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Some arithmetical properties of primitive numbers of power p[1]

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Abstract The main purpose of this paper is to study the arithmetical properties of the primitive numbers of power p by using the elementary method, and give some interesting identities and asymptotic formulae.

Keywords Primitive numbers of power p; Smarandache function; Asymptotic formula.

§1. Introduction

Let p be a fixed prime and n be a positive integer. The primitive numbers of power p, denoted as $S_p(n)$, is defined as following:

$$S_p(n) = \min\{m : m \in N, p^n | m!\}.$$

In problem 47,48 and 49 of [1], Professor F.Smarandache asked us to study the properties of the primitive numbers sequences $\{S_p(n)\}(n = 1, 2, \cdots)$. It is clear that $\{S_p(n)\}(n = 1, 2, \cdots)$ is the sequence of multiples of prime p and each number being repeated as many times as its exponent of power p is. What's more, there is a very close relationship between this sequence and the famous Smarandache function $S(n)$, where

$$S(n) = \min\{m : m \in N, n|m!\}.$$

Many scholars have studied the properties of $S(n)$, see [2], [3], [4], [5] and [6]. It is easily to show that $S(p) = p$ and $S(n) < n$ except for the cases $n = 4$ and $n = p$. Hence, the following relationship formula is obviously:

$$\pi(x) = -1 + \sum_{n=2}^{[x]} \left[\frac{S(n)}{n} \right],$$

where $\pi(x)$ denotes the number of primes up to x, and $[x]$ denotes the greatest integer less than or equal to x. However, it seems no one has given some nontrivial properties about the primitive number sequences before. In this paper, we studied the relationship between the Riemann zeta-function and an infinite series involving $S_p(n)$, and obtained some interesting identities and asymptotic formulae for $S_p(n)$. That is, we shall prove the following conclusions:

Theorem 1. For any prime p and complex number s, we have the identity:

$$\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1},$$

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where $\zeta(s)$ is the Riemann zeta-function.

Specially, taking $s = 2, 4$ and $p = 2, 3, 5$, we have the

Corollary. The following identities hold:

$$\sum_{n=1}^{\infty} \frac{1}{S_2^2(n)} = \frac{\pi^2}{18}; \qquad \sum_{n=1}^{\infty} \frac{1}{S_3^2(n)} = \frac{\pi^2}{48}; \qquad \sum_{n=1}^{\infty} \frac{1}{S_5^2(n)} = \frac{\pi^2}{144};$$

$$\sum_{n=1}^{\infty} \frac{1}{S_2^4(n)} = \frac{\pi^4}{1350}; \qquad \sum_{n=1}^{\infty} \frac{1}{S_3^4(n)} = \frac{\pi^4}{7200}; \qquad \sum_{n=1}^{\infty} \frac{1}{S_5^4(n)} = \frac{\pi^4}{56160}.$$

Theorem 2. Let p be any fixed prime. Then for any real number $x \geq 1$, we have the asymptotic formula:

$$\sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} \frac{1}{S_p(n)} = \frac{1}{p-1} \left(\ln x + \gamma + \frac{p \ln p}{p-1} \right) + O(x^{-\frac{1}{2}+\epsilon}),$$

where γ is the Euler constant, ϵ denotes any fixed positive number.

Theorem 3. Let k be any positive integer. Then for any prime p and real number $x \geq 1$, we have the asymptotic formula:

$$\sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} S_p^k(n) = \frac{x^{k+1}}{(k+1)(p-1)} + O(x^{k+\frac{1}{2}+\epsilon}).$$

§2. Proof of the theorems

To complete the proof of the theorems, we need a simple Lemma.

Lemma. Let b, T are two positive numbers, then we have

$$\frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{a^s}{s} ds = \begin{cases} 1 + O\left(a^b \min\left(1, \frac{1}{T \ln a}\right)\right), & if\ a > 1; \\ O\left(a^b \min\left(1, \frac{1}{T \ln a}\right)\right), & if\ 0 < a < 1; \\ \frac{1}{2} + O\left(\frac{b}{T}\right), & if\ a = 1. \end{cases}$$

Proof. See Lemma 6.5.1 of [8].

Now we prove the theorems. First, we prove Theorem 1. Let $m = S_p(n)$, if $p^\alpha \| m$, then the same number m will repeat α times in the sequence $S_p(n)$ $(n = 1, 2, \cdots)$. Noting that $S_p(n)$ $(n = 1, 2, \cdots)$ is the sequence of multiples of prime p, we can write

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} &= \sum_{\substack{m=1 \\ p^\alpha \| m}}^{\infty} \frac{\alpha}{m^s} = \sum_{p^\alpha} \sum_{\substack{m=1 \\ (m,p)=1}}^{\infty} \frac{\alpha}{p^{\alpha s} m^s} \\ &= \sum_{\alpha=1}^{\infty} \frac{\alpha}{p^{\alpha s}} \zeta(s) \left(1 - \frac{1}{p^s}\right) = \left(1 - \frac{1}{p^s}\right) \zeta(s) \sum_{\alpha=1}^{\infty} \frac{\alpha}{p^{\alpha s}}. \end{aligned}$$

Since

$$\left(1 - \frac{1}{p^s}\right) \sum_{\alpha=1}^{\infty} \frac{\alpha}{p^{\alpha s}} = \frac{1}{p^s} + \sum_{\alpha=1}^{\infty} \frac{1}{p^{(\alpha+1)s}} = \frac{1}{p^s} + \frac{1}{p^s} \left(\frac{1}{p^s - 1}\right) = \frac{1}{p^s - 1},$$

we have the identity

$$\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1}.$$

This completes the proof of Theorem 1.

Now we prove Theorem 2 and Theorem 3. Let $x \geq 1$ be any real number. If we set $a = \frac{x}{S_p(n)}$ in the lemma, then we can write

$$\begin{aligned} &\frac{1}{2\pi i} \int_{b-iT}^{b+iT} \sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} \frac{x^s}{S_p^{s-k}(n) s} ds \\ = &\sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} S_p^k(n) + O\left(\sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} \frac{x^b}{S_p^{b-k}(n)} \min\left(1, \frac{1}{T \ln\left(\frac{x}{S_p(n)}\right)}\right)\right), \qquad (1) \end{aligned}$$

$$\frac{1}{2\pi i} \int_{b-iT}^{b+iT} \sum_{\substack{n=1 \\ S_p(n) > x}}^{\infty} \frac{x^s}{S_p^{s-k}(n) s} ds = O\left(\sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} \frac{x^b}{S_p^{b-k}(n)} \min\left(1, \frac{1}{T \ln\left(\frac{x}{S_p(n)}\right)}\right)\right), \qquad (2)$$

where k is any integer. Combining (1) and (2), we find

$$\begin{aligned} &\frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{x^s}{s} \sum_{n=1}^{\infty} \frac{1}{S_p^{s-k}(n)} ds \\ = &\sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} S_p^k(n) + O\left(\sum_{n=1}^{\infty} \frac{x^b}{S_p^{b-k}(n)} \min\left(1, \frac{1}{T \ln\left(\frac{x}{S_p(n)}\right)}\right)\right). \qquad (3) \end{aligned}$$

Then from Theorem 1, we can get

$$\sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} S_p^k(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{\zeta(s-k) x^s}{(p^{s-k} - 1) s} ds + O\left(x^b \min\left(1, \frac{1}{T \ln\left(\frac{x}{S_p(n)}\right)}\right)\right) \qquad (4)$$

Now we calculate the first term in the right side of (4).

When $k = -1$, taking $b = \frac{1}{2}$ and $T = x$, we move the integral line from $s = \frac{1}{2} + iT$ to $s = -\frac{1}{2} + iT$. This time, the function

$$f(s) = \frac{\zeta(s+1) x^s}{(p^{s+1} - 1) s}$$

have a second order pole point at $s = 0$. Its residue is $\frac{1}{p-1}\left(\ln x + \gamma - \frac{p \ln p}{p-1}\right)$. Hence, we can write

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{\zeta(s+1) x^s}{(p^{s+1} - 1) s} ds \\ = &\frac{1}{p-1}\left(\ln x + \gamma - \frac{p \ln p}{p-1}\right) + \frac{1}{2\pi i}\left(\int_{\frac{1}{2}-iT}^{-\frac{1}{2}-iT} + \int_{-\frac{1}{2}-iT}^{-\frac{1}{2}+iT} + \int_{-\frac{1}{2}+iT}^{\frac{1}{2}+iT} \right) \frac{\zeta(s+1) x^s}{(p^{s+1} - 1) s} ds. \qquad (5) \end{aligned}$$

We can easily get the estimate

$$\left|\frac{1}{2\pi i}\left(\int_{\frac{1}{2}-iT}^{-\frac{1}{2}-iT}+\int_{-\frac{1}{2}+iT}^{\frac{1}{2}+iT}\right)\frac{\zeta(s+1)x^s}{(p^{s+1}-1)s}ds\right|$$
$$\ll \int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\zeta(\sigma+1+iT)x^{\frac{1}{2}}}{(p^{\sigma+1+iT}-1)T}\right|d\sigma \ll \frac{x^{\frac{1}{2}}}{T}=x^{-\frac{1}{2}}, \qquad (6)$$

and

$$\left|\frac{1}{2\pi i}\int_{-\frac{1}{2}-iT}^{-\frac{1}{2}+iT}\frac{\zeta(s+1)x^s}{(p^{s+1}-1)s}ds\right| \ll \int_0^T\left|\frac{\zeta(\frac{1}{2}+it)x^{-\frac{1}{2}}}{(p^{\frac{1}{2}+it}-1)(\frac{1}{2}+t)}\right|dt \ll x^{-\frac{1}{2}+\epsilon}. \qquad (7)$$

Combining $(4),(5),(6)$ and (7), we have

$$\sum_{\substack{n=1\\S_p(n)\le x}}^{\infty}\frac{1}{S_p(n)}=\frac{1}{p-1}\left(\ln x+\gamma+\frac{p\ln p}{p-1}\right)+O(x^{-\frac{1}{2}+\epsilon}).$$

This is the result of Theorem 2.

When $k\ge 1$, taking $b=k+\frac{3}{2}$ and $T=x$, we move the integral line of (4) from $s=k+\frac{3}{2}$ to $s=k+\frac{1}{2}$. Now the function

$$g(s)=\frac{\zeta(s-k)x^s}{(p^{s-k}-1)s}$$

have a simple pole point at $s=k+1$ with residue $\frac{x^{k+1}}{(p-1)(k+1)}$. Using the same method we can also get

$$\sum_{\substack{n=1\\S_p(n)\le x}}^{\infty}S_p^k(n)=\frac{x^{k+1}}{(k+1)(p-1)}+O(x^{k+\frac{1}{2}+\epsilon}).$$

This completes the proofs of the theorems.

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