On the Smarandache Lucas base and related counting function[1]

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1. Introduction and results

As usual, the Lucas sequence $\{L_n\}$ and the Fibonacci sequence $\{F_n\}$ ($n = 0, 1, 2, \ldots,$) are defined by the second-order linear recurrence sequences

$$L_{n+2} = L_{n+1} + L_n \qquad \text{and} \qquad F_{n+2} = F_{n+1} + F_n$$

for $n \geq 0$, $L_0 = 2$, $L_1 = 1$, $F_0 = 0$ and $F_1 = 1$. These sequences play a very important role in the studies of the theory and application of mathematics. Therefore, the various properties of L_n and F_n were investigated by many authors. For example, R. L. Duncan [1] and L. Kuipers [2] proved that $(\log F_n)$ is uniformly distributed mod 1. H.London and R.Finkelstein [3] studied the Fibonacci and Lucas numbers which are perfect powers. The author [4] obtained some identities involving the Fibonacci numbers.

In this paper, we introduce a new counting function $a(m)$ related to the Lucas numbers, then use elementary methods to give an exact calculating formula for its mean value. First we consider the Smarandache's generalized base, Professor F.Smarandach defined over the set of natural numbers the following infinite generalized base: $1 = g_0 < g_1 < \cdots < g_k < \cdots$. He proved that every positive integer N may be uniquely written in the Smarandache Generalized Base as:

$$N = \sum_{i=0}^{n} a_i g_i, \quad \text{with} \quad 0 \leq a_i \leq \left[\frac{g_{i+1} - 1}{g_i}\right]$$

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(integer part) for $i = 0, 1, \cdots, n$, and of course $a_n \geq 1$, in the following way: if $g_n \leq N < g_{n+1}$, then $N = g_n + r_1$; if $g_m \leq r_1 < g_{m+1}$, then $r_1 = g_m + r_2$, $m < n$; and so on untill one obtains a rest $r_j = 0$.

This base is important for partitions. If we take the g_i as the Lucas sequence, then we can get a particular base, for convenience, we refer to it as a Smarandache Lucas base. Then any positive integer m may be uniquely written in the Smarandache Lucas base as:

$$m = \sum_{i=1}^{n} a_i L_i, \quad \text{with all } a_i = 0 \text{ or } 1, \tag{1}$$

That is, any positive integer may be written as a sum of Lucas numbers. Now for an integer $m = \sum_{i=1}^{n} a_i L_i$, we define the counting function $a(m) = a_1 + a_2 + \cdots + a_n$. The main purpose of this paper is to study the distribution properties of $a(m)$, and present a calculating formula for the mean value

$$A_r(N) = \sum_{n<N} a^r(n), \; r = 1, \; 2. \tag{2}$$

That is, we prove the following two main conclusions:

Theorem 1. *For any positive integer k, we have the calculating formulae*

$$A_1(L_k) = \sum_{n<L_k} a(n) = kF_{k-1}$$

and

$$A_2(L_k) = \frac{1}{5}\left[(k-1)(k-2)L_{k-2} + 5(k-1)F_{k-2} + 7(k-1)F_{k-3} + 3F_{k-1}\right].$$

Theorem 2. *For any positive integer N, let $N = L_{k_1} + L_{k_2} + \cdots + L_{k_s}$ with $k_1 > k_2 > \cdots > k_s$ under the Smarandache Lucas base. Then we have*

$$A_1(N) = A_1(L_{k_1}) + N - L_{k_1} + A_1(N - L_{k_1})$$

and

$$A_2(N) = A_2(L_{k_1}) + N - L_{k_1} + A_2(N - L_{k_1}) + 2A_1(N - L_{k_1}).$$

Further,

$$A_1(N) = \sum_{i=1}^{s} [k_i F_{k_i-1} + (i-1)L_{k_i}].$$

For any positive integer $r \geq 3$, using our methods we can also give an exact calculating formula for $A_r(L_k)$. But in these cases, the computations are more complex.

2. Proof of the Theorems

In this section, we complete the proof of the Theorems. First we prove Theorem 1 by induction. For $k = 1, 2$, we have $A_1(L_1) = A_1(1) = 0$, $A_1(L_2) = A_1(3) = 2$ and $F_0 = 0$, $2F_1 = 2$. So that the identity

$$A_1(L_k) = \sum_{n<L_k} a(n) = kF_{k-1} \tag{3}$$

holds for $k = 1$ and 2. Assume (3) is true for all $k \leq m-1$. Then by the inductive assumption we have

$$\begin{aligned}
A_1(L_m) &= \sum_{n<L_{m-1}} a(n) + \sum_{L_{m-1}\leq n<L_m} a(n) \\
&= A_1(L_{m-1}) + \sum_{0\leq n<L_{m-2}} a(n+L_{m-1}) \\
&= A_1(L_{m-1}) + \sum_{0\leq n<L_{m-2}} (a(n)+1) \\
&= A_1(L_{m-1}) + L_{m-2} + \sum_{n<L_{m-2}} a(n) \\
&= A_1(L_{m-1}) + A_1(L_{m-2}) + L_{m-2} \\
&= (m-1)F_{m-2} + (m-2)F_{m-3} + L_{m-2} \\
&= m(F_{m-2} + F_{m-3}) - F_{m-2} - 2F_{m-3} + L_{m-2} \\
&= mF_{m-1} - F_{m-1} - F_{m-3} + L_{m-2} \\
&= mF_{m-1},
\end{aligned}$$

where we have used the identity $F_{m-1} + F_{m-3} = L_{m-2}$. That is, (3) is true for $k = m$. This proves the first part of Theorem 1.

Now we prove the second part of Theorem 1. For $k = 1, 2$, note that $1 = F_1 = F_0 + F_{-1}$ or $F_{-1} = 1$, we have $A_2(L_1) = A_2(1) = 0$, $A_2(L_2) = A_2(3) = 2$ and

$$\frac{1}{5}\left[(k-1)(k-2)L_{k-2} + 5(k-1)F_{k-2} + 7(k-1)F_{k-3} + 3F_{k-1}\right] = \begin{cases} 0, & \text{if } k = 1; \\ 2, & \text{if } k = 2. \end{cases}$$

So that the identity

$$A_2(L_k) = \frac{1}{5}\left[(k-1)(k-2)L_{k-2} + 5(k-1)F_{k-2} + 7(k-1)F_{k-3} + 3F_{k-1}\right] \quad (4)$$

holds for $k = 1, 2$. Assume (4) is true for all $k \le m-1$. Then by the inductive assumption, the first part of Theorem 1 and note that $L_{m-1} + 2L_{m-2} = 5F_{m-1}$ and $F_{m-1} + 2F_{m-2} = L_{m-1}$, we have

$$\begin{aligned}
A_2(L_m) &= \sum_{n<L_{m-1}} a^2(n) + \sum_{L_{m-1}\le n<L_m} a^2(n) \\
&= A_2(L_{m-1}) + \sum_{0\le n<L_{m-2}} a^2(n+L_{m-1}) \\
&= A_2(L_{m-1}) + \sum_{0\le n<L_{m-2}} (a(n)+1)^2 \\
&= A_2(L_{m-1}) + \sum_{0\le n<L_{m-2}} (a^2(n)+2a(n)+1) \\
&= A_2(L_{m-1}) + \sum_{n<L_{m-2}} a^2(n) + 2\sum_{n<L_{m-2}} a(n) + L_{m-2} \\
&= A_2(L_{m-1}) + A_2(L_{m-2}) + 2A_1(L_{m-2}) + L_{m-2} \\
&= \frac{1}{5}\left[(m-2)(m-3)L_{m-3} + 5(m-2)F_{m-3} + 7(m-2)F_{m-4} + 3F_{m-2}\right] \\
&\quad + \frac{1}{5}\left[(m-3)(m-4)L_{m-4} + 5(m-3)F_{m-4} + 7(m-3)F_{m-5} + 3F_{m-3}\right] \\
&\quad + 2(m-2)F_{m-3} + L_{m-2} \\
&= \frac{1}{5}\left[(m-1)(m-2)L_{m-3} + 5(m-1)F_{m-3} + 7(m-1)F_{m-4} + 3F_{m-2}\right] \\
&\quad + \frac{1}{5}\left[(m-1)(m-2)L_{m-4} + 5(m-1)F_{m-4} + 7(m-1)F_{m-5} + 3F_{m-3}\right] \\
&\quad - \frac{1}{5}\left[2(m-1)L_{m-3} + (4m-10)L_{m-4} + 5F_{m-3} + 7F_{m-4} + 10F_{m-4}\right. \\
&\quad \left. + 14F_{m-5}\right] + 2(m-2)F_{m-3} + L_{m-2} \\
&= \frac{1}{5}\left[(m-1)(m-2)L_{m-2} + 5(m-1)F_{m-2} + 7(m-1)F_{m-3} + 3F_{m-1}\right]
\end{aligned}$$

$$-\frac{1}{5}[2(m-2)(L_{m-3}+2L_{m-4})-2L_{m-4}+5(F_{m-3}+2F_{m-4})$$
$$+7(F_{m-4}+2F_{m-5})]+2(m-2)F_{m-3}+L_{m-2}$$
$$=\frac{1}{5}[(m-1)(m-2)L_{m-2}+5(m-1)F_{m-2}+7(m-1)F_{m-3}+3F_{m-1}]$$
$$-\frac{1}{5}[10(m-2)F_{m-3}-2L_{m-4}+5L_{m-3}+7L_{m-4}]$$
$$+2(m-2)F_{m-3}+L_{m-2}$$
$$=\frac{1}{5}[(m-1)(m-2)L_{m-2}+5(m-1)F_{m-2}+7(m-1)F_{m-3}+3F_{m-1}].$$

That is, (4) is true for $k=m$. This completes the proof of Theorem 1.

Proof of Theorem 2. Note that $N=L_{k_1}+L_{k_2}+\cdots+L_{k_s}$, applying Theorem 1 we have

$$\begin{aligned}
A_1(N) &= \sum_{n<L_{k_1}} a(n)+\sum_{L_{k_1}\le n<N} a(n)\\
&= A_1(L_{k_1})+\sum_{L_{k_1}\le n<N} a(n)\\
&= A_1(L_{k_1})+\sum_{0\le n<N-L_{k_1}} a(n+L_{k_1})\\
&= A_1(L_{k_1})+\sum_{0\le n<N-L_{k_1}} (a(n)+1)\\
&= A_1(L_{k_1})+A_1(N-L_{k_1})+N-L_{k_1}.
\end{aligned}$$

and

$$\begin{aligned}
A_2(N) &= \sum_{0\le n<L_{k_1}} a^2(n)+\sum_{L_{k_1}\le n<N} a^2(n)\\
&= A_2(L_{k_1})+\sum_{0\le n<N-L_{k_1}} a^2(n+L_{k_1})\\
&= A_2(L_{k_1})+\sum_{0\le n<N-L_{k_1}} (a^2(n)+2a(n)+1)\\
&= A_2(L_{k_1})+N-L_{k_1}+A_2(N-L_{k_1})+2A_1(N-L_{k_1}).
\end{aligned}$$

This proves the first part of Theorem 2.

The final formula in Theorem 2 can be proved using induction on s and the recursion formulae. This completes the proof of Theorem 2.

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