SMARANDACHE LATTICE AND PSEUDO COMPLEMENT

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Abstract. In this paper, we have introduced smarandache - 2 - Algebraic structure of lattice namely smarandache lattice. A smarandache 2- algebraic structure on a set N means a weak algebraic structure Ao on N such that there exists a proper subset M of N which is embedded with a stronger algebraic structure A1. Stronger algebraic structure means a structure which satisfies more axioms, by proper subset one can understand a subset different from the empty set, by the unit element if any, and from the whole set. We have defined smarandache lattice and obtained some of its characterization through Pseudo complemented .For the basic concept, we referred the PadilaRaul [4].

1. Introduction

New notions are introduced in algebra to study more about the congruence in number theory by Florentinsmarandache [1] .By <proper subset> of a set A, we consider a set P included in A, and different from A, different from the empty set, and from the unit element in A - if any they rank the algebraic structures using an order relationship.

The algebraic structures $S_1 \ll S_2$ if :both are defined on the same set; all S_1 laws are also S_2 laws ; all axioms of S_1 law are accomplished by the corresponding S_2 law; S_2 law strictly accomplishes more axioms than S_1 laws, or in other words S_2 laws has more laws than S_1.

For example : Semi group $\ll$ monoid $\ll$ group $\ll$ ring $\ll$ field, or Semi group $\ll$ commutative semi group, ring $\ll$ unitary ring, etc. they define a General special structure to be a structure SM on a set A, different from a structure SN, such that a proper subset of A is an SN structure, where SM $\ll$ SN.

2. Preliminaries

Definition 2.1. Let P be a lattice with 0.Let x $\in$ P , x* is a Pseudo complemented of x iff x* $\in$ P and x $\wedge$ x* = 0 and for every y $\in$ P: if x $\wedge$ y =0 then y $\leq$ x*.

Definition 2.2. Let P be a pseudo complemented lattice.$N_P = \{ x* : x \in P\}$ is the set of complements in P .N_P= { $N_P \leq N$, $\neg$ N, 0_N ,1_N , N , N } where:
(i) $\leq_N$, is defined by: for every x , y $\in$ N_P : x $\leq_N$y iff x $\leq_P$b
(ii)$\neg$ N is defined by :

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for every x $\in$ N_P : $\neg$ N (x) = x*
(iii)$\wedge_N$ is defined by:
for every x, y $\in$ N_P x $\wedge_N$ y = x $\wedge_P$ y
(iv) $\vee_N$ is defined by :
for every x, y $\in$ N_P : x $\vee_N$ y= (x * $\wedge_P$ y*)*

(v)1_N = 0_P*, 0_N = 0_P .

Definition 2.3. Let P be a lattice with 0.I_p is the set of all ideals in P
$I_P =< I_P, \leq_1, \wedge_1, \vee_1, 0_1, 1_1 >$ where:
$\leq = \subseteq, i \wedge_1 j = I \cap J, i \vee_1 j = (I \cup J], 0_1 = 0_A, 1_1 = A$

Definition 2.4. If P is a distributive lattice with 0,I_P is a complete Pseudo Complemented lattice. Let P be a lattice with 0. NI_P, the set of normal ideals in P, is given by NI_P= $\{ I * I_P : II_P\}$ Alternatively NI_P = $\{ II_P : I = I**\}$ .Thus NI_P = $\{ NI_P, \subseteq, \cap, \cup, \wedge_{NI}, \vee_{NI}\}$, NI_P is the set of Pseudo Complements in I_P.

Definition 2.5. A Pseudo complemented distributive lattice P is called a stone lattice if, for all a $\in$ P, it satisfies the property a* $\vee$ a** = 1.

Definition 2.6. Let P be a pseudo complemented distributive lattice. Then for any filter F of P, define the set δ(F) as follows δ(F) =$\{ a^* \in P/a^* \in F\}$.

Definition 2.7. Let P be a pseudo complemented distributive lattice.An ideal I of P is called a δ-ideal if I = δ(F) for some filter F of P.

Now we have introduced a definition by [**4**]:

Definition 2.8. A lattice S is said to be a Smarandache lattice. If there exist a proper subset L of S, Which is a Boolean algebra with respect to the same induced operations of S.

3. Characterizations

Theorem 3.1. *Let S be a lattice. If there exist a proper subset N_P of S defined in definition 2.2. Then S is a smarandache lattice.*

Proof. By hypothesis, let S be a lattice and whose proper subset $N_P = \{ x^* : x \in P\}$ is the set of all Pseudo complements in P. By definition, Let P is a pseudo complement lattice. $N_P = \{ x^* : x \in P\}$ is the set of complements in P.
$N_P = \{ N_P, \leq_N, \neg_N, 0_N, 1_N, \wedge_N, \vee_N\}$
where: (i) $\leq_N$, is defined by: for every x , y $\in$ N_P : x $\leq_N$y iff x $\leq_P$b
(ii)$\neg$ N is defined by :
for every x $\in$ N_P : $\neg$ N (x) = x*
(iii)$\wedge_N$ is defined by:
for every x, y $\in$ N_P x $\wedge_N$ y = x $\wedge_P$ y
(iv) $\vee_N$ is defined by :
for every x, y $\in$ N_P : x $\vee_N$ y= (x * $\wedge_P$ y*)*

(v)$1_N = 0_P{}^*$, $0_N = 0_P$.

It is enough to prove that N_P is a Boolean algebra.

(i) For every x, y $\in N_P$,
x $\wedge_N$ y N_P and $\wedge_N$ is meet under $\leq_N$.
If x, y $\in N_P$, then x=x** and y=y**.
Since x $\wedge_P$ y $\leq_P$ x, by result if x $\leq_N$ y then y* $\leq_N$ x*,
x* $\leq_P$(x $\wedge_P$ y)*, and, with by result if x $\leq_N$ y then y* $\leq_N$ x* ,
(x $\wedge_P$ y)** $\leq_P$ x. Similarly, (x $\wedge_P$ y)** $\leq_P$ y.
Hence (x $\wedge_P$ y)** P (x $\wedge_P$ y).
By result, x $\leq_N$ x** , (x $\wedge_P$ y) $\leq_P$ (x $\wedge_P$y)** ,
Hence (x $\wedge_P$ y) $\in N_P$, (x $\wedge_N$ y) $\in N_P$.
If a $\in N_P$ and a $\leq_N$ x and a $\leq_N$ y,
Then a $\leq_P$ x and a $\leq_P$ y, a $\leq_P$ (x $\wedge_P$ y),
Hence a $\leq_N$ (x $\wedge_N$ y).So indeed N is meet in $\leq_N$.

(ii) For every x,y $\in N_P$: x $\vee_N$ y $\in N_P$ and $\vee_N$ is join under $\leq_N$.
Let x,y v N_P Then x*, y* $\in N_P$. Then by (i) , (x* $\wedge_P$ y*) $\in N_P$
Hence (x* $\wedge_P$ y*)* $\in N_P$, and hence (x $\vee_N$ y) $\in N_P$
(x* $\wedge_P$ y*) $\leq_P$ x*, hence, by result x $\leq_N$ x**
x ** $\leq_P$(x* $\wedge_P$ y*)*,
By result N_P = { x$\in$ P; x = x**} , x $\leq_P$ (x* $\wedge_P$ y*)*.

Similarly, y $\leq_P$ (x* $\wedge_P$y*)*.
If a$\in N_P$ and x $\leq_N$ a and y $\leq_N$ a, then x $\leq_P$ a and y $\leq_P$ a ,then by result if x $\leq_N$ y
Then y* $\leq_N$ x*, a* $\leq_P$ x* and a* $\leq_P$ y*, hence a* $\leq_P$ (x* $\wedge_P$ y*)
Hence, by result if x $\leq_N$ y then y* $\leq_N$ x*,
(x* $\wedge_P$ y*)* $\leq_P$ a** hence, by result N_P = { x P; x = x**} ,
(x* $\wedge_P$ y*)* P a, hence x $\vee_N$ y $\leq_N$ a so, indeed $\vee_N$ is join in $\leq_N$.

(iii) 0_N, 1_N N_P and 0_N and 1_N are the bounds of N_P .
Obviously 1_N N_P, since $1_N = 0_P{}^*$ since for every a N_P, a$\wedge_P$ $0_P = 0_P$,
For every a N_P, a $\leq_P$ 0_P *, hence a $\leq_N$ 1_N.
$0_P{}^*$, $0_P{}^{**}$ N_P Hence $0_P{}^*$ $\wedge_P$ $0_P{}^{**}$ N_P But of course, $0_P{}^*$ $\wedge_P$
$0_P{}^{**}$=0_P Hence 0_P N_P , 0_N N_P ,Obviously, for every a N_P; 0_P $\leq_P$ a.
Hence for every a N_P : $0_N$$\leq_N$ a .So N_P is bounded lattice.

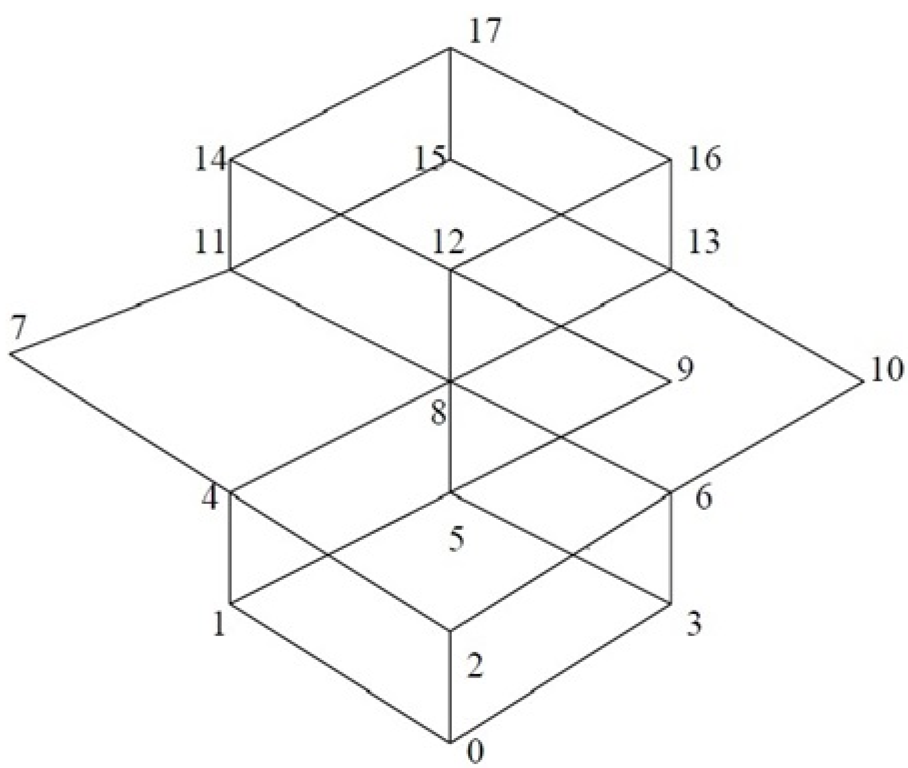
(iv) For every a $\in N_P$: neg_N (a) $\in N_P$ and
for every a $\in N_P$: a $\wedge_N$ $\neg_N$ (a) = 0_N and
For every a $\in N_P$: a $\vee_N$ $\neg_N$ (a) = 1_N .
Let a N_P , Obviously $\neg_N$ (a) $\in N_P$
a N $\neg_N$ (a) = a $\vee_N$ a* = ((a* $\wedge_P$ b**))* = (a* $\wedge_P$ a)* = $0_P{}^*$ = 1_N
a N $\neg_N$ (a) = a $\wedge_P$ a*= 0_P = 0_N .
So N_P is a bounded complemented lattice.

(v) Since x $\leq_N$ (x $\vee_N$ (y $\wedge_N$ z)), (x $\wedge_N$ z) $\leq_N$ x $\vee_N$ (y $\wedge_N$ z)
Also (y $\wedge_N$ z) $\leq_N$ x $\vee_N$ (y $\wedge_N$ z)
Obviously, if a $\leq_N$ b, then a $\wedge_N$ b* = 0_N, Since b $\wedge$b* = 0_N ,
Hence, (x $\wedge_N$ z) $\wedge_N$ (x $\vee_N$ (y $\wedge_N$ z))* =0_N and
(y $\wedge_N$ z) $\wedge_N$ (x $\vee_N$ (y $\wedge_N$ z))* =0_N,
x $\wedge_N$ (z $\wedge_N$(x $\vee_N$ (y $\wedge_N$ z))*) =0_N ,
y $\wedge_N$ (z $\wedge_N$ (x $\vee_N$ (y $\wedge_N$ z))*) =0_N .

By definition of Pseudo complement: z $\wedge_N$ (x $\vee_N$ (y$\wedge_N$z))* $\leq_N$ x*,
z$\wedge_N$ (x $\vee_N$ (y$\wedge_N$z))* $\leq_N$ y*,
Hence z $\wedge_N$ (x $\vee_N$ (y$\wedge_N$z))* $\leq_N$ x*$\wedge_N$ y*
Once again, If a $\leq_N$ b, then a$\wedge_N$b* = 0_N,
Hence, z $\wedge_N$ (x $\vee_N$ (y$\wedge_N$z))* $\wedge$(x*$\wedge_N$y*)* = 0_N
z$\wedge_N$ (x* $\wedge_N$ y*)*$\leq_N$(x$\vee_N$ (y$\wedge_N$z))**
Now, by definition of $\wedge_N$: z $\wedge_N$ (x*$\vee_N$y*)* = z $\wedge_N$ (x $\vee_N$ y)
And by N_P = { x $\in$ P, : x = x**} : (x $\vee_N$ (y$\wedge_N$ z))** = x $\vee_N$ (y $\wedge_N$ z),
Hence: z $\wedge_N$ (x$\vee_N$y) $\leq_N$ x$\vee_N$(y $\wedge_N$ z) .

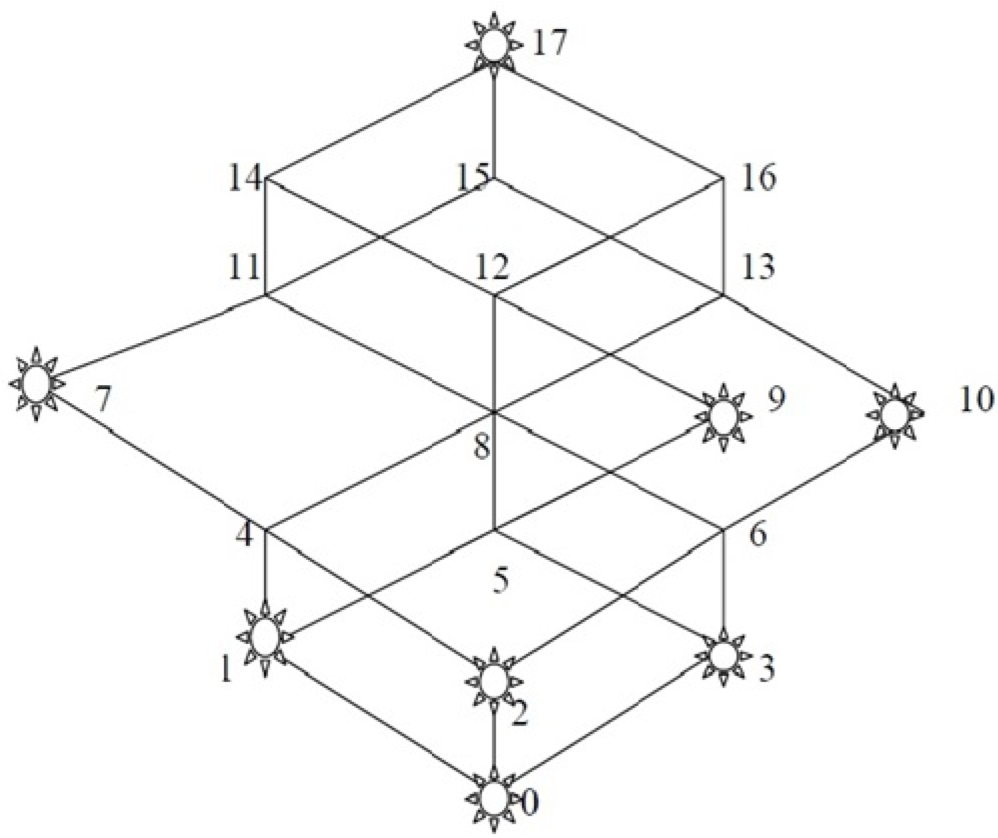
Hence, indeed N_P is a Boolean Algebra. Therefore by definition, S is a Smarandache lattice

Example: Distributive lattice D_3 Figure 1.1



D_3 is pseudocoplemented:
$0^* = 17$ $8^* = 11^* = 12^* = 13^* = 14^* = 15^* = 16^* = 17^* = 0$
$1^* = 10$ $6^* = 10^* = 1$
$2^* = 9$ $5^* = 9^* = 2$
$3^* = 7$ $4^* = 7^* = 3$

Figure: 1.2 Smarandache lattice



□

Theorem 3.2. *Let S be a distributive lattice with 0. If there exist a proper subset NI_P of S, defined Definition 2.4. Then S is a smarandache lattice.*

Proof. By hypothesis, let S be a distributive lattice with 0 and whose proper subset NI_P={ $I^* \in I_P$, $I \in I_P$ } is the set of normal ideals in P.
We claim that NI_P is Boolean algebra.
Since NI_P = { $I^* \in I_P$: $I \in I_P$ } is the set of normal ideals in P.
Alternatively NI_P= { $I \in I_P$:I= I^{**}} .
Let $I \in I_P$ Take I^* ={ y P: for every i I:y$\wedge$ i=0} ,$I^* \in I_P$

Namely if a$\in$ I* then for every i$\in$ I:a$\wedge$ i=0,
Let b$\leq$ a, Then, obviously, for every i$\in$ I:b$\wedge$ i=0 hence b$\in$ I*.
If a,b$\in$ I* then for every i$\in$ I:a $\wedge$ i=0,and for every i $\in$ I:b$\wedge$ i=0,

Hence for every i$\in$ I:(a$\wedge$ i)$\vee$ (b$\wedge$ i)=0.
With distributive, for every i$\in$ I: i$\wedge$ (a $\vee$ b)=0, hence a $\vee$ b$\in$ I* .
Hence I*$\in$ I_P,.I$\cap$ I* =I { y$\in$ P: for every i$\in$ I:y$\wedge$ i=0} ={ 0} .
Let I $\cap$ J={ 0} ,let j$\in$ J
Suppose that for some i$\in$ I:i$\wedge$ j $\neq$ 0. Then i$\wedge$ j$\in$ I$\cap$ J,
Since I and j are ideals, henceI$\cap$ J$\neq$ { 0} .
Hence for every i$\in$ I:j$\wedge$ i=0.,and hence j $\subseteq$ I .
Consequently, I* is a pseudo complement of I and I_P is a pseudo complemented.
Therefore I_P is a Boolean algebra.
Thus NI_P is the set of all Pseudo complements lattice in I_P.
In Theorem 3.1 we have proved that pseudo complemented form a Boolean algebra.
Therefore $NI_P$is a Boolean algebra.
Hence by definition, S is a smarandache lattice.

□

Theorem 3.3. *Let S be a lattice. If there exist a Pseudo complemented distributive lattice P, X*(P) is a sub lattice of the lattice* I^δ*(P) of all* δ*-ideals of P, which is the proper subset of S . Then S is a Smarandache lattice.*

Proof. By hypothesis, let S be a lattice and there exist a Pseudo complemented distributive lattice P, X*(P) is a sub lattice of the lattice I^δ(P) of all δ-ideals of P, which is the proper subset of S .

Let (a*],(b*] $\in$ X*(P),for some a,b$\in$ P.
Then clearly (a*]$\cap$ (b*]$\in$ X*(P).
Again, (a*] $\cup$ (b*] = δ([a)) $\cup$ δ([b))= δ[(a) $\cup$ ([b))= δ([a$\cap$ b))=((a$\cap$ b)*]$\in$ X*(P).
Hence X*(P) is a sub lattice of I^δ(P) and hence a distributive lattice.
Clearly (0**] and (0*] are the least and greatest elements of X*(P).

Now for any a $\in$ P, (a*] $\cap$ (a**] = (0] and
(a*] $\cup$ (b**] = δ([a)) $\cup$ δ([a*)) = δ([a)) $\cup$ ([a*)) = δ([a $\cap$ a*))= δ([0)) = δ(P) =P.
Hence (a**] is the complement of (a*] in X*(P).
Therefore { X*(P),$\cup$,$\cap$} is a bounded distributive lattice in which every element is complemented.
Thus X*(P) is a Boolean algebra.
By definition, S is a Smarandache lattice.

□

Theorem 3.4. *Let S be a latticeand P be a pseudo complemented distributive lattice. If S is a Smarandache lattice. Then the following conditions are equivalent:*
(a) P is a Boolean algebra.
(b) every element of P is closed,
(c) every principal ideal is a δ-ideal,
(d) for any ideal I, $a \in I$ implies $a^{**} \in I$,
(e) for any proper ideal I, $I \cap D(P) = \phi$,
(f) for any prime ideal A, $A \cap D(P) = \phi$,
(g) every prime ideal is a minimal prime ideal,
(h) every prime ideal is a δ-ideal,
(i) for any a, $b \in P$, $a^* = b^*$ implies $a = b$,
(j) D (P) is a singleton set.

Proof. Since S is a Smarandache lattice.
Then by definition and above theorem, we observe that, there exists a proper subset P of S such that which is a Boolean algebra.
Therefore P is a Boolean algebra.

Now to prove (a) $\Longrightarrow$ (b):
Then clearly P has a unique dense element, precisely the greatest element.
Let a P Then $a^* \wedge a=0 =a^* \wedge a^{**}$.
Also $a^* \vee a$, $a^* \vee a^{**} \in D$ (P).
Hence $a^* \vee a = a^* \vee a^{**}$.
By the cancellation property of P, we get $a =a^{**}$.
Therefore every element of P is closed.

(b) $\Longrightarrow$ (c): Let I be a principal ideal of P.
Then I = (a] for some $a \in P$.
Then by condition (b), $a = a^{**}$.
Now, $(a] = (a^{**}] = \delta([a^*))$.
Therefore (a)] is a δ-ideal.

(c) $\Longrightarrow$ (d): I be a proper ideal of P.
Let $a \in I$. Then $(a] = \delta(F)$ for some filter F of P.
Hence We get $a^{***}=a^* \ \delta$ F.
Therefore $a^{**} \in \delta$ (F) = (a] $\subseteq$ I.

(d) $\Longrightarrow$ (e): Let I be a proper ideal of P.
Suppose $a \in I \cap D$ (P).
Then $a^{**} \in P$ and $a^*=0$.
Therefore $1=0^* = a^{**} \in P$,
which is a contradiction.

(e) $\Longrightarrow$ (f): Let I be a proper ideal of P, I $\cap$ D(P) = ϕ,
then P be a prime ideal of P ,A $\cap$ D(P) = ϕ .

(f) $\Longrightarrow$ (g): Let A be a prime ideal of P such that A $\cap$ D(P) = ϕ.
Let a $\in$ A. Then clearly a $\wedge$ a*= 0 and a $\vee$ a* $\in$ D(P).
Hence a$\vee$a* $\notin$ A Thus a* $\notin$A
Therefore A is a minimal prime ideal of P.

(g) $\Longrightarrow$ (h): Let A be a minimal prime ideal of P.
Then clearly P-A is a filter of P. Let a $\in$ A.
Since A is minimal, there exists b $\notin$ A such that a $\wedge$ b = 0.
Hence a* $\wedge$ b = b. a* $\notin$ A.
Thus a* $\in$ (P-A) which yields a $\in$ δ(P-A).
Conversely, let a $\in$ δ(P-A).
Then we get a* $\notin$ A.
Hence we have a $\in$ A.
Thus P= δ(P-A) and therefore A is δ-ideal of P.

(h) $\Longrightarrow$ (i): Assume that every prime ideal of P is a δ-ideal.
Let a, b $\in$ P be such that a*=b*. Suppose a $\neq$ b.
Then there exists a prime ideal A of P such that a $\in$ A and b $\notin$ A.
By Hypothesis, A is a δ- ideal of P.
Hence A = δ(F) for some filter F of P.
Since a A= δ(F), We get b* = a* F.
Hence b (F) = A, which is a contradiction.
Therefore a=b.

(i) $\Longrightarrow$ (j): Suppose x, y be two elements of D (P).
Then x * = 0 = y *.
Hence x = y. Therefore D (P) is a singleton set.

(j) $\Longrightarrow$ (a): Assume that D (P) = { d} is singleton set. Let a P.
We have always a $\vee$ a* D (P).
Therefore a $\wedge$ a* = 0 and a $\wedge$ a*=d.
This true for all a $\in$ P. Also 0 $\leq$ a $\leq$ a $\vee$ a*=d.
Hence the above conditions are equivalent.

$\square$

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