SOME SMARANDACHE-TYPE MULTIPLICATIVE FUNCTIONS

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This note considers eleven particular families of interrelated multiplicative functions, many of which are listed in Smarandache's problems.

These are multiplicative in the sense that a function f(n) has the property that for any two coprime positive integers a and b, i.e. with a highest common factor (also known as greatest common divisor) of 1, then f(a*b)=f(a)*f(b). It immediately follows that f(1)=1 unless all other values of f(n) are 0. An example is d(n), the number of divisors of n. This multiplicative property allows such functions to be uniquely defined on the positive integers by describing the values for positive integer powers of primes. $d(p^i)=i+1$ if i>0; so $d(60) = d(2^2*3^1*5^1) = (2+1)*(1+1)*(1+1) = 12$.

Unlike d(n), the sequences described below are a restricted subset of all multiplicative functions. In all of the cases considered here, $f(p^i)=p^{g(i)}$ for some function g which does not depend on p.

	Definition	**Multiplicative with p^i->p^...**
$A_m(n)$	Number of solutions to $x^m \equiv 0 \pmod n$	i-ceiling[i/m]
$B_m(n)$	Largest m^{th} power dividing n	m*floor[i/m]
$C_m(n)$	m^{th} root of largest mth power dividing n	floor[i/m]
$D_m(n)$	m^{th} power-free part of n	i-m*floor[i/m]
$E_m(n)$	Smallest number x (x>0) such that n*x is a perfect m^{th} power (Smarandache m^{th} power complements)	m*ceiling[i/m]-i
$F_m(n)$	Smallest m^{th} power divisible by n divided by largest m^{th} power which divides n	m*(ceiling[i/m]-floor[i/m])
$G_m(n)$	m^{th} root of smallest m^{th} power divisible by n divided by largest m^{th} power which divides n	ceiling[i/m]-floor[i/m]
$H_m(n)$	Smallest m^{th} power divisible by n (Smarandache ^m function (numbers))	m*ceiling[i/m]
$J_m(n)$	m^{th} root of smallest mth power divisible by n (Smarandache Ceil Function of m^{th} Order)	ceiling[i/m]
$K_m(n)$	Largest m^{th} power-free number dividing n	min(i,m-1)

	(Smarandache m^{th} power residues)	
$L_m(n)$	n divided by largest m^{th} power-free number dividing n	max(0,i-m+1)

Relationships between the functions

Some of these definitions may appear to be similar; for example $D_m(n)$ and $K_m(n)$, but for m>2 all of the functions are distinct (for some values of n given m). The following relationships follow immediately from the definitions:

$$B_m(n)=C_m(n)^m \quad (1)$$

$$n=B_m(n)*D_m(n) \quad (2)$$

$$F_m(n)=D_m(n)*E_m(n) \quad (3)$$

$$F_m(n)=G_m(n)^m \quad (4)$$

$$H_m(n)=n*E_m(n) \quad (5)$$

$$H_m(n)=B_m(n)*F_m(n) \quad (6)$$

$$H_m(n)=J_m(n)^m \quad (7)$$

$$n=K_m(n)*L_m(n) \quad (8)$$

These can also be combined to form new relationships; for example from (1), (4) and (7) we have

$$J_m(n)=C_m(n)*G_m(n) \quad (9)$$

Further relationships can also be derived easily. For example, looking at $A_m(n)$, a number x has the property $x^m \equiv 0 \pmod n$ if and only if x^m is divisible by n or in other words a multiple of $H_m(n)$, i.e. x is a multiple of $J_m(n)$. But $J_m(n)$ divides into n, so we have the pretty result that:

$$n=J_m(n)*A_m(n) \quad (10)$$

We could go on to construct a wide variety of further relationships, such as using (5), (7) and (10) to produce:

$$n^{m-1}=E_m(n)*A_m(n)^m \quad (11)$$

but instead we will note that $C_m(n)$ and $J_m(n)$ are sufficient to produce all of the functions from $A_m(n)$ through to $J_m(n)$:

$$A_m(n)=n/J_m(n) \quad (12)$$

$$B_m(n)=C_m(n)^m$$

$C_m(n)=C_m(n)$

$D_m(n)=n/C_m(n)^m$ (13)

$E_m(n)=J_m(n)^m/n$ (14)

$F_m(n)=(J_m(n)/C_m(n))^m$ (15)

$G_m(n)=J_m(n)/C_m(n)$ (16)

$H_m(n)=J_m(n)^m$

$J_m(n)=J_m(n)$

Clearly we could have done something similar by choosing one element each from two of the sets {A,E,H,J}, {B,C,D}, and {F,G}. The choice of C and J is partly based on the following attractive property which further deepens the multiplicative nature of these functions.

If $m=a*b$ then:

$C_m(n)=C_a(C_b(n))$ (17)

$J_m(n)=J_a(J_b(n))$ (18)

Duplicate functions when m=2 ...

When m=2, $D_2(n)$ is square-free and $F_2(n)$ is the smallest square which is a multiple of $D_2(n)$, so

$$F_2(n)=D_2(n)^2 \quad (19)$$

Using (3) and (4) we then have:

$$D_2(n)=E_2(n)=G_2(n) \quad (20)$$

and from (13) and (14) we have

$$n=C_2(n)*J_2(n) \quad (21)$$

so from (10) we get

$$A_2(n)=C_2(n) \quad (22)$$

... and when m=1

If m=1, all the functions described either produce 1 or n. The analogue of (20) is still true with

$$D_1(n)=E_1(n)=G_1(n)=1 \quad (23)$$

but curiously the analogue of (22) is not, since:

$$A_1(n)=1 \quad (24)$$
$$C_1(n)=n \quad (25)$$

The two remaining functions

All this leaves two slightly different functions to be considered: $K_m(n)$ and $L_m(n)$. They have little connection with the other sequences except for the fact that since $G_m(n)$ is square-free, and divides $D_m(n)$, $E_m(n)$, $F_m(n)$, and $G_m(n)$, none of which have any factor which is a higher power than m, we get:

$$G_m(n)=J_m(D_m(n))=J_m(E_m(n))=J_m(F_m(n))=J_m(G_m(n))=K_2(D_m(n))=K_2(E_m(n))=K_2(F_m(n))=K_2(G_m(n)) \quad (26)$$

and so with (8) and (10)

$$n/G_m(n)=A_m(D_m(n))=A_m(E_m(n))=A_m(F_m(n))=A_m(G_m(n))=L_2(D_m(n))=L_2(E_m(n))=L_2(F_m(n))=L_2(G_m(n)) \quad (27)$$

We also have the related convergence property that for any y, there is a z (e.g. floor[$\log_2(n)$]) for which

$$G_m(n)=J_m(n)=K_2(n) \text{ for any } n<=y \text{ and any } m>z \quad (28)$$
$$A_m(n)=L_2(n) \text{ for any } n<=y \text{ and any } m>z \quad (29)$$

There is a simple relation where

$$L_m(n)=L_a(L_b(n)) \text{ if } m+1=a+b \text{ and } a,b>0 \quad (29)$$

and in particular

$$L_m(n)=L_{m-1}(L_2(n)) \text{ if } m>1 \quad (30)$$
$$L_3(n)=L_2(L_2(n)) \quad (31)$$
$$L_4(n)=L_2(L_2(L_2(n))) \quad (32)$$

so with (8) we also have

$$K_m(n)=K_b(n)*K_a(n/K_b(n)) \text{ if } m+1=a+b \text{ and } a,b>0 \quad (33)$$
$$K_m(n)=K_{m-1}(n)*K_2(n/K_{m-1}(n)) \text{ if } m>1 \quad (34)$$
$$K_3(n)=K_2(n)*K_2(n/K_2(n)) \quad (35)$$

$$K_4(n)=K_2(n)*K_2(n/K_2(n))*K_2(n/(K_2(n)*K_2(n/K_2(n))))\quad (36)$$

Recording the functions

The values of all these functions for n up from n=1 to about 70 or more are listed in Neil Sloane's Online Encyclopedia of Integer Sequences for m=2, 3 and 4:

	m=1	m=2	m=3	m=4	m>=x and n<2^x
$A_m(n)$	1	A000188	A000189	A000190	$L_2(n)$
$B_m(n)$	n	A008833	A008834	A008835	1
$C_m(n)$	n	A000188	A053150	A053164	1
$D_m(n)$	1	A007913	A050985	A053165	n
$E_m(n)$	1	A007913	A048798	A056555	$K_2(n)^m/n$
$F_m(n)$	1	A055491	A056551	A056553	$K_2(n)^m$
$G_m(n)$	1	A007913	A056552	A056554	$K_2(n)$
$H_m(n)$	n	A053143	A053149	A053167	$K_2(n)^m$
$J_m(n)$	n	A019554	A019555	A053166	$K_2(n)$
$K_m(n)$	1	A007947	A007948	A058035	n
$L_m(n)$	n	A003557	A062378	A062379	1

Further reading

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Multiplicative is not used here in the same sense as in S Tabirca, About Smarandache-Multiplicative Functions, American Research Press.