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Some identities involving function $U_t(n)$

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Abstract In this paper, we use the elementary method to study the properties of pseudo Smarandache function $U_t(n)$, and obtain some interesting identities involving function $U_t(n)$, and for any fixed integer n, offer a method of calculating of the infinite series $\sum_{i=1}^{\infty} U_t(n)$.

Keywords Pseudo Smarandache function, some identities, elementary method.

§1. Introduction and result

For any positive integer n and $U_t(n)$ fixed $t \geq 1$, we define function

$$U_t(n) = \min\left\{1^t + 2^t + 3^t + \cdots + n^t + k,\ n \mid m,\ k \in N^+,\ t \in N^+\right\},$$

where $n \in N^+,\ m \in N^+$. Wang Yu studied the properties of pseudo Smarandache function $U_t(n)$, and obtained calculation of the infinite series

$$\sum_{i=1}^{\infty} U_t(1),\ \sum_{i=1}^{\infty} U_t(2),\ \sum_{i=1}^{\infty} U_t(3).$$

In this paper we use the elementary method to study the calculating of the infinite series

$$\sum_{i=1}^{\infty} U_t(n),$$

where $n \geq 4$, obtain calculation of the infinite series $\sum_{i=1}^{\infty} U_t(4),\ \sum_{i=1}^{\infty} U_t(5)$, and for any fixed integer n, we offer a method of calculating the infinite series $\sum_{i=1}^{\infty} U_t(n)$.

Theorem 1. For any real positive integer s, we have

$$\sum_{i=1}^{\infty} \frac{1}{U_4^s}(n) = 1 + (2 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{5^s})(1 - \frac{1}{5^s}) + \frac{1}{7^s}(2 - \frac{1}{2^s} - \frac{1}{3^s}) + (1 - \frac{1}{2^s})(1 - \frac{1}{3^s})(2 - \frac{1}{5^s}).$$

Theorem 2. For any real positive integer s, we have

$$\sum_{i=1}^{\infty} \frac{1}{U_5^s}(n) = (2 - \frac{2}{2^s} + \frac{1}{4^s} - \frac{1}{6^s})\zeta(s).$$

§2. Some lemmas

Lemma 1. If $S_r(n) = \sum_{i=1}^{n} k^r$, then

$$S_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n,\ S_2(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n,\ S_3(n) = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2,$$

if $r \in Z^-$, let $S_r(n) = 0$, when $r \geq 4$, and $r \in Z$, we have

$$\begin{aligned} S_r(n) &= \frac{1}{r+1}n^{r+1} + \frac{1}{2}n^r + \frac{3r}{28}n^{r-1} + \frac{r(r-1)}{84}n^{r-2} \\ &+\frac{r(r-1)(r-2)}{8\cdot 7\cdot 6\cdot 5}n^{r-3} - \frac{r(r-1)}{42}S_{r-2}(n) \\ &-\frac{1}{8\cdot 7\cdot 6\cdot 5}\sum_{i=1}^{r-n}(-1)^j \begin{pmatrix} r \\ 8+j \end{pmatrix} \frac{(j+1)(j+2)(j+3)(j+4)}{j+9}S_{r-8-j}(n). \end{aligned}$$

Proof. See reference [1].

Lemma 2. For any positive integer n, we have

$$\sum_{i=1}^{\infty}\frac{1}{U_4^s}(n) = \begin{cases} \frac{n}{30}, & \text{if } n \equiv 0(\text{mod } 30), \\ n, & \text{if } n \equiv 1(\text{mod } 30),\ \text{or } n \equiv 29(\text{mod } 30),\ \text{or } n \equiv 7(\text{mod } 30), \\ & \text{or } n \equiv 23(\text{mod } 30),\ \text{or } n \equiv 11(\text{mod } 30),\ \text{or } n \equiv 19(\text{mod } 30), \\ & \text{or } n \equiv 13(\text{mod } 30),\ \text{or } n \equiv 17(\text{mod } 30), \\ \frac{n}{2}, & \text{if } n \equiv 2(\text{mod } 30),\ n \equiv 28(\text{mod } 30),\ \text{or } n \equiv 4(\text{mod } 30), \\ & \text{or } n \equiv 26(\text{mod } 30),\ \text{or } n \equiv 8(\text{mod } 30)\ \text{or } n \equiv 22(\text{mod } 30), \\ & \text{or } n \equiv 14(\text{mod } 30),\ \text{or } n \equiv 16(\text{mod } 30), \\ \frac{n}{3}, & \text{if } n \equiv 3(\text{mod } 30),\ \text{or } n \equiv 27(\text{mod } 30),\ \text{or } n \equiv 9(\text{mod } 30), \\ & \text{or } n \equiv 21(\text{mod } 30), \\ \frac{n}{5}, & \text{if } n \equiv 5(\text{mod } 30),\ \text{or } n \equiv 25(\text{mod } 30), \\ \frac{5n}{6}, & \text{if } n \equiv 6(\text{mod } 30),\ \text{or } n \equiv 24(\text{mod } 30),\ \text{or } n \equiv 12(\text{mod } 30), \\ & \text{or } n \equiv 18(\text{mod } 30), \\ \frac{7n}{10}, & \text{if } n \equiv 10(\text{mod } 30),\ \text{or } n \equiv 20(\text{mod } 30), \\ \frac{7n}{15}, & \text{if } n \equiv 15(\text{mod } 30), \end{cases}$$

Proof. (1) If $n \equiv 0(\text{mod } 30)$, then we have $n = 30h_1$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (2700h_1^3+90h_1^2)(30h_1+1)(60h_1+1)-(1800h_1^3+60h_1^2+31h_1),$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{30}$.

(2) If $n \equiv 1(\text{mod } 30)$, then we have $n = 30h_1 + 1$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+1)(15h_1+1)(20h_1+1)(540h_1^2+54h_1+1),$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = n$.

If $n \equiv 29 (\bmod\ 30)$, then we have $n = 30h_1 + 29$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+29)(h_1+1)(60h_1+59)(2700h_1^2+5310h_1+2609),$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = n$.

(3) If $n \equiv 2 (\bmod\ 30)$, then we have $n = 30h_1 + 2$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} &= (90h_1+9)(15h_1+1)(10h_1+1)(12h_1+1)(30h_1+2)\\ &\quad -5h_1(30h_1+2)(12h_1+1)+6h_1(30h_1+2)+(15h_1+1),\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{2}$.

If $n \equiv 28 (\bmod\ 30)$, then we have $n = 30h_1 + 28$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\ =\ &(15h_1+14)(30h_1+29)(20h_1+19)(30h_1+28)(18h_1+17)\\ &+(15h_1+14)(30h_1+28)(20h_1+19)(12h_1+11)+(30h_1+28)(12h_1+11)(10h_1+9)\\ &+(6h_1+5)(30h_1+28)+(15h_1+14),\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{2}$.

(4) If $n \equiv 3 (\bmod\ 30)$, then we have $n = 30h_1 + 3$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\ =\ &(18h_1+2)(15h_1+2)(10h_1+1)(60h_1+7)(30h_1+3)\\ &+2(15h_1+2)(10h_1+1)(12h_1+1)(30h_1+3)+5h_1(12h_1+1)(30h_1+3)\\ &+8h_1(30h_1+3)+(20h_1+2),\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{3}$.

If $n \equiv 27 (\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 27$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & (10h_1+9)(15h_1+14)(12h_1+11)(30h_1+27)(90h_1+84) \\ & -(12h_1+11)(30h_1+27)(5h_1+4)-(30h_1+27)(8h_1-7)-(10h_1+9), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{3}$.

(5) If $n \equiv 4 (\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 4$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & (15h_1+2)(6h_1+1)(20h_1+3)(90h_1+15)(30h_1+4) \\ & -(6h_1+1)(30h_1+4)(10h_1+1)-3h_1(30h_1+4)-(15h_1+2), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{2}$.

If $n \equiv 26 (\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 26$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & (15h_1+13)(10h_1+9)(60h_1+53)(18h_1+16)(30h_1+26) \\ & +2(6h_1+9)(15h_1+13)(10h_1+9)(30h_1+26)+(5h_1+4)(30h_1+26)(6h_1+5) \\ & +(3h_1+2)(30h_1+3)+(15h_1+13), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{2}$.

(6) If $n \equiv 5 (\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 5$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & 2(6h_1+1)(5h_1+1)(2700h_1^2+990h_1+89)(30h_1+5) \\ & +(6h_1+1)(5h_1+1)(90h_1+18)(30h_1+5)-h_1(30h_1+5)-(6h_1+1), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_5(n),$$

if and only if $U_4(n) = \frac{n}{5}$.

If $n \equiv 25 (\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 25$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & (6h_1+5)(15h_1+13)(20h_1+17)(90h_1+78)(30h_1+25) \\ & -(6h_1+5)(10h_1+8)(30h_1+25)-(5h_1+4)(30h_1+25)-(6h_1+5), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_5(n),$$

if and only if $U_4(n) = \frac{n}{5}$.

(7) If $n \equiv 6(\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 6$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}
&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\
= \ & (5h_1+1)(30h_1+6)(60h_1+13)(540h_1^2+234h_1+25)\\
&+2(5h_1+1)(30h_1+6)(540h_1^2+234h_1+25)+(5h_1+1)(18h_1+4)(30h_1+6)\\
&+(30h_1+6)h_1+(5h_1+1),
\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_5(n),$$

if and only if $U_4(n) = \frac{5n}{6}$.

If $n \equiv 24(\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 24$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}
&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\
= \ & 2(5h_1+4)(6h_1+5)(30h_1+24)(2700h_1^2+4410h_1+1799)\\
&+(5h_1+4)(6h_1+5)(90h_1+75)(30h_1+24)-h_1(30h_1+24)-(25h_1+20),
\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_5(n),$$

if and only if $U_4(n) = \frac{5n}{6}$.

(8) If $n \equiv 7(\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 7$, $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+7)(15h_1+4)(4h_1+1)(2700h_1^2+1350h_1+167),$$

if and only if $U_4(n) = n$.

If $n \equiv 23(\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 23$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+23)(5h_1+4)(60h_1+47)(540h_1^2+846h_1+333),$$

if and only if $U_4(n) = n$.

(9) If $n \equiv 8(\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 8$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}
&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\
= \ & (15h_1+4)(10h_1+3)(60h_1+17)(30h_1+8)(18h_1+5)\\
&+2(15h_1+4)(10h_1+3)(30h_1+18)(12h_1+29)+(15h_1+4)(4h_1+9)(30h_1+8)\\
&+(12h_1+7)(30h_1+8)+(15h_1+4),
\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{2}$.

If $n \equiv 22 (\bmod\ 30)$, then we have $n = 30h_1 + 22$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\ =\ &(15h_1+11)(4h_1+3)(30h_1+22)(2700h_1^2+4050h_1+1517)\\ &+(15h_1+11)(4h_1+3)(90h_1+69)(30h_1+22)-(2h_1+1)(30h_1+22)-(15h_1+11),\end{aligned}$$

so if and only if $U_4(n) = \frac{n}{2}$.

(10) If $n \equiv 9 (\bmod\ 30)$, then we have $n = 30h_1 + 9$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\ =\ &2(10h_1+3)(3h_1+1)(30h_1+9)(2700h_1^2+1710h_1+269)\\ &+(3h_1+1)(10h_1+3)(90h_1+30)(30h_1+9)-h_1(30h_1+9)-(10h_1+3),\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{3}$.

If $n \equiv 21 (\bmod\ 30)$, then we have $n = 30h_1 + 21$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\ =\ &2(10h_1+7)(15h_1+11)(30h_1+21)(540h_1^2+774h_1+277)\\ &+(10h_1+7)(15h_1+11)(18h_1+13)(30h_1+21)+(5h_1+3)(6h_1+4)(30h_1+21)\\ &+(4h_1+2)(30h_1+21)+(20h_1+14),\end{aligned}$$

so if and only if $U_4(n) = \frac{n}{3}$.

(11) If $n \equiv 10 (\bmod\ 30)$, then we have $n = 30h_1 + 10$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\ =\ &(3h_1+1)(30h_1+11)(20h_1+7)(90h_1+33)(30h_1+10)\\ &-(3h_1+1)(20h_1+7)(30h_1+10)-2h_1(30h_1+10)-(21h_1+7),\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{7n}{10}$.

If $n \equiv 20 (\bmod\ 30)$, then we have $n = 30h_1 + 20$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\ =\ &(10h_1+7)(60h_1+41)(3h_1+2)(90h_1+63)(30h_1+20)\\ &-2(10h_1+7)(3h_1+2)(30h_1+20)-h_1(30h_1+20)-(20h_1+14),\end{aligned}$$

so if and only if $U_4(n) = \frac{7n}{10}$.

(12) If $n \equiv 11 \pmod{30}$, then we have $n = 30h_1 + 11$ $(h_1 = 1, 2, 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+11)(2h_1+2)(60h_1+23)(540h_1^2+378h_1+79),$$

if and only if $U_4(n) = n$.

If $n \equiv 19 \pmod{30}$, then we have $n = 30h_1 + 19$ $(h_1 = 1, 2, 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+19)(3h_1+2)(20h_1+13)(2700h_1^2+3510h_1+1139),$$

if and only if $U_4(n) = n$.

(13) If $n \equiv 12 \pmod{30}$, then we have $n = 30h_1 + 12$ $(h_1 = 1, 2, 3, \cdots)$,

$$\begin{aligned}&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\ = \ &(5h_1+2)(30h_1+12)(12h_1+5)(2700h_1^2+2250h_1+467)\\ &+(5h_1+2)(12h_1+5)(90h_1+39)(30h_1+12)-2h_1(30h_1+12)-(25h_1+10),\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{5n}{6}$.

If $n \equiv 18 \pmod{30}$, then we have $n = 30h_1 + 18$ $(h_1 = 1, 2, 3, \cdots)$,

$$\begin{aligned}&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\ = \ &(5h_1+3)(30h_1+18)(60h_1+39)(540h_1^2+666h_1+205)\\ &+(5h_1+3)(18h_1+11)(30h_1+18)+(2h_1+1)(30h_1+18)+(5h_1+3),\end{aligned}$$

so if and only if $U_4(n) = \frac{5n}{6}$.

(14) If $n \equiv 13 \pmod{30}$, then we have $n = 30h_1 + 13$ $(h_1 = 1, 2, 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+13)(15h_1+7)(20h_1+19)(540h_1^2+486h_1+109),$$

if and only if $U_4(n) = n$.

If $n \equiv 17 \pmod{30}$, then we have $n = 30h_1 + 17$ $(h_1 = 1, 2, 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+17)(5h_1+3)(12h_1+7)(2700h_1^2+3510h_1+917),$$

if and only if $U_4(n) = n$.

(15) If $n \equiv 14 \pmod{30}$, then we have $n = 30h_1 + 14$ $(h_1 = 1, 2, 3, \cdots)$,

$$\begin{aligned}&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\ = \ &2(15h_1+7)(2h_1+1)(30h_1+14)(2700h_1^2+2610h_1+629)\\ &+(15h_1+7)(2h_1+1)(90h_1+45)(30h_1+14)-h_1(30h_1+14)-(15h_1+7),\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n)=\frac{n}{2}$.

If $n\equiv 16(\text{mod }30)$, then we have $n=30h_1+16\ (h_1=1,\ 2,\ 3,\cdots)$,

$$\begin{aligned}&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\ =\ &(15h_1+8)(30h_1+16)(20h_1+11)(540h_1^2+594h_1+163)\\ &+(15h_1+8)(20h_1+11)(18h_1+10)(30h_1+16)+(10h_1+5)(30h_1+16)(6h_1+3)\\ &+(3h_1+1)(30h_1+16)+(15h_1+8),\end{aligned}$$

so if and only if $U_4(n)=\frac{n}{2}$.

(16) If $n\equiv 15(\text{mod }30)$, then we have $n=30h_1+15\ (h_1=1,\ 2,\ 3,\cdots)$,

$$\begin{aligned}&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\ =\ &2(15h_1+8)(2h_1+1)(30h_1+15)(2700h_1^2+2790h_1+719)\\ &+(2h_1+1)(15h_1+8)(90h_1+48)(30h_1+15)-h_1(30h_1+15)-(16h_1+8),\end{aligned}$$

so if and only if $U_4(n)=\frac{7n}{15}$.

Lemma 3. For any positive integer n, we have

$$\sum_{i=1}^{\infty}\frac{1}{U_4^s}(n)=\begin{cases}n, & \text{if } n\equiv 0(\text{mod }12),\ \text{or } n\equiv 1(\text{mod }12), or\ n\equiv 11(\text{mod }12),\\ & \text{or } n\equiv 3(\text{mod }12),\ \text{or } n\equiv 9(\text{mod }12),\ \text{or } n\equiv 4(\text{mod }12),\\ & \text{or } n\equiv 8(\text{mod }12),\ \text{or } n\equiv 5(\text{mod }12),\ \text{or } n\equiv 7(\text{mod }12),\\ \frac{n}{2}, & \text{if } n\equiv 2(\text{mod }12),\ n\equiv 10(\text{mod }12),\ \text{or } n\equiv 6(\text{mod }12),\end{cases}$$

Proof. Using the same method of Lemma 1, we can complete the proof of Lemma 2.

§3. Proof of the Theorem 1

In this section, we shall use the Lemma 1 to complete the proof of the theorems. First we prove Theorem 1. For any real number $s>1$, we have

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{U_4^s}(n)\ =\ &\sum_{\substack{i=1\\ n=30h_1}}^{\infty}\frac{1}{(\frac{n}{30})^s}+\sum_{\substack{i=1\\ n=30h_1+1}}^{\infty}\frac{1}{n^s}+\sum_{\substack{i=1\\ n=30h_1+2}}^{\infty}\frac{1}{(\frac{n}{2})^s}+\sum_{\substack{i=1\\ n=30h_1+3}}^{\infty}\frac{1}{(\frac{n}{3})^s}\\ &+\sum_{\substack{i=1\\ n=30h_1+4}}^{\infty}\frac{1}{(\frac{n}{2})^s}+\sum_{\substack{i=1\\ n=30h_1+5}}^{\infty}\frac{1}{(\frac{n}{5})^s}+\sum_{\substack{i=1\\ n=30h_1+6}}^{\infty}\frac{1}{(\frac{5n}{6})^s}+\sum_{\substack{i=1\\ n=30h_1+7}}^{\infty}\frac{1}{n^s}\\ &+\sum_{\substack{i=1\\ n=30h_1+8}}^{\infty}\frac{1}{(\frac{n}{2})^s}+\sum_{\substack{i=1\\ n=30h_1+9}}^{\infty}\frac{1}{(\frac{n}{3})^s}+\sum_{\substack{i=1\\ n=30h_1+10}}^{\infty}\frac{1}{(\frac{7n}{10})^s}+\sum_{\substack{i=1\\ n=30h_1+11}}^{\infty}\frac{1}{n^s}\end{aligned}$$

$$+ \sum_{\substack{i=1 \\ n=30h_1+12}}^{\infty} \frac{1}{(\frac{5n}{6})^s} + \sum_{\substack{i=1 \\ n=30h_1+13}}^{\infty} \frac{1}{n^s} + \sum_{\substack{i=1 \\ n=30h_1+14}}^{\infty} \frac{1}{(\frac{n}{2})^s} + \sum_{\substack{i=1 \\ n=30h_1+15}}^{\infty} \frac{1}{(\frac{7n}{15})^s}$$
$$+ \sum_{\substack{i=1 \\ n=30h_1+16}}^{\infty} \frac{1}{(\frac{n}{2})^s} + \sum_{\substack{i=1 \\ n=30h_1+17}}^{\infty} \frac{1}{n^s} + \sum_{\substack{i=1 \\ n=30h_1+18}}^{\infty} \frac{1}{(\frac{5n}{6})^s} + \sum_{\substack{i=1 \\ n=30h_1+19}}^{\infty} \frac{1}{n^s}$$
$$+ \sum_{\substack{i=1 \\ n=30h_1+20}}^{\infty} \frac{1}{(\frac{7n}{10})^s} + \sum_{\substack{i=1 \\ n=30h_1+21}}^{\infty} \frac{1}{(\frac{n}{3})^s} + \sum_{\substack{i=1 \\ n=30h_1+22}}^{\infty} \frac{1}{(\frac{n}{2})^s} + \sum_{\substack{i=1 \\ n=30h_1+23}}^{\infty} \frac{1}{n^s}$$
$$+ \sum_{\substack{i=1 \\ n=30h_1+24}}^{\infty} \frac{1}{(\frac{5n}{6})^s} + \sum_{\substack{i=1 \\ n=30h_1+25}}^{\infty} \frac{1}{(\frac{n}{5})^s} + \sum_{\substack{i=1 \\ n=30h_1+26}}^{\infty} \frac{1}{(\frac{n}{2})^s} + \sum_{\substack{i=1 \\ n=30h_1+27}}^{\infty} \frac{1}{(\frac{n}{3})^s}$$
$$+ \sum_{\substack{i=1 \\ n=30h_1+28}}^{\infty} \frac{1}{(\frac{n}{2})^s} + \sum_{\substack{i=1 \\ n=30h_1+29}}^{\infty} \frac{1}{n^s},$$

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{1}{U_4^s}(n) &= 1 + (1-\frac{1}{3^s})(1-\frac{1}{5^s}) + (1-\frac{1}{2^s})(1-\frac{1}{3^s})(1-\frac{1}{5^s}) + \frac{1}{5^s}(1-\frac{1}{5^s}) \\
&\quad +(1-\frac{1}{2^s})(1-\frac{1}{5^s}) + \frac{1}{7^s}(1-\frac{1}{3^s}) + \frac{1}{7^s}(1-\frac{1}{2^s}) + (1-\frac{1}{2^s})(1-\frac{1}{3^s}) \\
&= 1 + (2-\frac{1}{2^s}-\frac{1}{3^s}+\frac{1}{5^s})(1-\frac{1}{5^s}) + \frac{1}{7^s}(2-\frac{1}{2^s}-\frac{1}{3^s}) + (1-\frac{1}{2^s})(1-\frac{1}{3^s})(2-\frac{1}{5^s}).
\end{aligned}$$

This completes the proof of Theorem 1.

Using the same method, we can completes the proof of Theorem 2. In addition, by Theorem 1 and Lemma 1, and for any fixed integer n, we can obtain the calculating of the infinite series $\sum_{i=1}^{\infty} U_t(n)$.

References

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