1 to surface topology. Now we will introduce some concepts and terminologies in graph embedding theory, which are related with *map geometries*, i.e., *Smarandache 2-dimensional manifolds.*

Let G be a connected multigraph. An *embedding* of G is a pair $\Pi = (\pi, \lambda)$ where $\pi = \{\pi_v \mid v \in V(G)\}$ is a rotation system and λ is a signature mapping which assigns to each edge $e \in E(G)$ a sign $\lambda(e) \in \{-1, 1\}$. If e is an edge incident with $v \in V(G)$, then the cyclic sequence $e, \pi_v(e), \pi_v^2(e), \cdots$ is called the Π-*clockwise ordering* around v(or the local rotation at v). Given an embedding Π of G we say that G is Π-*embedded.*

We define the Π-*facial walks* as the closed walks in G that are determined by the face traversal procedure. The edges that are contained(twice) in only one facial walk are called *singular*.

A cycle C of a Π-embedded graph G is Π-*onesided* if it has an odd number of edges with negative sign. Otherwise C is Π-*twosided.*

Let H be a subgraph of G. An H-*bridge* in G is a subgraph of G which is either an edge not in H but with both ends in H, or a connected component of $G - V(H)$ together with all edges which have one end in this component and other end in H.

Let $C = v_0e_1v_1e_2\cdots v_{l-1}e_lv_l$ be a Π-twosided cycle of a Π-embedded graph G. Suppose that the signature of Π is positive on C. We define the *left graph* and the *right graph* of C as follows. For $i = 1, \cdots, l$, if $e_{i+1} = \pi_{v_i}^{k_i}(e_i)$, then all edges $\pi_{v_i}(e_i), \pi_{v_i}^2(e_i), \cdots, \pi_{v_i}^{k_i-1}(e_i)$ are said to be on the left side of C. Now, the left graph of C, denoted by $G_l(C,\Pi)$(or just $G_l(C)$), is defined as the union of all C-bridges that contain an edge on the left side of C. The right graph $G_r(C,\Pi)$(or just $G_r(C)$) is defined analogously. If the signature is not positive on C, then there is an embedding Π' equivalent to Π whose signature is positive on C(since C is Π-twosided). Now we define $G_l(C,\Pi)$ and $G_r(C,\Pi)$ as the left and the right graph of C with respect to the embedding Π'. Note that a different choice of Π' gives rise to the same pair $\{G_l(C,\Pi), G_r(C,\Pi)\}$ but the left and the right graphs may interchange.

A cycle C of a Π-embedded graph G is Π-*separating* if C is Π-twosided and $G_l(C,\Pi)$ and $G_r(C,\Pi)$ have no edges in common.

Given an embedding $\Pi = (\pi, \lambda)$ of a connected multigraph G, we define the geometric dual multigraph G^* and its embedding $\Pi^* = (\pi^*, \lambda^*)$, called the *dual embedding* of Π,as follows. The vertices of G^* correspond to the Π-facial walks. The edges of G^* are in bijective correspondence $e \longmapsto e^*$ with the edges of G, and the edge e^* joins the vertices corresponding to the Π-facial walks containing e.(If e is singular, then e^* is a loop.) If $W = e_1, \cdots, e_k$ is a Π-facial walk and w its vertex of G^*, then $\pi_w^* = (e_1^*, \cdots, e_k^*)$. For $e^* = ww'$ we set $\lambda^*(e^*) = 1$ if the Π-facial walks W and W' used to define π_w^* and $\pi_{w'}^*$ traverse the edge e in opposite direction; otherwise $\lambda^*(e^*) = -1$.

Let H be a subgraph of G. H^* is the union of edges e^* in G^*, where e is an edge of H.

Lemma 7 *Let G be a Π-embedded graph and G^* its geometric dual multigraph. C is a cycle of G. Then C is a Π-separating cycle if and only if C^* is a co-cycle of G^*, where C^* is the set of edges corresponding those of C.*

Proof First, we prove the necessity of the condition. Since C is a Π-separating cycle, C is Π-twosided and $G_l(C,\Pi)$ and $G_r(C,\Pi)$ have no edges in common. Assume that $C = v_0e_1v_1e_2\cdots v_{l-1}e_lv_l$, and$\lambda(e_i) = 1, i = 1, \cdots, l$. We divide the vertex set of G^* into two parts V_l^* and V_r^*, such that for any vertex w in $V_l^*(V_r^*)$, w corresponds to a facial walk W containing an edge in $G_l(C)(G_r(C))$.

Claim 1. $V_l^* \cap V_r^* = \Phi$, i.e. each Π-facial walk of G is either in $G_l(C) \cup C$ or in $G_r(C) \cup C$.

Otherwise, there is a Π-facial walk W of G, such that W has some edges in $G_l(C)$ and some in $G_r(C)$. Let $W = P_1Q_1\cdots P_kQ_k$, where P_i is a walk in which none of the edges is in $C(i = 1, \cdots, k)$,and Q_i is a walk in which all the edges are in $C(j = 1, \cdots, k)$.Since each P_i is contained in exactly one C-bridge, there exist $t \in \{1, \cdots, k\}$ such that $P_t \subseteq G_l(C), P_{t+1} \subseteq G_r(C)$(Note $P_{t+1} = P_1$). Let $Q_t = v_pe_{p+1}\cdots e_qv_q$. Then $W = \cdots e^tv_pe_{p+1}\cdots e_qv_qe^{t+1}\cdots$, where $e^t \in P_t, e^{t+1} \in P_{t+1}$. Since e^t and e^{t+1} are, respectively, on the left and right side of C, $\pi_{v_p}(e^t) = e_{p+1}$ and $\pi_{v_q}(e^{t+1}) = e_q$. As W is a Π-facial walk, there exist an edge e in Q_t such that $\lambda(e) = -1$, a contradiction with the assumption of C.

Next we prove that $[V_l^*, V_r^*] = C^*$.

Let $e^* = w_1w_2$ be an edge in G^*, where w_1 and w_2 are, respectively, corresponding to the

Π-facial walks W_1 and W_2 containing e in common.

If $e^* \in [V_l^*, V_r^*]$ where $w_1 \in V_l^*, w_2 \in V_r^*$. Then $W_1 \subseteq G_l(C) \cup C$ and $W_2 \subseteq G_r(C) \cup C$. As $G_l(C, \Pi)$ and $G_r(C, \Pi)$ have no edges in common, we have $e \in C$ i.e. $e^* \in C^*$. So $[V_l^*, V_r^*] \subseteq C^*$.

Claim 2. If $e^* = w_1 w_2 \in C^*$, i.e., $e \in C$, then $W_1 \neq W_2$, and W_1, W_2 can't be contained in $G_l(C) \cup C$(or $G_r(C) \cup C$) at the same time.

Suppose that $W_1 = W_2$. Let $W_1 = u_0 e u_1 \widetilde{e_1} u_2 \widetilde{e_2} \cdots u_k \widetilde{e_k} u_1 e u_0 \cdots$. Clearly, $\{\widetilde{e_1}, \cdots, \widetilde{e_k}\}$ is not a subset of $E(C)$, otherwise C isn't a cycle. So we may assume that $\widetilde{e_s} \notin C, \widetilde{e_t} \notin C, (1 \leq s \leq t \leq k)$ such that $\widetilde{e_i} \in C, i = 1, \cdots, s-1$ and $\widetilde{e_j} \in C, j = t+1, \cdots, k$. Let $C = u_0 e u_1 \widetilde{e_1} \cdots \widetilde{e_{s-1}} \cdots = u_0 e u_1 \widetilde{e_k} u_k \cdots \widetilde{e_{t+1}} \cdots$. Since W_1 is a Π-facial walk, assume that $\widetilde{e_1} = \pi_{u_1}(e)$ and $\widetilde{e_k} = \pi_{u_1}^{-1}(e)$. As the sign of edges on C is 1,we get $\widetilde{e_s} = \pi_{u_s}(\widetilde{e_{s-1}})$ and $\widetilde{e_t} = \pi_{u_{t+1}}^{-1}(\widetilde{e_{t+1}})$. So $\widetilde{e_s} \in G_l(C)$ and $\widetilde{e_t} \in G_r(C)$, a contradiction with **Claim 1**.

Suppose $W_1 \neq W_2$ and $W_1, W_2 \subseteq G_l(C) \cup C$. Let $W_1 = v_0 e v_1 e_1^1 v_2^1 e_2^1 \cdots v_0$ and $W_2 = v_0 e v_1 e_1^2 v_2^2 e_2^2 \cdots v_0$. Assume that $e_1^1 \neq e_1^2$, otherwise we consider e_2^1 and e_2^2.

Case 1. $e_1^1 \in C$ and $e_1^2 \in C$. Then $e_1^1 = e_1^2$.

Case 2. $e_1^1 \notin C$ and $e_1^2 \notin C$. By claim 1, $\pi_{v_1}(e) = e_1^1$ and $\pi_{v_1}(e) = e_1^2$, then $e_1^1 = e_1^2$.

Case 3. $e_1^1 \notin C$ and $e_1^2 \in C$. By claim 1, $\pi_{v_1}(e) = e_1^1$. As $e_1^1 \neq e_1^2$, we get $\pi_{v_1}^{-1}(e) = e_1^2$. Let $e_t^2 \notin C$, and $e_1^2, \cdots, e_{t-1}^2 \in C$. Since $\lambda(e_i^2) = 1$, $\pi_{v_{i+1}^2}^{-1}(e_i^2) = e_{i+1}^2$ $(i = 1, \cdots, t-1)$. Then $\pi_{v_t^2}^{-1}(e_{t-1}^2) = e_t^2$,i.e. $e_t^2 \in G_r(C)$. So $W_2 \subseteq G_r(C) \cup C$, a contradiction with **Claim 1**.

Case 4. $e_1^1 \in C$ and $e_1^2 \notin C$.Like case 3,it's impossible.

So claim 2 is valid. And by claim 2, $C^* \subseteq [V_l^*, V_r^*]$.

Summing up the above discussion, we get that C^* is a co-cycle of G^*.

Next, we prove the sufficiency of the condition. Since C^* is a co-cycle of G^*, let $C^* = [V_l^*, V_r^*]$, where $V_l^* \cap V_r^* = \Phi$. Then all the Π-facial walks are divided into two parts F_l and F_r, where for any Π-facial walk W in $F_l(F_r)$ corresponding to a vertex w in $V_l^*(V_r^*)$. Firstly, we prove that C is twosided.Let $C = v_0 e_1 v_1 e_2 \cdots v_{l-1} e_l v_l$. Supposed that C is onesided, with $\lambda(e_1) = -1$ and $\lambda(e_i) = 1, i = 2 \cdots, l$. Then $\lambda^*(e_1^*) = -1$ and $\lambda^*(e_i^*) = 1, i = 2 \cdots, l$. Let $e_1^* = \widetilde{w_1} \widetilde{w_2}$, where $\widetilde{w_1} \in V_l^*, \widetilde{w_2} \in V_r^*$. Suppose that $\widetilde{w_1}$ and $\widetilde{w_2}$ are, respectively, corresponding to the Π-facial walks $\widetilde{W_1}$ and $\widetilde{W_2}$ containing e_1.Then $\widetilde{W_1} \in F_l, \widetilde{W_2} \in F_r$. Since $\widetilde{W_1}$ is a Π-facial walk, there must be another edge $\widetilde{e_2}$ with negative sign appearing once in $\widetilde{W_1}$. We change the signature of $\widetilde{e_2}$ into 1.(Here we don't consider the embedding) Suppose W_2 is the other Π-facial walk containing $\widetilde{e_2}$. Like $\widetilde{W_1}$, there must be an edge $\widetilde{e_3}$ with negative sign appearing once in W_2.Then change the signature of $\widetilde{e_3}$ into 1. So similarly we got a sequence $\widetilde{W_1}, \widetilde{e_2}, W_2, \widetilde{e_3}, W_3, \cdots$,where the signature of $\widetilde{e_2}, \widetilde{e_3}, \cdots$ in Π are -1, and $W_2, W_3, \cdots$ are all in W_l. Since the number of edges with negative sign is finite, $\widetilde{W_2}$ must in the sequence, a contradiction with $V_l^* \cap V_r^* = \Phi$.

Secondly, we prove that $G_l(C)$ and $G_r(C)$ have no edge in common.

Let $C = v_0 e_1 v_1 e_2 \cdots v_{l-1} e_l v_l$, and $\lambda(e_i) = 1, i = 1, \cdots, l$. Let $\pi_{v_1} = (e_1^1, e_2^1, \cdots, e_s^1)$ and $\pi_{v_2} = (e_1^2, e_2^2, \cdots, e_t^2)$, where $e_1^1 = e_1, e_p^1 = e_2 (1 < p \leq s)$ and $e_1^2 = e_2, e_q^2 = e_3 (1 < q \leq t)$. Then we have some Π-facial walks $W_i^1 = e_i^1 v_1 e_{i+1}^1 \cdots$ $(i = 1, \cdots, s)$ and $W_j^2 = e_j^2 v_2 e_{j+1}^2 \cdots$

$(j = 1, \cdots, t)$.Note that $W_{p-1}^1 = W_1^2 = e_{p-1}^1 v_1 e_2 v_2 e_2^2 \cdots$. Suppose that $W_1^1 \in F_l$. Then $W_i^1 \in F_l$, by $e_i^1 \notin C(i = 2, \cdots, p-1)$. Further more $W_p^1 \in F_r$, as $e_p^1 \in C$. Then $W_j^1 \in F_r$, since $e_j^1 \notin C(j = p+1, \cdots, s)$. Similarly, as $W_1^2 = W_{p-1}^1 \in F_l$, we get $W_i^2 \in F_l, i = 1, \cdots, q-1$ and $W_j^2 \in F_r, j = q, \cdots, t$. And then consider $v_3, v_4, \cdots$. It's clearly that for any facial walk W, if W contain an edge on the left(right) side of C, then $W \in F_l(F_r)$.

Let $V_l = V(F_l) - V(C)$ and $V_r = V(F_r) - V(C)$.

Claim 3. $V_l \cap V_r = \Phi$. If $v \notin C$, let $\pi_v = (e^1, e^2, \cdots, e^k)$, $W^i = e^i v e^{i+1} \cdots$ be a Π-facial walk$(i = 1, \cdots, k)$, where $e^{k+1} = e^1$. Suppose $W^1 \in F_l$, then $W^i \in F_l$, since $e^i \notin C$ $(i = 2, \cdots, k)$. So we say $v \in V_l$. Similarly, if all the Π-facial walks are in F_r, we say $v \in V_r$.

Suppose B is a C-bridge containing an edge in $G_l(C)$ and an edge in $G_r(C)$. Then $V(B) \cap V_l \neq \Phi$ and $V(B) \cap V_r \neq \Phi$ On the other hand, since B is connected there is an edge $v_l v_r$, where $v_l \in V_l$ and $v_r \in V_r$. Clearly $v_l v_r \notin C$, then $v_l v_r \in F_l$(or $v_l v_r \in F_r$). So $V_l \cap V_r \neq \Phi$, a contradiction with claim 3. This completes the proof of lemma 7. □

Lemma 8 *Let C be a cycle in a Π-embedded graph G which is generated by a collection of separating cycles(i.e., $C = C_1 \oplus C_2 \oplus \cdots \oplus C_k$).Then the edge set C^* which is determined by edges in C is generated by $\{C_1^*, C_2^*, \cdots, C_k^*\}$ i.e.$C^* = C_1^* \oplus C_2^* \oplus \cdots \oplus C_k^*$, where C_i^* corresponds to C_i in G^*.*

Proof For any edge e^* in C^*,$e \in C = C_1 \oplus C_2 \oplus \cdots \oplus C_k$. So there are odd number of C_i containing e, i.e. there are odd number of C_i^* containing e^*.So $e^* \in C_1^* \oplus C_2^* \oplus \cdots \oplus C_k^*$.Thus $C^* \subseteq C_1^* \oplus C_2^* \oplus \cdots \oplus C_k^*$.

For any edge e^* in $C_1^* \oplus C_2^* \oplus \cdots \oplus C_k^*$,$e^*$ appears odd times in $\{C_1^*, C_2^*, \cdots, C_k^*\}$, i.e. e appears odd times in $\{C_1, C_2, \cdots, C_k\}$.So $e \in C_1 \oplus C_2 \oplus \cdots \oplus C_k = C$.Then $e^* \in C^*$.Thus $C_1^* \oplus C_2^* \oplus \cdots \oplus C_k^* \subseteq C^*$. □

Lemma 9 *Let $[S, \overline{S}]$ and $[T, \overline{T}]$ be a pair of co-cycle of G. Then $[S, \overline{S}] \oplus [T, \overline{T}]$ is also a co-cycle of G.*

Proof Let $A = S \cap T, B = S \cap \overline{T}, C = \overline{S} \cap T, D = \overline{S} \cap \overline{T}$. Then

$$\begin{aligned} & [S, \overline{S}] \oplus [T, \overline{T}] \\ = & ([A, C] \oplus [A, D] \oplus [B, C] \oplus [B, D]) \oplus ([A, B] \oplus [A, D] \oplus [C, B] \oplus [C, D]) \\ = & [A, C] \oplus [B, D] \oplus [A, B] \oplus [C, D] \\ = & [A \cup D, B \cup C] = [A \cup D, \overline{A \cup D}] \end{aligned}$$

So $[S, \overline{S}] \oplus [T, \overline{T}]$ is also a co-cycle. □

Theorem 10 *Separating cycles can't span any nonseparating cycle.*

Proof Let G be a connected Π-embedded multigraph and G^* its geometric dual multigraph. Suppose $C = C_1 \oplus \cdots \oplus C_k$ is a nonseparating cycle of G, where $C_1, \cdots, C_k$ are separating cycles. Then $C^* = C_1^* \oplus \cdots \oplus C_k^*$,where C^* and C_i^* are, respectively, the geometric dual graph of C and C_i, for any $i = 1, \cdots, k$. By lemma 1, C^* isn't a co-cycle while C_i is a nonseparating cycle of G. Thus, some co-cycles could span a nonco-cycle, a contradiction with lemma 3. $\square$

A cycle of a graph is *induced* if it has no chord. A famous result in cycle space theory is due to W.Tutte which states that in a 3-connected graph, the set of induced cycles (each of which can't separated the graph) generates the whole cycle space[4]. If we consider the case of embedded graphs, we have the following

Theorem 11 *Let G be a 2-connected graph embedded in a nonspherical surface such that its facial walks are all cycles. Then there is a cycle base consists of induced nonseparating cycles.*

Remark Tutte's definition of nonseparating cycle differs from ours. The former defined a cycle which can't separate the graph, while the latter define a cycle which can't separate the surface in which the graph is embedded. So, Theorem 11 and Tutte's result are different. From our proof one may see that this base is determined simply by shortest nonseparating cycles. As for the structure of such bases, we may modify the condition of Theorem 2 and obtain another condition for bases consisting of shortest nonseparating cycles.

Proof Notice that any cycle base consists of two parts: the first part is determined by nonseparating cycles while the second part is composed of separating cycles. So, what we have to do is to show that any facial cycle may be generated by nonseparating cycles. Our proof depends on two steps.

Step 1. Let x be a vertex of G. Then there is a nonseparating cycle passing through x.

Let C' be a nonseparating cycle of G which avoids x. Then by Menger's theorem, there are two inner disjoint paths P_1 and P_2 connecting x and C'. Let $P_1 \cap C' = \{u\}$,$P_2 \cap C' = \{v\}$. Suppose further that $u\overrightarrow{C'}v$ and $v\overrightarrow{C'}u$ are two segments of C', where $\overrightarrow{C}$ is an orientation of C. Then there are three inner disjoint paths connecting u and v:

$$Q_1 = u\overrightarrow{C}v, \qquad Q_2 = v\overrightarrow{C}u, \qquad Q_3 = P_1 \cup P_2.$$

Since $C' = Q_1 \cup Q_2$ is non separating, at least one of cycles $Q_2 \cup Q_3$ is nonseparating by Theorem 10.

Step 2. Let ∂f be any facial cycle. Then there exist two nonseparating cycles C_1 and C_2 which span ∂f.

In fact, we add a new vertex x into the inner region of ∂f(i.e. Int(∂f)) and join new edges to each vertex of ∂f. Then the resulting graph also satisfies the condition of Theorem 11. By Step 1, there is a nonseparating C passing through x. Let u and v be two vertices of $C \cap \partial f$. Then $u\overrightarrow{C}v$ together with two segments of ∂f connecting u and v forms a pair of nonseparating cycles. $\square$

Theorem 12 *Let G be a 2-connected graph embedded in a nonspherical surface such that all of its facial walks are cycles. Let $\mathcal{B}$ be a base consisting of nonsepareting cycles. Then $\mathcal{B}$ is* *shortest iff for every nonseparating cycle C,*

$$\forall\alpha \in \mathrm{Int}(C) \Rightarrow |C| \geq |\alpha|,$$

where $\mathrm{Int}(C)$ *is the subset of cycles of* $\mathcal{B}$ *which span* C.

Theorem 13 *Let G be a 2-connected graph embedded in some nonspherical surface with all its facial walks are cycles. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of shortest nonseparating cycle bases. Then there exists a 1-1 correspondence φ between elements of $\mathcal{B}_1$ and $\mathcal{B}_2$ such that for every element $\alpha \in \mathcal{B}_1 : |\alpha| = |\varphi(\alpha)|$.*

Proof: It follows from the proving procedure of Theorems 2 and 5. □

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