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On the mean value of the F.Smarandache simple divisor function

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Abstract In this paper, we introduce a new arithmetic function $\tau_{sp}(n)$ which we called the simple divisor function. The main purpose of this paper is to study the asymptotic properties of the mean value of $\tau_{sp}(n)$ by using the elementary methods, and obtain an interesting asymptotic formula for it.

Keywords Smarandache simple number divisor; Simple divisor function; Asymptotic formula.

§1. Introduction

A positive integer n is called simple number if the product of its all proper divisors is less than or equal to n. In problem 23 of [1], Professor F.Smarandache asked us to study the properties of the sequence of the simple numbers. About this problem, many scholars have studied it before. For example, in [2], Liu Hongyan and Zhang Wenpeng studied the mean value properties of $1/n$ and $1/\phi(n)$ (where n is a simple number), and obtained two asymptotic formulae for them. For convenient, let $\mathcal{A}$ denotes the set of all simple numbers, they proved that

$$\sum_{\substack{n\in\mathcal{A}\\ n\leq x}} \frac{1}{n} = (\ln\ln x)^2 + B_1\ln\ln x + B_2 + O\left(\frac{\ln\ln x}{\ln x}\right)$$

and

$$\sum_{\substack{n\in\mathcal{A}\\ n\leq x}} \frac{1}{\phi(n)} = (\ln\ln x)^2 + C_1\ln\ln x + C_2 + O\left(\frac{\ln\ln x}{\ln x}\right),$$

where B_1, B_2, C_1, C_2 are constants, and $\phi(n)$ is the Euler function.

For $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers. If one of the divisor d of n satisfing $\tau(d) \leq 4$ (where $\tau(n)$ denotes the numbers of all divisors of n), then we call d as a simple number divisor. In this paper, we introduce a new arithmetic function

$$\tau_{sp}(n) = \sum_{\substack{d|n\\ \tau(d)\leq 4}} 1,$$

which we called the simple divisor function. The main purpose of this paper is to study the asymptotic property of the mean value of $\tau_{sp}(n)$ by using the elementary methods, and obtain an interesting asymptotic formula for it. That is, we will prove the following:

Theorem. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} \tau_{sp}(n) = \frac{1}{2}x(\log\log x)^2 + \frac{1}{2}(a+1)x\log\log x + \left(\frac{b+A}{2} + B + C\right)x + O\left(\frac{x\log\log x}{\log x}\right),$$

where a and b are two computable constants, $A = \gamma + \sum_p (\log(1-1/p) + 1/p)$, γ is the Euler constant, $B = \sum_p \frac{1}{p^2}$ and $C = \sum_p \frac{1}{p^3}$.

§2. Two Lemmas

Before the proof of Theorem, two useful Lemmas will be introduced which we will use subsequently.

Lemma 1. For any real number $x \geq 1$, we have the asymptotic formula

$$(a) \qquad \sum_{n \leq x} \omega(n) = x\log\log x + Ax + O\left(\frac{x}{\log x}\right),$$

$$(b) \qquad \sum_{n \leq x} \omega^2(n) = x(\log\log x)^2 + ax\log\log x + bx + O\left(\frac{x\log\log x}{\log x}\right),$$

where $A = \gamma + \sum_p (\log(1-1/p) + 1/p)$, γ is the Euler constant, a and b are two computable constants.

Proof. See references [3] and [4].

Lemma 2. For any positive integer $n \geq 1$, we have

$$\tau_{sp}(n) = \frac{1}{2}\omega^2(n) + \frac{1}{2}\omega(n) + \sum_{p^2|n} 1 + \sum_{p^3|n} 1,$$

where $\omega(n)$ denotes the number of all different prime divisors of n, $\sum_{p^2|n} 1$ denotes the number of all primes such that $p^2 \mid n$, $\sum_{p^3|n} 1$ denotes the number of all primes such that $p^3 \mid n$.

Proof. Let $n > 1$, we can write $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then from the definition of $\tau_{sp}(n)$ we know that there are only four kinds of divisors d such that the number of the divisors of d less or equal to 4. That is $p \mid n$, $p_i p_j \mid n$, $p^2 \mid n$ and $p^3 \mid n$, where $p_i \neq p_j$.

Hence, we have

$$\begin{aligned} \tau_{sp}(n) &= \sum_{p|n} 1 + \sum_{\substack{p_i p_j | n \\ p_i \neq p_j}} 1 + \sum_{p^2|n} 1 + \sum_{p^3|n} 1 \\ &= \omega(n) + \frac{1}{2}\omega(n)(\omega(n)-1) + \sum_{p^2|n} 1 + \sum_{p^3|n} 1 \\ &= \frac{1}{2}\omega^2(n) + \frac{1}{2}\omega(n) + \sum_{p^2|n} 1 + \sum_{p^3|n} 1 \end{aligned}$$

This proves Lemma 2.

§3. Proof of the theorem

Now we completes the proof of Theorem. From the definition of the simple divisor function, Lemma 1 and Lemma 2, we can write

$$
\begin{aligned}
\sum_{n\le x}\tau_{sp}(n) &= \sum_{n\le x}\left(\frac{1}{2}\omega^2(n)+\frac{1}{2}\omega(n)+\sum_{p^2|n}1+\sum_{p^3|n}1\right)\\
&= \frac{1}{2}\sum_{n\le x}\omega^2(n)+\frac{1}{2}\sum_{n\le x}\omega(n)+\sum_{n\le x}\sum_{p^2|n}1+\sum_{n\le x}\sum_{p^3|n}1\\
&= \frac{1}{2}\left(x(\log\log x)^2+ax\log\log x+bx+O\left(\frac{x\log\log x}{\log x}\right)\right)\\
&+ \frac{1}{2}\left(x\log\log x+Ax+O\left(\frac{x}{\log x}\right)\right)+\sum_{p\le x}\left[\frac{x}{p^2}\right]+\sum_{p\le x}\left[\frac{x}{p^3}\right]\\
&= \frac{1}{2}\left(x(\log\log x)^2+(a+1)x\log\log x+(b+A)x+O\left(\frac{x\log\log x}{\log x}\right)\right)\\
&+ x\sum_{p\le x}\frac{1}{p^2}+O\left(\frac{x}{\log x}\right)+x\sum_{p\le x}\frac{1}{p^3}+O\left(\frac{x}{\log x}\right)\\
&= \frac{1}{2}\left(x(\log\log x)^2+(a+1)x\log\log x+(b+A)x+O\left(\frac{x\log\log x}{\log x}\right)\right)\\
&+ (B+C)x+O\left(\frac{x}{\log x}\right)\\
&= \frac{1}{2}x(\log\log x)^2+\frac{1}{2}(a+1)x\log\log x+\left(\frac{b+A}{2}+B+C\right)x+O\left(\frac{x\log\log x}{\log x}\right).
\end{aligned}
$$

where $B=\sum_{p}\frac{1}{p^2}$ and $C=\sum_{p}\frac{1}{p^3}$.

This completes the proof of Theorem.

References

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