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On Smarandache friendly numbers

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Abstract The Smarandache friendly numbers have been defined by Murthy [1]. This paper finds the Smarandache friendly numbers by solving the associated Pell's equation.

Keywords Smarandache friendly numbers, pell's equation, recurrence relation.

§1. Introduction

Murthy [1] defines the Smarandache friendly numbers as follows :

Definition 1.1. A pair of positive integers $(m,\ n)$ (with $n > m$) is called the Smarandache friendly numbers if

$$m + (m + 1) + \cdots + n = mn.$$

For example, $(3, 6)$ is a Smarandache friendly pair, since

$$3 + 4 + 5 + 6 = 18 = 3 \times 6.$$

Recently, Khainar, Vyawahare and Salunke [2] have treated the problem of finding Smarandache friendly pairs and Smarandache friendly primes.

In this paper, we show that the problem of finding the Smarandache friendly pairs can be reduced to solving a particular type of Pell's equation, which can then be used to find the sequence of all Smarandache friendly pairs.

In section 2, we give some preliminary results that would be necessary in the next section which gives the main results of this paper. It is conjectured that, if (m, n) is a Smarandache friendly pair of numbers, then $(m + 2n, 2m + 5n - 1)$ is also a friendly pair. We also prove this conjecture in the affirmative.

§2. Some preliminary results

The following result is well-known (see, for example, Hardy and Wright [3]).

Lemma 2.1. The general solution of the Diophantine equation $x^2 - 2y^2 = -1$ is

$$x + \sqrt{2}y = (1 + \sqrt{2})^{2\nu+1}; \quad \nu \geq 0. \tag{1}$$

Note that, the Diophantine equation

$$x^2 - 2y^2 = -1 \tag{2}$$

is a particular type of Pell's equation.

Lemma 2.2. Denoting by $(x_\nu,\ y_\nu)$ the ν-th solution of the Diophantine equation (2), $(x_\nu,\ y_\nu)$ satisfies the following recurrence relation:

$$x_{\nu+1} = 3x_\nu + 4y_\nu, \quad y_{\nu+1} = 2x_\nu + 3y_\nu; \quad \nu \geq 1, \tag{3}$$

with

$$x_1 = 7, \quad y_1 = 9. \tag{4}$$

Proof. Since

$$\begin{aligned} x_{\nu+1} + \sqrt{2}\, y_{\nu+1} &= (1+\sqrt{2})^{2\nu+3} \\ &= (x_\nu + \sqrt{2}\, y_\nu)(1+\sqrt{2})^2 \\ &= (x_\nu + \sqrt{2}\, y_\nu)(3+2\sqrt{2}) \\ &= (3x_\nu + 4\, y_\nu) + \sqrt{2}(2x_\nu + 3\, y_\nu), \end{aligned}$$

the result follows.

Lemma 2.2 enables us to calculate the solutions of the Diophantine equation (2) recursively, starting with $x_1 = 7,\ y_1 = 9$.

§3. Main results

We now consider the problem of finding the pair of integers $(m,\ n)$, with $n > m > 0$, such that

$$m + (m+1) + \cdots + n = mn. \tag{5}$$

Writing

$$n = m + k \quad \text{for some integer} \quad k > 0, \tag{6}$$

(5) takes the form

$$m + (m+1) + \cdots + (m+k) = m(m+k),$$

which, after some simple algebraic manipulations, gives

$$k(k+1) = 2m(m-1). \tag{7}$$

In Eq.(7), we substitute

$$k = K + \frac{1}{2}, \quad m = M + \frac{1}{2},$$

to get

$$K^2 - \frac{1}{4} = 2(M^2 - \frac{1}{4}), \tag{8}$$

that is,

$$4K^2 - 8M^2 = -1,$$

that is,

$$x^2 - 2y^2 = -1, \tag{9}$$

where

$$x = 2K, \quad y = 2M. \tag{10}$$

Note that, though K and M are not integers, each of x and y is a positive integer.

Lemma 3.1. The sequence of Smarandache friendly pair of numbers, $\{(m_\nu,\ n_\nu\}_{\nu=1}^{\infty}$, is given by

$$\begin{aligned} m_\nu &= M_\nu + \frac{1}{2} = \frac{1}{2}(y_\nu + 1), \\ k_\nu &= K_\nu - \frac{1}{2} = \frac{1}{2}(x_\nu - 1), \\ n_\nu &= m_\nu + k_\nu = \frac{1}{2}(x_\nu + y_\nu), \end{aligned} \tag{11}$$

where

$$x_\nu + \sqrt{2}\, y_\nu = (1 + \sqrt{2})^{2\nu+1};\ \nu \geq 1. \tag{12}$$

Proof. Since x and y satisfy the Diophantine equation (9), with solutions given by (12), the result follows.

Lemma 3.2. The sequence of Smarandache friendly pair of numbers $\{(m_\nu,\ n_\nu\}_{\nu=1}^{\infty}$ satisfies the following recurrence relation:

$$m_{\nu+1} = m_\nu + 2n_\nu,\ n_{\nu+1} = 2m_\nu + 5n_\nu - 1;\ \nu \geq 1,$$

with

$$m_1 = 3,\ n_1 = 6.$$

Proof. By Lemma 3.1, together with Lemma 2.2,

$$m_{\nu+1} = \frac{1}{2}(y_{\nu+1} + 1) = \frac{1}{2}(2x_\nu + 3y_\nu + 1) = x_\nu + y_\nu + \frac{1}{2}(y_\nu + 1) = 2n_\nu + m_\nu,$$

$$\begin{aligned} n_{\nu+1} &= \frac{1}{2}(x_{\nu+1} + y_{\nu+1}) = \frac{1}{2}[(3x_\nu + 4y_\nu) + (2x_\nu + 3y_\nu)] \\ &= \frac{5}{2}(x_\nu + y_\nu) + y_\nu = 5n_\nu + (2m_\nu - 1), \end{aligned}$$

and we get the desired results.

Lemma 3.2 shows that, if $(x_\nu,\ y_\nu)$ is a Smarandache friendly pair of numbers, so is the pair $(m_\nu + 2n_\nu,\ 2m_\nu + 5n_\nu - 1)$, which is the result conjectured in [2]; moreover, it is the next pair in the sequence. Thus, starting with the smallest friendly pair (3, 6), the other pairs can be obtained recursively, using Lemma 3.2.

§4. Open problems

A pair of primes $(p,\ q)$ with $q > p \geq 2$ is called a pair of Smarandache friendly primes if the sum of the primes from p through q is equal to pq.

Open problem 1. Find all the pairs of Smarandache friendly primes.

Open problem 2. Is the sequence of pairs of Smarandache primes finite?

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References

[1] Amarnath Murthy, Some Notions on Least Common Multiples, Smarandache Notions Journal, **12**(2004), 307-308.

[2] S. M. Khairnar, A. W. Vyawahare and J. N. Salunke, Smarandache Friendly Numbers-Another Approach, Scientia Magna, **5**(2009), 32-39.

[3] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, UK, **5**(2002), 343.