SOME EXPRESSIONS OF THE SMARANDACHE PRIME FUNCTION

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Abstract The main purpose of this paper is using elementary arithmetical functions to give some expressions of the Smarandache Prime Function $P(n)$.

In this article we gave some expressions of the Smarandache Prime Function $P(n)$ (see reference [1]), using elementary arithmetical functions. The Smarandache Prime Function is the complementary of the Prime Characteristic Function:

$$P(n) = \begin{cases} 0 & if \quad n \quad is \quad a \quad prime, \\ 1 & if \quad n \quad is \quad a \quad composite. \end{cases}$$

Expression 1.

$$P(n) = 1 \left\lfloor \frac{lcm(1,2,\cdots,n)}{n \cdot lcm(1,2,\cdots,n-1)} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ is the floor function (see reference [2]).

Proof. We consider three cases:

Case 1: If $n = p$ with p prime, then we have

$$lcm(1,2,\cdots,p) = lcm(lcm(1,2,\cdots p-1),p) = p \cdot lcm(1,2,\cdots,p-1)$$

Therefore we have: $P(n) = 0$.

Case 2: If $n = p^\alpha$ with p is prime and α is a positive integer greater than one, we may have

$$\begin{aligned} & \left\lfloor \frac{lcm(1,2,\cdots,n)}{n \cdot lcm(1,2,\cdots,n-1)} \right\rfloor \\ = & \left\lfloor \frac{lcm(1,2,\cdots,p^\alpha)}{n \cdot lcm(1,2,\cdots,p^\alpha - 1)} \right\rfloor \\ = & \left\lfloor \frac{lcm(lcm(1,2,\cdots,p^{\alpha-1},\cdots,p^\alpha - 1),p^\alpha}{n \cdot lcm(1,2,\cdots,p^\alpha - 1)} \right\rfloor \end{aligned}$$

$$= \left\lfloor \frac{p \cdot lcm(1, 2, \cdots, p^{\alpha-1}, \cdots, p^{\alpha} - 1)}{n \cdot (1, 2, \cdots, p^{\alpha} - 1)} \right\rfloor$$
$$= \left\lfloor \frac{p}{n} \right\rfloor = 0.$$

So we have: $P(n) = 1$.

Case 3: If $n = a \cdot b$ with $gcd(a, b) = 1$ and $a, b > 1$. We can suppose $a < b$, then we have

$$\begin{aligned} & lcm(1, 2, \cdots, a, \cdots, b, \cdots, n) \\ & = lcm(1, 2, \cdots, a, \cdots, b, \cdots, n-1, a \cdot b) \\ & = lcm(1, 2, \cdots, a, \cdots, b, \cdots, n-1) \end{aligned}$$

and therefore we have:

$$\begin{aligned} P(n) &= 1 - \left\lfloor \frac{lcm(1, 2, \cdots, n)}{n \cdot lcm(1, 2, \cdots, n-1)} \right\rfloor \\ &= 1 - \left\lfloor \frac{1}{n} \right\rfloor = 1 - 0 = 1 \end{aligned}$$

With this the expression one is proven.

Expression 2. [3],[4]

$$P(n) = -\left\lfloor \frac{2 - \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor}{n} \right\rfloor$$

Proof. We consider $d(n) = \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor$ is the number of divisors of n because:

$$\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor = \begin{cases} 1 & if \quad i \quad divides \quad n, \\ 0 & if \quad i \quad not \quad divide \quad n. \end{cases}$$

If $n = p$ prime we have $d(n) = 2$ and therefore $P(n) = 0$.
If n is composite we have $d(n) > 2$ and therefore:

$$-1 < \frac{2 - d(n)}{n} < 0 \Longrightarrow P(n) = 1.$$

Expression 3.
We can also prove the following expression:

$$P(n) = 1 - \left\lfloor \frac{1}{n} \cdot GCD\left(\binom{n}{1}, \binom{n}{2}, \cdots, \binom{n}{n-1}\right)\right\rfloor,$$

where $\binom{n}{i}$ is the binomial coefficient.

Can the reader prove this last expression?

References

[1] E. Burton, Smarandache Prime and Coprime Functions, http://www.gallup.unm.edu/ smarandache/primfnct.txt.

[2] S. M. Ruiz, A New Formula for the n-th Prime, Smarandache Notions Journal **15** 2005.

[3] S. M. Ruiz, A Functional Recurrence to Obtain the Prime Numbers Using the Smarandache Prime Function, Smarandache Notions Journal **11** (2000), 56.

[4] S. M. Ruiz, The General Term of the Prime Number Sequence and the Smarandache Prime Function, Smarandache Notions Journal, **11** (2000), 59.