Hindawi
Journal of Mathematics
Volume 2021, Article ID 9937647, 3 pages
https://doi.org/10.1155/2021/9937647

Research Article

On the Distribution Properties of the Smarandache Prime Part

Yahui Yu[1] and Jiayuan Hu[2]

[1]Department of Mathematics and Physics, Luoyang Institute of Science and Technology, Luoyang 471023, Henan, China
[2]Department of Mathematics and Computer Science, Hetao College, Bayannur 015000, China

Correspondence should be addressed to Jiayuan Hu; hujiayuan1986@163.com

Received 27 March 2021; Accepted 5 August 2021; Published 17 August 2021

Academic Editor: Ghulam Shabbir

For each integer n, denote by $p_p(n)$ the largest prime $\leq n$ and by $P_p(n)$ the smallest prime $\geq n$, called as the Smarandache inferior prime part and superior prime part of n, respectively. Define $I_n := (1/n)\sum_{m\leq n} p_p(m)$ and $S_n := (1/n)\sum_{m\leq n} P_p(m)$. In this short note, we proved some estimates on $I_n - S_n$ and I_n/S_n, answering a question proposed by Kashihara and improving a result of Yan.

1. Introduction

The properties of primes play a fundamental role in number theory; especially, the distribution of prime numbers in certain sequences has always been a hot topic. It is well known that if $(h,k)=1$, the arithmetic progression

$$kn+h, \quad n=0,1,2,\ldots, \tag{1}$$

contains infinitely many primes. This result is now known as Dirichlet's theorem. Take another example; for any positive real numbers α and β, the set

$$B(\alpha,\beta)=\{\lfloor n\alpha+\beta\rfloor\colon n\in N\}, \tag{2}$$

is called the associated Betty sequence (or Beatty set). For irrational α, it follows from a classical result of Ivan Vinogradov that the number $\pi_{B(\alpha,\beta)} \sim (1/\alpha)\pi(x)$. In 2020, Janyarak Tongsomporn and Jorn Steuding showed that, for positive real irrational number α of finite type, arbitrary real number β, and every $b>2$, there exist a positive constant $\delta_b>0$ and arbitrarily large x and x' such that

$$\begin{aligned}\pi_{B(\alpha,\beta)}(x+y)-\pi_{B(\alpha,\beta)}(x) &> \frac{1+\delta_b}{\alpha}\frac{y}{\log x},\\ \pi_{B(\alpha,\beta)}(x'+y')-\pi_{B(\alpha,\beta)}(x') &> \frac{1+\delta_b}{\alpha}\frac{y'}{\log x'},\end{aligned} \tag{3}$$

where $y=y(x)=(\log x)^b$ and $y'=y(x')$.

In 1993, American-Romanian number theorist Florentin Smarandache published a book named *Only problems, not solutions!* In this book, he presented 105 unsolved arithmetical problems and conjectures about special sequences and functions, which can help us to analyze the properties of primes and the factors of integers. In 2019, Liu Miaohua studied the Smarandache dual function $S^*(m)$ is defined as n, that n factorial divide m, namely, is

$$S^*(m)=\max\{n\colon n\in N, n!|m\}. \tag{4}$$

They derived a formula of $\prod_{d|n} S^*(d)$; that is, for any positive integer m,

$$\prod_{d|n} S^*(d)=\begin{cases}1, & m=p_1^{\beta_1}\cdot\ldots\cdot p_j^{\beta_j}, p_i\geq 3,\\ 2^{\beta(\beta_1+1)(\beta_2+1),\ldots,(\beta_j+1)}, & m=2^{\beta}p_1^{\beta_1},\ldots,p_j^{\beta_j}, p_i\geq 3,\\ 2^{\beta(\beta_2+1)(\beta_3+1),\ldots,(\beta_j+1)}3^{\beta\beta_1(\beta_2+1)(\beta_3+1),\ldots,(\beta_j+1)}, & m=2^{\beta}3^{\beta_1}p_1^{\beta_2},\ldots,p_{j-1}^{\beta_j}, p_i\geq 5, \beta=1 \text{ or } 2,\end{cases} \tag{5}$$

where p_i is different odd prime numbers and β_i and β are the positive integers.

Based on this, we now consider the following Smarandache problems. As usual, let p_n be the nth prime. For each integer n, denote by $p_p(n)$ the largest prime $\leq n$ and by $P_p(n)$ the smallest prime $\geq n$ (see [1]), where $p_p(n)$ and $P_p(n)$ are called as the Smarandache inferior prime part and superior prime part of n, respectively. According to these definitions, the following are obvious:

(a) For each integer n, we have $p_p(n) \leq n \leq P_p(n)$.

(b) The integer n is a prime if and only if $p_p(n) = P_p(n)$.

(c) We have

$$\begin{cases} p_p(m) = p_n \\ P_p(m) = p_{n+1} \end{cases}, \quad (p_n < m < p_{n+1}). \tag{6}$$

Define

$$I_n := \frac{1}{n} \sum_{m \leq n} p_p(m), \qquad S_n := \frac{1}{n} \sum_{m \leq n} P_p(m). \tag{7}$$

In [2], Problem 10, Kashihara proposed two problems about I_n and S_n:

(A) To determinate whether $\lim_{n\longrightarrow\infty}(I_n - S_n)$ converges or diverges. If it converges, find the limit.

(B) To determinate whether $\lim_{n\longrightarrow\infty} I_n/S_n$ converges or diverges. If it converges, find the limit.

Yan [3] proved that

$$\frac{I_n}{S_n} = 1 + O\left(n^{-1/3}\right), \tag{8}$$

for $n \longrightarrow \infty$, which implies an answer to problem (B):

$$\lim_{n\longrightarrow\infty} \frac{I_n}{S_n} = 1. \tag{9}$$

In this short note, we shall continue a such study. Our aims is double. Firstly, we shall give a complete answer for question (A). Our proof shows that this problem is closely related to the quantity $D(X) := \sum_{x<pn\leq 2x} (p_{n+1} - p_n)^2$. According to [4], it is conjectured that $D(X) \sim X \log X$ as $X \longrightarrow \infty$. Our first result shows that $\lim_{n\longrightarrow\infty}(I_n - S_n)$ diverges. Precisely, we have

This short note will focus on these two problems. The first result in this paper shows that $\lim_{n\longrightarrow\infty}(I_n - S_n)$ diverges. Precisely, we have the following theorem.

Theorem 1. *For any $\varepsilon > 0$, there is a positive constant $C_\varepsilon > 0$ depending on ε such that the following inequalities*

$$\left(\frac{193}{192} - \varepsilon\right)\log n - C_\varepsilon \leq I_n - S_n \leq C_\varepsilon n^{(5/18)+\varepsilon}, \tag{10}$$

hold for $n \geq 1$. In particular, we have $\lim_{n\longrightarrow\infty}(I_n - S_n) = \infty$.

The second result in this paper improves Yan's equality (8) considerably.

Theorem 2. *For any $\varepsilon > 0$, we have*

$$\frac{I_n}{S_n} = 1 + O_\varepsilon\left(n^{(-13/18)+\varepsilon}\right), \tag{11}$$

as $n \longrightarrow \infty$.

For comparison, we have $(13/18) = 0.722, \ldots$ and $(1/3) = 0.333, \ldots$.

2. Proof of Theorem 1

In view of facts (a), (b), and (c) mentioned above, we have

$$I_n = \frac{1}{n}\sum_{m\leq n} p_p(m) = \frac{1}{n}\sum_{p_k\leq n} p_k \sum_{p_p(m)=p_k} 1 = \frac{1}{n}\sum_{p_k\leq n} p_k(p_k - p_{k-1}), \tag{12}$$

$$S_n = \frac{1}{n}\sum_{m\leq n} P_p(m) = \frac{1}{n}\sum_{p_k\leq n} p_{k-1} \sum_{P_p(m)=p_{k-1}} 1 = \frac{1}{n}\sum_{p_k\leq n} p_{k-1}(p_k - p_{k-1}), \tag{13}$$

where we have made the convention that $p_0 = 0$. From (12) and (13), we can deduce that

$$\begin{aligned} I_n - S_n &= \frac{1}{n}\sum_{p_k\leq n} p_k(p_k - p_{k-1}) - \frac{1}{n}\sum_{p_k\leq n} p_{k-1}(p_k - p_{k-1}) \\ &= \frac{1}{n}\left(\sum_{p_k\leq n} p_k^2 + \sum_{p_k\leq n} p_{k-1}^2 - 2\sum_{p_k\leq n} p_{k-1}p_k\right) \\ &= \frac{1}{n}\sum_{p_k\leq n}(p_k - p_{k-1})^2. \end{aligned} \tag{14}$$

According to Theorem 1, for any $\varepsilon > 0$, we have

$$\sum_{X\leq p_k\leq 2X} (p_k - p_{k-1})^2 \geq \left(\frac{193}{192} - \varepsilon\right) X \log X, \tag{15}$$

for all $X \geq X_0(\varepsilon)$. Combining (14) and (15), we can derive that

$$\begin{aligned} I_n - S_n &\geq \frac{1}{n}\sum_{1\leq d\leq \log(n/X_0(\varepsilon))/\log 2}\ \sum_{n/2^d < p_k \leq n/2^{d-1}} (p_k - p_{k-1})^2 \\ &\geq \frac{1}{n}\sum_{1\leq d\leq \log(n/X_0(\varepsilon))/\log 2} \left(\frac{193}{192} - \varepsilon\right)\frac{n}{2^d}\log\frac{n}{2^d} \\ &= \left(\frac{193}{192} - \varepsilon\right)(M\log n - E\log 2), \end{aligned} \tag{16}$$

where

$$M := \sum_{1 \le d \le \log(n/X_0(\varepsilon))/\log 2} \frac{1}{2^d},$$
$$E := \sum_{1 \le d \le \log(n/X_0(\varepsilon))/\log 2} \frac{d}{2^d}. \quad (17)$$

By elementary computation, we have

$$M = 1 + O_\varepsilon(n^{-1}),$$
$$E \ll 1. \quad (18)$$

Then, inserting (18) into (16), we can obtain the first inequality in (10).

On the contrary, according to [5], Heath-Brown proved that

$$\sum_{p_k \le n} (p_k - p_{k-1})^2 \ll_\varepsilon n^{(23/18)+\varepsilon}, \quad (19)$$

for all $n \ge 1$. The second inequality in (10) follows from (14) and (19).

The second assertion is an immediate consequence of (10). This completes the proof of Theorem 1.

3. Proof of Theorem 2

In view of (14) and (19), we derive that

$$\frac{I_n}{S_n} = 1 + \frac{1}{S_n n} \sum_{p_k \le n} (p_k - p_{k-1})^2 = 1 + O\left(\frac{n^{(5/18)+\varepsilon}}{S_n}\right). \quad (20)$$

On the contrary, formula (13) can be rewritten as

$$S_n \ge \frac{1}{n} \sum_{p_k \le n} p_{k-1} = \frac{1}{n} \sum_{p_k \le n} p_k - \frac{1}{n} \sum_{p_k \le n} (p_k - p_{k-1}). \quad (21)$$

By the prime number theorem, we have

$$\frac{1}{n} \sum_{p_k \le n} p_k \gg \frac{n}{\log n} \quad (n \ge 2). \quad (22)$$

By the Cauchy–Schwarz inequality and (19), it follows that

$$\frac{1}{n} \sum_{p_k \le n} (p_k - p_{k-1}) \le \frac{1}{n} \left(\sum_{p_k \le n} 1 \cdot \sum_{p_k \le n} (p_k - p_{k-1})^2 \right)^{1/2}$$
$$\ll_\varepsilon \frac{1}{n} \left(n^{1+(23/18)+\varepsilon}\right)^{1/2}$$
$$\ll_\varepsilon n^{(5/36)+\varepsilon}. \quad (23)$$

From (21)–(23), we deduce that

$$S_n \gg \frac{n}{\log n} \quad (n \ge 2). \quad (24)$$

Inserting this into (20), we obtain

$$\frac{I_n}{S_n} = 1 + O_\varepsilon\left(n^{(-13/18)+\varepsilon}\right). \quad (25)$$

This completes the proof.

4. Conclusion

The main results of this paper are two theorems involving Smarandache inferior prime part $p_p(n)$ and superior prime part $P_p(n)$. Theorem 1 establishes an inequality for $I_n - S_n$ and shows that $\lim_{n \longrightarrow \infty}(I_n - S_n) = \infty$. Theorem 2 establishes an estimate on I_n/S_n and obtains that $(I_n/S_n) = 1 + O_\varepsilon(n^{(-13/18)+\varepsilon})$ as $n \longrightarrow \infty$. These results represent new contributions to the research on the distribution of the Smarandache inferior prime part and improve the related results before. Of course, our methods can also be generalized to other problems of primes in integer sets. However, there are also many unsolved Smarandache problems to be explored, such as the properties of the determinant of Smarandache prime numbers. These open problems remain to be further studied.

Data Availability

All the data generated or analysed during this study are included within this published article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Both authors read and approved the final manuscript.

Acknowledgments

The authors would like to express their sincere appreciation to Wu Jie for his enthusiastic guidance and generous help. This work was supported by the N. S. F (11871193) of P. R. China, Young Talent-Training Plan for College Teachers (2019GGJS241) of Henan Province, Key Research Projects for Universities and Colleges (20A110027) of Henan Province, Startup Foundation for Introducing Talent (HYRC2019007) of Hetao College (CN), and N. S. F. (2021MS01003) of Inner Mongolia (CN).

References

[1] F. Smarandache, *Only Problems, Not Solutions*, Xiquan Publishing House, Chicago, IL, USA, 1993.

[2] K. Kashihara, *Comments and Topics on Smarandache Notions and Problems*, Erhus University Press, Aarhus, Denmark, 1996.

[3] X. Yan, "On the smarandache prime part," *Scientia Magna*, vol. 3, no. 3, pp. 74–77, 2007.

[4] A. Y. Cheer and D. A. Goldston, "Longer than average intervals containing no primes," *Transactions of the American Mathematical Society*, vol. 304, no. 2, pp. 469–486, 1987.

[5] D. R. Heath-Brown, "The differences between consecutive primes, III," *Journal of the London Mathematical Society*, vol. 20, no. 2, pp. 177-178, 1979.