Some works on the Generalized Andrica's Conjectures and Variants

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The Andrica's conjecture affirm that for every prime number $\sqrt[2]{p_{n+1}} - \sqrt[2]{p_n} < 1$ (I.1) and more precisely, that the upper bound of this function is reached for $p_n = 7$. Another conjecture, called generalized Andrica's conjecture, affirm that, for every prime number, there is an only real number x, such as : $(p_{n+1})^x - (p_n)^x = 1$ (I.2), and the minimum value of $x=x(p_n)$ is obtained for $p_n = 113$. Florentin Smarandache in 2000 [8] , has listed a set of conjectures linked to Andrica's conjecture such as $(p_{n+1})^{1/k} - (p_n)^{1/k} < 1/k$ (I.3) for all p_n and all k>2. David Lowry-Duda in 2017 [4], has formulated the following open question : For any constant C, what is the smallest x such as $(p_{n+1})^x - (p_n)^x = C$ (I.4). In 2018 Matt Wisser [1],[2] from Victoria University of Wellington has proposed a variant of Andrica's conjecture and presents an original approach to prove that $Ln^k(p_{n+1}) - Ln^k(p_n)$ has an upper bound and gives some numerical values for $k \leq 5$.

In the present work, we are going to establish a set of results regarding :

Part 1 : the generalized Andrica's conjecture : $F_n(x) = (p_{n+1})^x - (p_n)^x = 1$.

We shall build some lower and upper bound functions $X^-(p_n)$ et $X^+(p_n)$ such as, for all p_n , $X^-(p_n) \leq x(p_n) \leq X^+(p_n)$.

We shall study the behavior of these functions $X^-(p_n)$ et $X^+(p_n)$ and show under a weaker condition than Cramer's conjecture that x is minimal for $p_n = 113$.

We'll give the proof that $(p_{n+1})^{1/k} - (p_n)^{1/k} < 1/k$ for all k > 2.

We'll show that all x < 1, $F_x(p_n) = (p_{n+1})^x - (p_n)^x$ has an upper bound for all $x \in [0,0.475[$ and Cramer's conjecture for all $x \in [0.475,1[$.

Part 2 : We propose an alternative approach to prove that for all integer k, $L_k(p_n) = Ln^k(p_{n+1}) - Ln^k(p_n)$ has an upper bound and we'll build some lower and upper bound functions $L_k^-(p_n)$ and $L_k^+(p_n)$. We'll try to estimate the integer sequence $(p_{n,k})$ such as $Max_n (L_k(p_n)) = L_k(p_{n,k})$. We'll propose some upper bound values of $L_k(p_n)$ for all k<25. We'll deduce that for all integers k>0 and r>0, $G_{k,r}(p_n) = Ln^k(p_{n+1})^r - Ln^k(p_n)^r$ has an upper bound value. We'll prove that $L_y(p_n) = Ln^y(p_{n+1}) - Ln^y(p_n)=1$ has a solution $y(p_n) \in \mathbb{R}$ for all p_n and we will propose an approximation of $y(p_n)$.

Part 3 ; We'll generalize the different results obtained in part 1 and 2 to establish that the following functions on prime number accept upper values :

$H_{k,x}(p_n) = (p_{n+1})^x.Ln^k(p_{n+1}) - (p_n)^x.Ln^k(p_n)$ for all integer k>0 and real $0 \leq x < 1$

$H_{x,y}(p_n) = (p_{n+1})^x.Ln^y(p_{n+1}) - (p_n)^x.Ln^y(p_n)$ for all real numbers (x,y) such as $0 \leq x < 1$and y>0

We'll study the relation between x and y which verify $H_{x,y}(p_n) = (p_{n+1})^x.Ln^y(p_{n+1}) - (p_n)^x.Ln^y(p_n)=1$

Part 1 : The Generalised Andrica's conjecture

I.1 - Study of the equation $(p_{n+1})^x - (p_n)^x = 1$

We define the function $F_n(x) = p_{n+1}^{\,x} - p_n^{\,x}$ where $x \in \mathbb{R}$ and p_n is the n^{st} prime number and $g_n = p_{n+1} - p_n$ the gap between the two consecutive primes p_n and p_{n+1}.

We can express $F_n(x)$ as following

$$F_n(x) = (p_n + g_n)^x - p_n^{\,x} = p_n^{\,x}.(1 + \frac{g_n}{p_n})^x - p_n^{\,x} = p_n^{\,x}.\left((1 + \frac{g_n}{p_n})^x - 1\right)$$

We have obviously , $F_n(0) = 0$ and $F_n(1) = p_{n+1} - p_n > 1$ for all p_n

According to the theorem of intermediate values, as $F_n(x)$ is a continuous function in [0,1], there is at least one x such as $F_n(x) = 1$ and $0 < x < 1$. And as $F_n(x)$ is a growing function, x is unique. x is a function of p_n, $x=x(p_n)$, but to simplify the notation in the following, we'll keep the notation x.

We can make a taylor serie expansion of $F_n(x)$

$$F_n(x) = p_n^{\,x}.(1 + \frac{g_n}{p_n})^x - p_n^{\,x} = p_n^{\,x}.(1 + \frac{x.g_n}{p_n} + \frac{x.(x-1)}{2}.\left(\frac{g_n}{p_n}\right)^2 + O\left(\frac{g_n}{p_n}\right)^3) - p_n^{\,x}$$

$$F_n(x) = x\,.\,p_n^{\,x}.\frac{g_n}{p_n} + \frac{x.(x-1)}{2}.p_n^{\,x}.\left(\frac{g_n}{p_n}\right)^2 + p_n^{\,x}.O\left(\frac{g_n}{p_n}\right)^3) \qquad \text{(I.5)}$$

As $0 < x < 1$, $x-1 < 0$ and $\frac{x.(x-1)}{2}.p_n^{\,x}.\left(\frac{g_n}{p_n}\right)^2 < 0$ and $p_n^{\,x}.O\left(\frac{g_n}{p_n}\right)^3 > 0$

So we obtain a upper bound of $F_n(x)$ given by : $M_n(x) = x\,.\,p_n^{\,x}.\frac{g_n}{p_n}$

We are going to find x such as $\qquad M_n(x) = x\,.\,p_n^{\,x}.\frac{g_n}{p_n} = 1 \qquad$ (I.6)

which gives the equation $\qquad x\,.\,p_n^{\,x} = \frac{p_n}{g_n} \qquad$ (I.7)

Before trying to estimate x, we are going first to study the function $F_{1/4}(x) = p_{n+1}^{\,1/4} - p_n^{\,1/4}$

$$F_{1/4}(x) = p_{n+1}^{\,1/4} - p_n^{\,1/4} = \frac{p_{n+1}^{\,1/2} - p_n^{\,1/2}}{p_{n+1}^{\,1/4} + p_n^{\,1/4}} = \frac{p_{n+1} - p_n}{(p_{n+1}^{\,1/4} + p_n^{\,1/4}).(p_{n+1}^{\,1/2} + p_n^{\,1/2})} < \frac{p_{n+1} - p_n}{4.p_n^{\,3/4}} = \frac{g_n}{4.p_n^{\,3/4}}$$

According to the best known result due to Baker, Harman and Pintz (BHP), stating that for all $p_n > 113$, $g_n << (p_n)^a$ with a=0,525

$$F_{1/4}(x) = \frac{g_n}{4.p_n^{\,3/4}} < \frac{p_n^{\,0,525}}{4.p_n^{\,3/4}} = \frac{p_n^{\,21/40}}{4.p_n^{\,3/4}} = \frac{1}{4.p_n^{\,9/40}} < \frac{1}{4}$$

So we can limit our approach to the values of $x \in [\frac{1}{4}, 1]$ and we can write :

$$\frac{1}{4}p_n^{\,x} < x.p_n^{\,x} < p_n^{\,x}$$

We replace by (I.7) : $\qquad \frac{1}{4}p_n^{\,x} < \frac{p_n}{g_n} < p_n^{\,x} \qquad$ (I.8)

Which gives two inequalities to estimate x :

$$\frac{p_n}{g_n} < p_n^{\,x} \Leftrightarrow x > \frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln(p_n)} = 1 - \frac{\ln(g_n)}{\ln(p_n)} = x_o \quad \text{and} \quad \frac{p_n}{g_n} > \frac{1}{4}.p_n^{\,x} \Leftrightarrow x < \frac{\ln\left(\frac{4p_n}{g_n}\right)}{\ln(p_n)} = 1 - \frac{\ln(g_n/4)}{\ln(p_n)} = x_1$$

So we obtain a lower x_o and upper x_1 values for x and we are going to show that :

$$F_n(x_o) < 1 < F_n(x_1)$$

$F_n(x_o) < x_o . p_n^{x_o} . \frac{g_n}{p_n} = \frac{\ln(\frac{p_n}{g_n})}{\ln(p_n)} . p_n^{\frac{\ln(\frac{p_n}{g_n})}{\ln(p_n)}} . \frac{g_n}{p_n} = \frac{\ln(\frac{p_n}{g_n})}{\ln(p_n)} < 1$ for all p_n

$F_n(x_1) > x_1 . p_n^{x_1} . \frac{g_n}{p_n} + \frac{x_1.(x_1-1)}{2} . p_n^{x_1} . \left(\frac{g_n}{p_n}\right)^2 = \frac{\ln(\frac{4p_n}{g_n})}{\ln(p_n)} . p_n^{\frac{\ln(\frac{4p_n}{g_n})}{\ln(p_n)}} . \frac{g_n}{p_n} - \frac{\ln\left(\frac{4p_n}{g_n}\right)}{2\ln(p_n)} . \frac{\ln\left(\frac{g_n}{4}\right)}{\ln(p_n)} . p_n^{\frac{\ln\left(\frac{4p_n}{g_n}\right)}{\ln(p_n)}} . \left(\frac{g_n}{p_n}\right)^2$

$F_n(x_1) > \frac{4\ln(\frac{4p_n}{g_n})}{\ln(p_n)} - \frac{2\ln\left(\frac{4p_n}{g_n}\right)}{\ln(p_n)} . \frac{\ln\left(\frac{g_n}{4}\right)}{\ln(p_n)} . \frac{g_n}{p_n} = \frac{4\ln(\frac{4p_n}{g_n})}{\ln(p_n)} . (1 - \frac{g_n.\ln\left(\frac{g_n}{4}\right)}{2p_n.\ln(p_n)}) = 4(1 - \frac{\ln(\frac{g_n}{4})}{\ln(p_n)}) . (1 - \frac{g_n.\ln\left(\frac{g_n}{4}\right)}{2p_n.\ln(p_n)})$

Using the Baker, Harman and Pintz (BHP) result, stating that for all $p_n > 113$, $g_n << (p_n)^a$ with a=0,525

$\frac{\ln(\frac{g_n}{4})}{\ln(p_n)} = \frac{\ln(g_n)-\ln(4)}{\ln(p_n)} < \frac{0.525.\ln(p_n)-\ln(4)}{\ln(p_n)} < \frac{21}{40}$ and for $p_n \geq 2$ $\frac{g_n.\ln\left(\frac{g_n}{4}\right)}{2p_n.\ln(p_n)} < \frac{a.p_n^a}{2p_n} < \frac{0.2625}{p_n^{0.475}} < \frac{1}{5}$

In these conditions : $F_n(x_1) > 4.(1 - \frac{21}{40}).(1 - \frac{1}{5}) = 1.52 > 1$ for all $p_n > 113$

So we have proven that : $F_n(x_o) < 1 < F_n(x_1)$ for all p_n for all $p_n > 113$ (I.9)

We are going to build a sequence (x_i) such as $x_{i+1} < x_i$ and find an integer k such as $F(x_{k+1}) < 1 < F(x_k)$

We define $x_2 = \frac{x_o + x_1}{2} = \frac{\ln\left(\frac{p_n}{g_n}\right)+\ln\left(\frac{4p_n}{g_n}\right)}{2\ln(p_n)} = \frac{\ln\left(4.\left(\frac{p_n}{g_n}\right)^2\right)}{2\ln(p_n)} = \frac{\ln\left(2.\frac{p_n}{g_n}\right)}{\ln(p_n)}$

We define $x_3 = \frac{x_o + x_2}{2} = \frac{\ln\left(\frac{p_n}{g_n}\right)+\ln\left(\frac{2p_n}{g_n}\right)}{2\ln(p_n)} = \frac{\ln\left(2.\left(\frac{p_n}{g_n}\right)^2\right)}{2\ln(p_n)} = \frac{\ln\left(\sqrt{2}.\frac{p_n}{g_n}\right)}{\ln(p_n)}$

And we can generalize to x_{k+2} defined by

$$x_{k+2} = \frac{x_o + x_{k+1}}{2} = \frac{\ln\left(\frac{p_n}{g_n}\right)+\ln\left(\frac{2^{1/2^{k-1}}.p_n}{g_n}\right)}{2\ln(p_n)} = \frac{\ln\left(2^{1/2^{k-1}}.\left(\frac{p_n}{g_n}\right)^2\right)}{2\ln(p_n)} = \frac{\ln\left(2^{1/2^k}.\frac{p_n}{g_n}\right)}{\ln(p_n)}$$

We must find k such as $F(x_{k+2}) = 1$

As $2^{1/2^k} \to 1^+$ *when* $k \to \infty$, we can replace $2^{1/2^k}$ by 1+ε to simplify our analysis :

$F_n(x_{k+2}) < x_{k+2}. p_n^{x_{k+2}}. \frac{g_n}{p_n} = \frac{\ln\left((1+\varepsilon).\frac{p_n}{g_n}\right)}{\ln(p_n)} . p_n^{\frac{\ln((1+\varepsilon).\frac{p_n}{g_n})}{\ln(p_n)}} . \frac{g_n}{p_n} = (1+\varepsilon). \frac{\ln\left((1+\varepsilon).\frac{p_n}{g_n}\right)}{\ln(p_n)}$

To solve the equation $F_n(x_{k+2}) \approx 1$ we have to solve the following equation :

$$(1+\varepsilon).\left(\ln(1+\varepsilon) + \ln\left(\frac{p_n}{g_n}\right)\right) = \ln(p_n) \qquad \text{(I.10)}$$

For k high, ε is small and we can make a taylor serie expansion of $\ln(1+\varepsilon) = \varepsilon + O(\varepsilon^2)$

$$(1+\varepsilon).\left(\varepsilon + O(\varepsilon^2) + \ln\left(\frac{p_n}{g_n}\right)\right) = \ln(p_n)$$

We can develop to obtain an estimation of ε

$\varepsilon + O(\varepsilon^2) + \ln\left(\frac{p_n}{g_n}\right) + \varepsilon.\ln\left(\frac{p_n}{g_n}\right) = \ln(p_n)$ so $\varepsilon \approx \frac{\ln(g_n)}{\ln\left(\frac{e.p_n}{g_n}\right)}$ and $1+\varepsilon \approx \frac{\ln(e.p_n)}{\ln\left(\frac{e.p_n}{g_n}\right)}$

Finally we obtain a first evaluation of x, noted X^+ :

$$X^+(p_n) \approx \frac{\ln\left(\frac{p_n}{g_n}\cdot\frac{\ln(e.p_n)}{\ln\left(\frac{e.p_n}{g_n}\right)}\right)}{\ln(p_n)} = \frac{\ln\left(\frac{p_n}{g_n}\cdot\left(1-\frac{\ln(g_n)}{\ln\left(\frac{e.p_n}{g_n}\right)}\right)\right)}{\ln(p_n)} = 1 - \frac{\ln(g_n)}{\ln(p_n)} - \frac{\ln\left(1-\frac{\ln(g_n)}{\ln(e.p_n)}\right)}{\ln(p_n)} \quad \text{(I.11)}$$

Using a Taylor serie expansion, we can write (I.11) as follows :

$$X^+(p_n) = 1 - \frac{\ln(g_n)}{\ln(p_n)} + \frac{1}{\ln(p_n)}.\left(\left(\frac{\ln(g_n)}{\ln(e.p_n)}\right) - \frac{1}{2}\left(\frac{\ln(g_n)}{\ln(e.p_n)}\right)^2 + O\left(\frac{\ln(g_n)}{\ln(e.p_n)}\right)^3\right) \quad \text{with } O\left(\frac{\ln(g_n)}{\ln(e.p_n)}\right)^3 > 0 \quad \text{(I.12)}$$

We can extract from this formulation different functions given some lower bounds of x.

$$X^-(p_n) = 1 - \frac{\ln(g_n)}{\ln(e.p_n)}.\left(1 - \frac{\ln(g_n)}{2.\ln(p_n).\ln(e.p_n)}\right) \quad \text{(I.13)}$$

$$X^{2-}(p_n) = 1 - \frac{\ln(g_n)}{\ln(p_n)} + \frac{\ln(g_n)}{\ln(p_n).\ln(e.p_n)} = 1 - \frac{\ln(g_n)}{\ln(e.p_n)} \quad \text{(I.14)}$$

$X^{3-}(p_n) = 1 - \frac{\ln(g_n)}{\ln(p_n)}$ which correspond to x_o where we have already prove that $F(x_o) < 1$

So we are going to demonstrate that $F(X^{2-}) < 1$ for all pn.

$$F_n(X^{2-}) < X^{2-}.p_n^{X^{2-}}.\frac{g_n}{p_n} = \left(1 - \frac{\ln(g_n)}{\ln(e.p_n)}\right).p_n^{\left(1-\frac{\ln(g_n)}{\ln(e.p_n)}\right)}.\frac{g_n}{p_n} = \left(1 - \frac{\ln(g_n)}{\ln(e.p_n)}\right).p_n^{-\frac{\ln(g_n)}{\ln(e.p_n)}}.g_n$$

$$F_n(X^{2-}) < \left(1 - \frac{\ln(g_n)}{\ln(e.p_n)}\right).e^{-\frac{\ln(g_n).\ln(p_n)}{\ln(e.p_n)}}.e^{\ln(g_n)} = \left(1 - \frac{\ln(g_n)}{\ln(e.p_n)}\right).e^{\frac{\ln(g_n)}{\ln(e.p_n)}}$$

We have only to show that for every positive y such a $0 \le y < 1$, the function $g(y) = (1- y).e^y < 1$
This is obvious because $g(1)=0$, $g(0)= 1$ and $g'(y) = -y.\ e^y < 0$ for all $y > 0$.

So if we take $y = \frac{\ln(g_n)}{\ln(e.p_n)}$, then $F_n(X^{2-}) < 1$ and $X^{2-}(p_n)$ gives a good lower bound for x.

We present below in table I.1, the numerical values of X^+, $F(X^+)$, X^{2-}, $F(X^{2-})$, X^- and $F(X^-)$ for a set of the 300 first prime numbers. We see that the local lower bounds are reach for the prime numbers corresponding to the first occurrence of the maximal gaps. It means that, to find the lower bound of x, we can limit our study to this subset of prime numbers, noted $\mathbb{P}_{G_n}$. Effectively, if we consider $\mathbb{P}_{G_n} = \{p_i \in \mathbb{P} \ / \ g_i = G_n\}$, for all $p_i \in \mathbb{P}_G$, if $P_n = \mathrm{Min}\{p_i \in \mathbb{P}_{G_n}\}$ then $X^{2-}(p_i) > X^{2-}(P_n)$. In table I.2, we presents the numerical values of X^+, $F(X^+)$, X^{2-}, $F(X^{2-})$, X^- and $F(X^-)$ for all prime numbers corresponding to the first occurrence of the maximal gaps $G_n = \mathrm{Max}_{i \le n}(g_i)$, up to $p_n =$ 218 209 405 436 543 corresponding to the maximal gap $G_n = 906$.

Primes	Gaps	$X^-(P_n)$	$F(X^-(P_n))$	$X^{2-}(P_n)$	$F(X^{2-}(P_n))$	$X^+(P_n)$	$F(X^+(P_n))$
2	1	1	1	1	1	1	1
3	2	0,669711655	0,851328799	0,719719798	0,979732227	0,733992458	1,018887569
5	2	0,734369162	0,913957339	0,756327983	0,978907397	0,761156522	0,993683791
7	4	0,529417297	0,757405268	0,58650709	0,950360565	0,614413829	1,058090378
11	2	0,796006903	0,959235065	0,804687007	0,990823929	0,806080038	0,995978635
13	4	0,611132103	0,854182017	0,640641635	0,96977175	0,65151272	1,01563719
17	2	0,819173336	0,97155371	0,824944888	0,994853341	0,825749936	0,99814351
19	4	0,648544605	0,890454187	0,669532525	0,979875239	0,67624737	1,01012675
23	6	0,566736314	0,829998067	0,59669363	0,963347411	0,609663343	1,02671729
29	2	0,841286872	0,980732053	0,845027558	0,997705074	0,845477012	0,999762777
31	6	0,595903329	0,860562023	0,61969188	0,973172523	0,628978628	1,020617914
37	4	0,699345252	0,929961567	0,711865052	0,991030592	0,715118524	1,00749506
41	2	0,852946519	0,98458737	0,855858261	0,998758836	0,856179242	1,000332774
43	4	0,708835099	0,936055362	0,720107404	0,99262813	0,722917695	1,007218142
47	6	0,630576301	0,892033379	0,648305109	0,982959573	0,654392898	1,016096755
53	6	0,63950619	0,899263623	0,65587674	0,985093601	0,661309415	1,015208888
59	2	0,863487529	0,987527879	0,865772774	0,999449078	0,866004576	1,000665917
61	6	0,649422092	0,90687384	0,66437439	0,98726838	0,669148918	1,014302542
67	4	0,733645296	0,950124554	0,742082767	0,99600428	0,74396113	1,006498093
71	2	0,868290073	0,988707403	0,870324947	0,999686399	0,870523295	1,000762774
73	6	0,66132252	0,915432903	0,674692296	0,989606412	0,678764977	1,013293769
79	4	0,741818079	0,954191216	0,749446632	0,996866919	0,751079964	1,006235718
83	6	0,669346342	0,920856432	0,681719573	0,991016748	0,685368643	1,012646942
89	8	0,621136945	0,892404769	0,63712936	0,984190579	0,642822962	1,018941877
97	4	0,751324442	0,958585794	0,758083885	0,997722624	0,759465317	1,005905712
101	2	0,876557025	0,990520698	0,878207956	0,999992617	0,878357746	1,0008563
103	4	0,753973197	0,959747982	0,760503701	0,997933869	0,761820831	1,005808709
107	2	0,877812781	0,990773432	0,879410316	1,000028695	0,879553637	1,000863072
109	4	0,75642073	0,960798364	0,762744664	0,998118895	0,764004535	1,005717102
113	**14**	**0,539221472**	**0,831800413**	**0,561683725**	**0,964797943**	**0,57240055**	**1,034955938**
127	4	0,7627909	0,963428556	0,768599099	0,998556077	0,769719252	1,005469944
131	6	0,695029908	0,936409558	0,704569735	0,994667638	0,707097175	1,010669337
137	2	0,88291395	0,991742404	0,884307184	1,000149681	0,884426453	1,000872585
139	10	0,611998449	0,889916672	0,627255531	0,983946511	0,632876861	1,020888288
149	2	0,884551402	0,992034558	0,885883209	1,000180362	0,885995474	1,000869959
151	6	0,702230988	0,940299	0,711067952	0,995469009	0,71333738	1,010126917
157	6	0,704146838	0,941300687	0,712803196	0,995666927	0,715007704	1,00998303
163	4	0,772505548	0,96716211	0,777585945	0,999106284	0,778517158	1,005070231
167	6	0,707132841	0,942834691	0,715512963	0,995962963	0,717619289	1,009759075
173	6	0,708812846	0,943683353	0,717040334	0,996122975	0,719092994	1,009633229
179	2	0,887974146	0,992617131	0,889183806	1,000232819	0,889282473	1,000856487
181	10	0,628525257	0,902158096	0,64179952	0,987379747	0,646396638	1,018630613
191	2	0,889136777	0,992806655	0,890306823	1,000247261	0,890401178	1,000849636
193	4	0,778642353	0,969356219	0,783297911	0,999386971	0,784123543	1,004805318
197	2	0,889682523	0,992894198	0,890834295	1,000253482	0,890926677	1,000846049
199	12	0,605150756	0,887737822	0,619879636	0,983504692	0,625445908	1,022176909
211	12	0,608790594	0,89050073	0,623090885	0,984302072	0,628423311	1,021616198
223	4	0,783633964	0,971051668	0,787963048	0,999580332	0,788710039	1,004584036
227	2	0,892116331	0,993273821	0,89318906	1,000277083	0,893273037	1,000827337
229	4	0,784526848	0,971346766	0,788799294	0,999611799	0,78953287	1,004543987
233	6	0,722252552	0,950110752	0,729329077	0,997241903	0,730990879	1,008630809
239	2	0,892974434	0,993403562	0,894020235	1,000283863	0,894101395	1,000819744
241	10	0,644925644	0,913196615	0,65642027	0,990190948	0,660157912	1,016532036
251	6	0,725419908	0,951535051	0,732242786	0,997466235	0,733821846	1,008395949
257	6	0,726410345	0,951973595	0,733155245	0,997533484	0,734709123	1,008322648
263	6	0,727371048	0,952395895	0,734040915	0,99759742	0,735570677	1,008251616
269	2	0,894893477	0,993686211	0,895880792	1,000296307	0,895955921	1,000801044
271	6	0,728608418	0,952935373	0,735182516	0,997677911	0,73668166	1,008160234
277	4	0,790716984	0,973326565	0,794611078	0,99980549	0,795256827	1,004263202
281	2	0,895584491	0,993785504	0,896551324	1,00029991	0,896624369	1,000793765
283	10	0,653509368	0,918555794	0,664143345	0,991444209	0,667486611	1,015479812
293	14	0,604941746	0,889725659	0,618681633	0,984111231	0,623878645	1,02223408

Table I.1

I.2 - Behaviour of the lower bound function $X^{2-}(p_n)$:

We see in tables 1 and 2 that the minimum value of X^{2-}, x, X^+ are reached for $p_n = 113$. To analyse the behavior of the function $X^{2-}(p_n)$, we compute the derivative. We suppose that $X^{2-}(p)$ is a derivable function of p on $\mathbb{R}$ and g=g(p) is a derivable function of p

$$\frac{dX^{2-}}{dp} = \frac{\ln(g)}{p.\ln(e.p)^2} - \frac{g'}{g..\ln(e.p)} = \frac{\ln(g)}{p.\ln(e.p)^2}.\left(1 - \frac{g'.p.\ln(e.p)}{g.\ln(g)}\right)$$

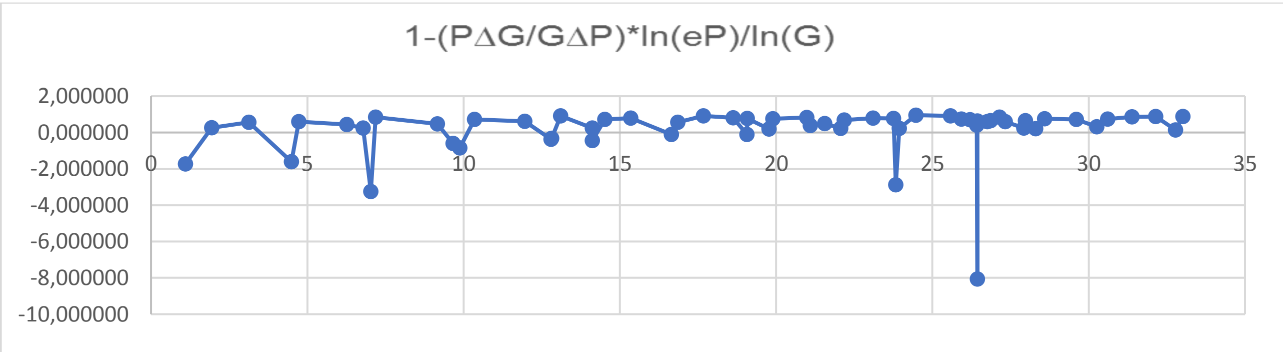
$\frac{\partial X^{2-}}{\partial p} = \frac{\ln(g)}{p.\ln(e.p)^2} > 0$ means, considering an even number $g=g_n$, for all $p_i \in \mathbb{P}_{g_n}$, $X^{2-}(p_i)$ is a growing function for all $p_i \in \mathbb{P}_{g_n}$ and the smallest value of $X^{2-}(p_i)$ is obtained for the smallest $p_i \in \mathbb{P}_{g_n}$, represented by p_{g_n}. If we consider the integer sequence (G_n) defined previously and P_n, the prime number corresponding to the first occurrence of G_n, we can insure that for all $p_i \in \mathbb{P}_{G_n}$, $\mathrm{Min}_{pi \in \mathbb{P}_{G_n}}\left(X^{2-}(p_i)\right) = X^{2-}(P_n)$. It means that we can limit the research of the lower value of x to the prime numbers belonging to the integer sequence (P_n). In the following of this work P_n will systematically represent the prime number associated to the first occurrence of the maximal gap G_n .

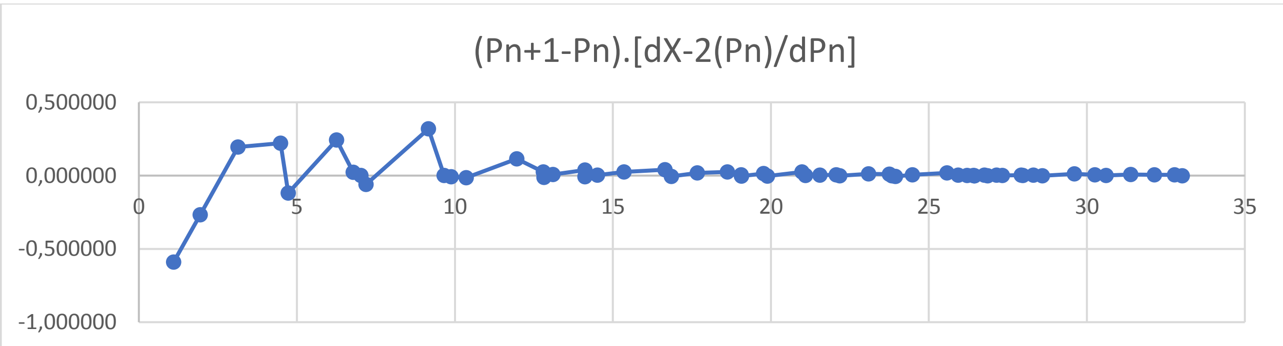
$\frac{\partial X^{2-}}{\partial g} = -\frac{g'}{g.\ln(e.p)}$ can be positive or negative depending on the values of two consecutive prime gaps. If $g_{n+1} \leq g_n$ then $g' \approx g_{n+1} - g_n \leq 0$ and $\frac{\partial X^{2-}}{\partial g} > 0$ which means that $X^{2-}(p_n) \leq X^{2-}(p_{n+1})$. But in contrary, if $g_{n+1} > g_n$ then we'll not be able to conclude. It means that we have only to look at the situation where g_n is strictly growing. It is exactly what's happen in the integer sequence (G_n).

In reality, we must find the first prime number P_n such as :

$$1 - \frac{G'_{n-1}.P_{n-1}.\ln(e.P_{n-1})}{G_{n-1}.\ln(G_{n-1})} \geq 0 \quad \text{and} \quad 1 - \frac{G'_{n-1}.P_{n-1}.\ln(e.P_{n-1})}{G_{n-1}.\ln(G_{n-1})} < 0 \qquad (I.15)$$

If we look at in table I.2 and the two graphics below, the first prime P_n which satisfy (I.15) is P_n=113 for which $X^{2-}(P_n) = 0{,}539221472$ is the minimal value of X^{2-}.





On these two graphics, the horizontal axis correspond to the values of $\ln(P_n)$. We can see that the first P_n for which $\frac{dX^{2-}}{dp} = 0$ and positive before and negative after is for $4 < \ln(P_n) < 5$ which correspond to $P_n = 113$ is table I.2.

We give below the behavior of the lower and upper bound functions of X^+, $F(X^+)$, X^{2-}, $F(X^{2-})$, X^- and $F(X^-)$ for all p_n corresponding to the first occurrence of G_n.

Primes	Gaps	$X^{2-}(P_n)$	$F(X^{2-}(P_n))$	$X^-(P_n)$	$F(X^-(P_n))$	$X^+(P_n)$	$F(X^+(P_n))$
2	1	1,000000	1,000000	1,000000	1,000000	1,000000	1,000000
3	2	0,669712	0,851329	0,719720	0,979732	0,733992	1,018888
7	4	0,529417	0,757405	0,586507	0,950361	0,614414	1,058090
23	6	0,566736	0,829998	0,596694	0,963347	0,609663	1,026717
89	8	0,621137	0,892405	0,637129	0,984191	0,642823	1,018942
113	14	0,539221	0,831800	0,561684	0,964798	0,572401	1,034956
523	18	0,601854	0,890154	0,614518	0,984072	0,619362	1,022454
887	20	0,615332	0,900121	0,626233	0,986536	0,630201	1,019937
1 129	22	0,615019	0,900464	0,625563	0,986457	0,629406	1,019729
1 327	34	0,569466	0,871111	0,582356	0,977503	0,587896	1,026978
9 551	36	0,647444	0,920507	0,654226	0,989804	0,656410	1,013193
15 683	44	0,645021	0,919440	0,651544	0,989145	0,653664	1,012908
19 609	52	0,636959	0,915312	0,643627	0,987907	0,645864	1,013504
31 397	72	0,623349	0,908074	0,630200	0,985549	0,632621	1,014446
155 921	86	0,656224	0,925360	0,661166	0,989084	0,662703	1,009780
360 653	96	0,669146	0,931515	0,673424	0,990210	0,674687	1,008237
370 261	112	0,658623	0,926558	0,663168	0,988930	0,664568	1,008975
492 113	114	0,664253	0,929244	0,668554	0,989489	0,669849	1,008386
1 349 533	118	0,684380	0,938347	0,687908	0,991353	0,688887	1,006572
1 357 201	132	0,677083	0,935139	0,680776	0,990561	0,681831	1,006997
2 010 733	148	0,677890	0,935502	0,681465	0,990510	0,682483	1,006763
4 652 353	154	0,691984	0,941594	0,695074	0,991745	0,695903	1,005659
17 051 707	180	0,705811	0,947223	0,708410	0,992755	0,709067	1,004612
20 831 323	210	0,700475	0,945092	0,703137	0,992208	0,703826	1,004782
47 326 693	220	0,711147	0,949306	0,713508	0,993033	0,714091	1,004139
122 164 747	222	0,724647	0,954356	0,726682	0,994015	0,727155	1,003462
189 695 659	234	0,728062	0,955585	0,730002	0,994222	0,730446	1,003274
191 912 783	248	0,725325	0,954602	0,727303	0,994004	0,727761	1,003358
387 096 133	250	0,734215	0,957750	0,736001	0,994597	0,736398	1,002973
436 273 009	282	0,729972	0,956264	0,731805	0,994260	0,732220	1,003079
1 294 268 491	288	0,742373	0,960524	0,743954	0,995050	0,744292	1,002586
1 453 168 141	292	0,743099	0,960765	0,744663	0,995089	0,744996	1,002553
2 300 942 549	320	0,744273	0,961154	0,745790	0,995126	0,746111	1,002469
3 842 610 773	336	0,747843	0,962323	0,749284	0,995322	0,749583	1,002327
4 302 407 359	354	0,746821	0,961991	0,748266	0,995243	0,748568	1,002339
10 726 904 659	382	0,753261	0,964057	0,754579	0,995594	0,754846	1,002104
20 678 048 297	384	0,759593	0,966025	0,760809	0,995941	0,761048	1,001916
22 367 084 959	394	0,759318	0,965941	0,760533	0,995919	0,760772	1,001915
25 056 082 087	456	0,754554	0,964464	0,755812	0,995615	0,756065	1,002003
42 652 618 343	464	0,758997	0,965843	0,760183	0,995857	0,760417	1,001871
127 976 334 671	468	0,768638	0,968726	0,769685	0,996357	0,769880	1,001616
182 226 896 239	474	0,771201	0,969468	0,772211	0,996481	0,772397	1,001551
241 160 624 143	486	0,772639	0,969882	0,773625	0,996546	0,773806	1,001512
297 501 075 799	490	0,774081	0,970290	0,775047	0,996613	0,775222	1,001477
303 371 455 241	500	0,773505	0,970130	0,774476	0,996584	0,774653	1,001485
304 599 508 537	514	0,772532	0,969850	0,773511	0,996529	0,773690	1,001499
416 608 695 821	516	0,774959	0,970536	0,775905	0,996645	0,776077	1,001441
461 690 510 011	532	0,774693	0,970460	0,775638	0,996628	0,775809	1,001441
614 487 453 523	534	0,776848	0,971073	0,777765	0,996728	0,777930	1,001394
738 832 927 927	540	0,777905	0,971364	0,778808	0,996770	0,778969	1,001364
1 346 294 310 749	582	0,779923	0,971921	0,780790	0,996847	0,780943	1,001308
1 408 695 493 609	588	0,779913	0,971918	0,780779	0,996853	0,780931	1,001312
1 968 188 556 461	602	0,781622	0,972376	0,782464	0,996921	0,782611	1,001257
2 614 941 710 599	652	0,781022	0,972208	0,781861	0,996888	0,782008	1,001265
7 177 162 611 713	674	0,787163	0,973907	0,787928	0,997128	0,788057	1,001148
13 829 048 559 701	716	0,789695	0,974480	0,790426	0,997223	0,790547	1,001087
19 581 334 192 423	766	0,789873	0,974651	0,790595	0,997211	0,790715	1,001003
42 842 283 925 351	778	0,794473	0,976006	0,795146	0,997574	0,795255	1,001007
90 874 329 411 493	804	0,798144	0,976349	0,798778	0,997467	0,798879	1,000671
171 231 342 420 521	806	0,801857	0,977203	0,802456	0,996582	0,802549	1,000580
218 209 405 436 543	906	0,799831	0,976318	0,800438	0,997131	0,800533	1,000244

Table I.2

We are going to try to estimate the minimum value of $X^{2-}(p_n)$ using different upper bound of g_n and/or G_n.

I.2.1 - Baker, Harman and Pintz result

The best known result due to Baker, Harman and Pintz giving an upper bound value of g_n : $g_n < G_n << (p_n)^a$ with $a = 0{,}525$

$$X^{2-}_{BHP}(p_n) == 1 - \frac{\ln(g_n)}{\ln(e.p_n)} > 1 - \frac{\ln({p_n}^a)}{\ln(e.p_n)} = \frac{(1-a)\ln(p_n)+1}{\ln(p_n)+1} > 1 - a = 0{,}475$$

This approach can only let us conclude that x is always higher than 0,475.

I.2.2 – Cramer's conjecture (under RH)

Cramer stated that $g_n \approx O(\ln^2(p_n))$.

It means that for p_n enough high, $g_n < G_n < O(\ln^2(p_n)) < \ln^{2+\varepsilon}(p_n)$ with $\varepsilon>0$

$$X^{2-}_{Crm}(p_n) == 1 - \frac{\ln(g_n)}{\ln(e.p_n)} > 1 - \frac{\ln(\ln^{2+\varepsilon}(p_n))}{\ln(e.p_n)} = 1 - \frac{(2+\varepsilon).\ln(\ln(p_n))}{\ln(e.p_n)} = X_{m_1}(p_n)$$

We are going to calculate the derivative of the function $M_1(p_n)$ using (15)

$$\frac{dX_{m_1}(p_n)}{dp_n} = \frac{(2+\varepsilon).\ln(\ln(p_n))}{p_n.\ln(e.p_n)^2} - \frac{(2+\varepsilon).\ln^2(p_n)}{\ln^3(p_n).\ln(e.p_n)} = \frac{(2+\varepsilon).(\ln(\ln(p_n)).\ln(p_n) - \ln(e.p_n)}{p_n.\ln(p_n).\ln(e.p_n)^2}$$

To find the lower bound of $X_{m_1}(p_n)$, we have to solve the equation $\frac{dX_{m_1}(p_n)}{dp_n} = 0$

$$\ln(\ln(p_n)).\ln(p_n) - \ln(e.p_n) = 0 \qquad (I.16)$$

To simplify (I.16), we replace by $y=\ln(p_n)$. So we have to find Y which verify :

$$\ln(y).y - y - 1 = 0$$

An estimation of Y is given by $\ln(p_n) = y = (e^2 + 2.e)^{1/2} \approx 3.58$ and $g_n \approx \ln^2(p_n) \approx 12.82$

On table I.4 below, we gives the numerical values of $X_{m_1}(p_n)$ for all P_n corresponding to the first occurrence of a maximal gap and with $\varepsilon=0.5$. We see that the minimal value of $X_{m_1}(p_n)$ is reached for $p_n = 23$ which is the closest prime of the solution of (I.16).

For $p_n > 10\,726\,904\,659$, we have $x > X^{2-}(p_n) > m_1(p_n) > 0.6$ and we know that for all p_n up to 10^{15} (as stated in table I.2) the lowest value of x is given for $p_n = 113$ (and $g_n = 14$) , of course under the condition of Cramer's conjecture.

I.2.3 – Weaker hypothesis on maximal gap upper bound

Cramer has conjectured that $g_n < O(\ln^2(p_n))$ under Riemann hypothesis, and this statement hasn't never been contradicted by numerical computation up to very large numbers. If nobody up to now, has succeed on this point, we can think that the probability to find any prime number such as $g_n > \ln^r(p_n)$ for any real r>2, is very low. In these conditions, we are going to show that if there is at least a real r, such that $g_n < \ln^r(p_n)$ for all prime numbers, then we'll prove that we can build a lower bound function $X_{m_r}(p_n)$ of $X^{2-}(p_n)$, which is growing and higher than 0.6 for all p_n enough large.

If we suppose that there is a real r, such that $g_n < \ln^r(p_n)$, we can write :

$$X^{2-}(p_n) == 1 - \frac{\ln(g_n)}{\ln(e.p_n)} > 1 - \frac{\ln(\ln^r(p_n))}{\ln(e.p_n)} = 1 - \frac{r\ln(\ln(p_n))}{\ln(e.p_n)} = X_{m_r}(p_n)$$

We easily see that the functions $X_{m_r}(p_n)$ has a lower bound for $p_n = 23$ for all real r>0 and that the limit of $m_r(p_n)$ when p_n tends toward ∞, is 1. So, there is for each real r, a prime $p_{n,r}$ such as for any prime number p_n higher than $p_{n,r}$, $X^{2-}(p_n) > 0.6$. As it has been calculated on all prime numbers below 218 209 405 436 543 that the lowest value of x has been reached for $p_n = 113$, we can conclude that if there is a real r, such that $g_n < \ln^r(p_n)$, which is a much more weakest hypothesis than Cramer's conjecture, then the generalized Andrica's conjecture will be verified, if and only if we can prove that for $p_n >$ 218 209 405 436 543, $X^{2-}(P_n) > 0{,}6$.

To insure this condition, we have to find $p_n <$ 218 209 405 436 543 such as $X^{2-}(P_n) > X_{m_r}(p_n) > 0{,}6$.

$$1 - \frac{r\ln(\ln(p_n))}{\ln(e.p_n)} > 0{,}6 \quad \Leftrightarrow \quad \frac{r\ln(\ln(p_n))}{\ln(e.p_n)} > 0{,}4$$

Which is satisfy for all $p_n > e^{16{,}533787} \approx$ 41 193 733 which is coherent the numerical values of $X_{m_r}(p_n)$ in table 3. This finish to establish that under the $g_n < \ln^r(p_n)$ with r=2.5, the minimum value of x is reached for $p_n = 113$.

I.2.4 – Upper bound of x thanks to Lower bound on maximal gaps

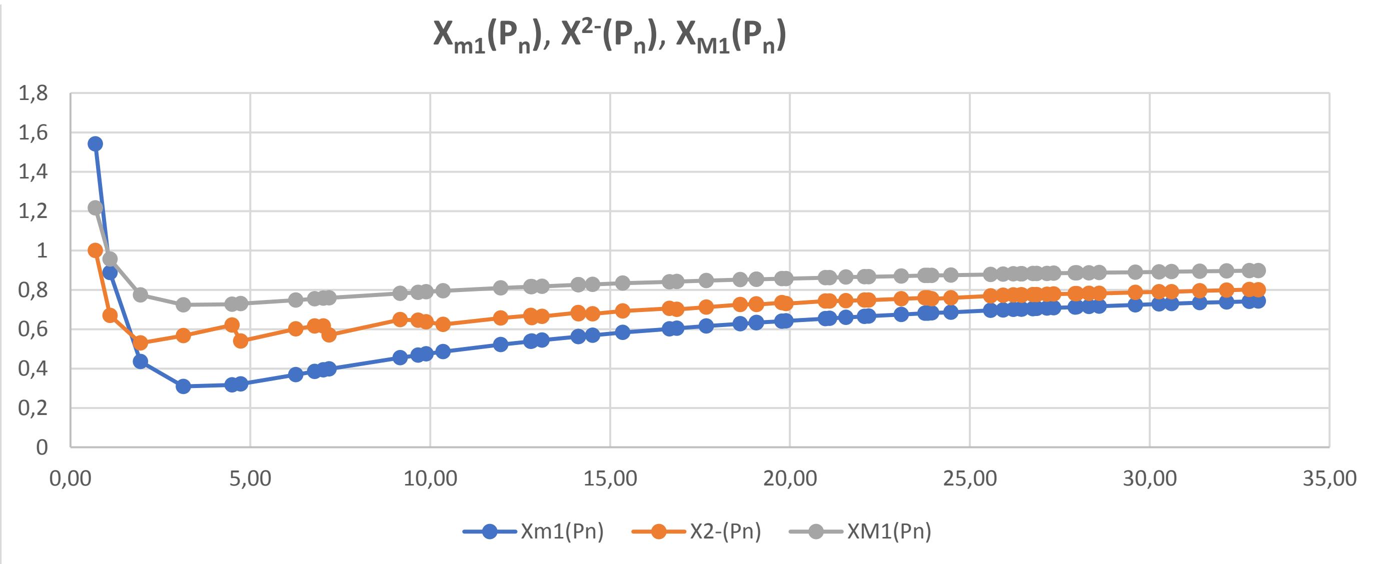
In 1931, Erik Westzynthius [13] proved that maximal gaps grows more than logarithmically.

$$\text{Limsup } \frac{G_n}{\ln(p_n)} = \infty \quad \text{which imply that for n enough large} \quad G_n > \ln(p_n)$$

In these conditions we can define for all P_n, an upper bound function of $X^{2-}(P_n)$ independent of G_n.

$$X^{2-}(P_n) = 1 - \frac{\ln(G_n)}{\ln(e.P_n)} < 1 - \frac{\ln(\ln(P_n))}{\ln(e.P_n)} = X_{M1}(P_n)$$

The Graphic below shows the evolution of $X^{2-}(P_n)$ surrounded by the lower and upper bound functions $X_{m_1}(P_n)$ and $X_{M_1}(P_n)$



We give below the behavior of the lower and upper bound functions X_{m1}, $F(X_{m1})$, X_{M1}, $F(X_{M1})$ of X^{2-}, $F(X^{2-})$, for all p_n corresponding to the first occurrence of G_n.

Primes	Gaps	$Ln(P_n)$	$X_{m1}(P_n)$	$F(X_{m1}(P_n))$	$X^{2-}(P_n)$	$F(X^{2-}(P_n))$	$X_{M1}(P_n)$	$F(X_{M1}(P_n))$
2	1	0,69	1,541171088	0,944224262	1	1	1,216468435	1,013547875
3	2	1,10	0,887964266	0,972790415	0,669711655	0,851328799	0,955185706	0,991653064
7	4	1,95	0,435038938	0,646085363	0,529417297	0,757405268	0,774015575	0,942802451
23	6	3,14	0,30915947	0,501750404	0,566736314	0,829998067	0,723663788	0,93699799
89	8	4,49	0,31606466	0,544656826	0,621136945	0,892404769	0,726425864	0,94854812
113	14	4,73	0,321954074	0,559151858	0,539221472	0,831800413	0,728781629	0,950513065
523	18	6,26	0,368381864	0,655514074	0,601854215	0,890153941	0,747352746	0,960716153
887	20	6,79	0,385217119	0,684714704	0,615332318	0,900120577	0,754086848	0,963413106
1129	22	7,03	0,392814909	0,696991051	0,615019462	0,900463798	0,757125964	0,964538042
1327	34	7,19	0,397856421	0,704838152	0,569466487	0,871110994	0,759142569	0,965258269
9551	36	9,16	0,455126154	0,779200356	0,647444164	0,92050724	0,782050462	0,972397117
15683	44	9,66	0,468115076	0,792913799	0,645021429	0,919439877	0,78724603	0,973822721
19609	52	9,88	0,473781404	0,798606188	0,636959143	0,915312439	0,789512562	0,974427525
31397	72	10,35	0,48535278	0,809735209	0,623349481	0,908074421	0,794141112	0,975632633
155921	86	11,96	0,521242266	0,840682918	0,656223564	0,925360302	0,808496907	0,979138011
360653	96	12,80	0,538060372	0,853662182	0,669146357	0,931515118	0,815224149	0,980669673
370261	112	12,82	0,538567801	0,854040818	0,658623112	0,926558073	0,81542712	0,98071484
492113	114	13,11	0,543984662	0,858038	0,664253312	0,92924402	0,817593865	0,981193239
1349533	118	14,12	0,562155146	0,870877496	0,684379777	0,938346626	0,824862058	0,982748459
1357201	132	14,12	0,562252856	0,870944278	0,677083335	0,935139132	0,824901142	0,982756618
2010733	148	14,51	0,568919576	0,875446017	0,677890351	0,935502058	0,827567831	0,98330822
4652353	154	15,35	0,582443168	0,884256345	0,69198383	0,941593507	0,832977267	0,984396545
17051707	180	16,65	0,601666374	0,896074454	0,705810832	0,94722331	0,840666549	0,985873904
20831323	210	16,85	0,604459938	0,897725542	0,700475184	0,945091969	0,841783975	0,986081864
47326693	220	17,67	0,615477106	0,904077702	0,711147249	0,949306378	0,846190843	0,986885518
122164747	222	18,62	0,627401596	0,910672158	0,724646546	0,954356428	0,850960638	0,987725887
189695659	234	19,06	0,632664021	0,913491391	0,728062423	0,955585458	0,853065608	0,988087026
191912783	248	19,07	0,632800768	0,913563916	0,725324968	0,954602135	0,853120307	0,988096394
387096133	250	19,77	0,640855053	0,917770579	0,734215264	0,957750126	0,856342021	0,988637526
436273009	282	19,89	0,642189298	0,918455169	0,729971915	0,956264032	0,856875719	0,988725927
1294268491	288	20,98	0,653837626	0,924285628	0,742372686	0,960523725	0,86153505	0,989481196
1453168141	292	21,10	0,655028989	0,924867293	0,743098577	0,960765261	0,862011596	0,989556879
2300942549	320	21,56	0,659669076	0,927107083	0,744273299	0,961154306	0,86386763	0,98984848
3842610773	336	22,07	0,664686723	0,929483461	0,747843175	0,962322969	0,865874689	0,990159333
4302407359	354	22,18	0,665770629	0,929990609	0,74682144	0,961990597	0,866308252	0,990225822
10726904659	382	23,10	0,674255309	0,933885169	0,753261311	0,964057438	0,869702124	0,990736961
20678048297	384	23,75	0,680062453	0,93647434	0,759592713	0,966024682	0,872024981	0,991078734
22367084959	394	23,83	0,680741857	0,936773278	0,759317567	0,965940595	0,872296743	0,991113901
25056082087	456	23,94	0,681718544	0,937201517	0,754554244	0,964464486	0,872687418	0,991178036
42652618343	464	24,48	0,686208291	0,93914777	0,758996737	0,965842515	0,874483316	0,991422653
127976334671	468	25,58	0,695051167	0,942875706	0,768638097	0,968725502	0,878020467	0,991942406
182226896239	474	25,93	0,69777919	0,943997636	0,7712014	0,969467998	0,879111676	0,992058754
241160624143	486	26,21	0,699903979	0,944862366	0,772638824	0,969881535	0,879961592	0,992191315
297501075799	490	26,42	0,701474411	0,945496231	0,774080862	0,970290184	0,880589765	0,992284775
303371455241	500	26,44	0,70161964	0,945554867	0,773505452	0,970129848	0,880647856	0,9923172
303 371 455 241	500	26,44	0,70161964	0,945554867	0,773505452	0,970129848	0,880647856	0,9923172
304 599 508 537	514	26,44	0,701649646	0,945566833	0,772532494	0,969849825	0,880659859	0,992296219
416 608 695 821	516	26,76	0,703955334	0,946489245	0,774958971	0,970535994	0,881582134	0,992393494
461 690 510 011	532	26,86	0,704703251	0,946786523	0,774692824	0,970460176	0,8818813	0,992500305
614 487 453 523	534	27,14	0,706762396	0,947600275	0,776848219	0,971072674	0,882704958	0,992622375
738 832 927 927	540	27,33	0,708072866	0,94811371	0,777905459	0,97136426	0,883229146	0,992671967
1 346 294 310 749	582	27,93	0,712251117	0,949732125	0,779922999	0,971920967	0,884900447	0,992843628
1 408 695 493 609	588	27,97	0,712561223	0,949851453	0,779913156	0,971918106	0,885024489	0,992889404
1 968 188 556 461	602	28,31	0,714827535	0,950713396	0,781621809	0,972376347	0,885931014	0,992797852
2 614 941 710 599	652	28,59	0,716721888	0,951429009	0,781022338	0,972208023	0,886688755	0,993087769
7 177 162 611 713	674	29,60	0,723233148	0,953845501	0,787162734	0,973907471	0,889293259	0,99395752
13 829 048 559 701	716	30,26	0,727287703	0,955310345	0,789694676	0,974479675	0,890915081	0,992370605
19 581 334 192 423	766	30,61	0,729384733	0,956071854	0,789873231	0,974651337	0,891753893	0,992919922
42 842 283 925 351	778	31,39	0,733976712	0,957698822	0,794472824	0,976005554	0,893590685	0,997802734
90 874 329 411 493	804	32,14	0,738226866	0,959186554	0,798144279	0,976348877	0,895290747	0,992675781
171 231 342 420 521	806	32,77	0,741692363	0,960258484	0,801857154	0,977203369	0,896676945	1,008789063
218 209 405 436 543	906	33,02	0,742991698	0,960548401	0,799831139	0,976318359	0,897196679	0,989257813

Table I.3

I.3 – Proof of the inequation $(p_{n+1})^{1/k} - (p_n)^{1/k} < 1/k$ and $(p_{n+1})^{p/q} - (p_n)^{p/q} < p/q$

Florentin Smarandache, has listed some conjectures derived from the Andrica's conjecture, such as the fact that $(p_{n+1})^{1/k} - (p_n)^{1/k} < 2/k$ for k≥2. We are going to prove that for all integer k>2, this assertion is true.
We'll start by using the Taylor serie expansion of (I.5) :

$$F_n(x) = x.p_n^x.\frac{g_n}{p_n} + \frac{x.(x-1)}{2}.p_n^x.\left(\frac{g_n}{p_n}\right)^2 + p_n^x.O\left(\frac{g_n}{p_n}\right)^3)$$

As x-1 < 0, $F_n(x) = x.p_n^x.\frac{g_n}{p_n} - p_n^x.O\left(\frac{g_n}{p_n}\right)^2$ and as according to Baker, Harman and Pintz result : $g_n << (p_n)^{0.525}$

$F_n(x) = x.p_n^x.\frac{g_n}{p_n} < x.p_n^{x-0,475}$

If we replace x by 1/k with k>2, then $p_n^{x-0,475} < 1$ and $F_n(1/k) < 1/k$.
So for any k>2, we can conclude that $(p_{n+1})^{1/k} - (p_n)^{1/k} < 1/k$. But we cannot conclude for k=2 except if we use the Cramer's conjecture. The same approach applied to $F_n\left(\frac{p}{q}\right) = p_{n+1}^{\frac{p}{q}} - p_n^{\frac{p}{q}}$ let us to show that $p_{n+1}^{\frac{p}{q}} - p_n^{\frac{p}{q}} < \frac{p}{q}$ for all rational numbers such as $\frac{p}{q} < 0,475$.

I.4 - Study of the equation $(p_{n+1})^x - (p_n)^x = C$

In 2017, David Lowry-Duda [4], has formulated the following question : for any constant C, what is the smallest x such as $(p_{n+1})^x - (p_n)^x = C$ With the same approach developed before, we are going to build some lower and upper bound functions of x satisfying the condition $F_n(x) = (p_{n+1})^x - (p_n)^x = C$

We can start from the Taylor serie expansion of (5) :

$$F_n(x) = x.p_n^x.\frac{g_n}{p_n} + \frac{x.(x-1)}{2}.p_n^x.\left(\frac{g_n}{p_n}\right)^2 + p_n^x.O\left(\frac{g_n}{p_n}\right)^3)$$

Resolving $F_n(x) = C$ correspond to solve, at the first degree of the Taylor serie expansion the following equality:

$$F_n(x) = x.p_n^x.\frac{g_n}{p_n} = C$$

Which can writen : $x.p_n^x.\frac{h_n}{p_n} = 1$ *with* $h_n = \frac{g_n}{C}$ (I.18)

By analogy to the previous development, the following function give a good upper bound of x :

$$X_c^+(p_n) = 1 - \frac{\ln(h_n)}{\ln(p_n)} - \frac{\ln\left(1-\frac{\ln(h_n)}{\ln(e.p_n)}\right)}{\ln(p_n)} \quad (I.19)$$

And with the same Taylor serie expansion, we build the following lower bound function :

$$X_c^-(p_n) = 1 - \frac{\ln(h_n)}{\ln(e.p_n)}\left(1 - \frac{\ln(h_n)}{2.\ln(p_n)\ln(e.p_n)}\right) \quad (I.20)$$

$$X_c^{2-}(p_n) = 1 - \frac{\ln(h_n)}{\ln(e.p_n)} \quad (I.21)$$

$$X_c^{3-}(p_n) = 1 - \frac{\ln(h_n)}{\ln(p_n)} \quad (I.22)$$

Several situations can be studied :

If C is an even number, and p_n enough large :

for all p_n such as $g_n = C$, $x = X_C^{+}(p_n) = X_C^{1-}(p_n) = X_C^{2-}(p_n) = X_C^{3-}(p_n) = 1$

for all p_n such as $g_n < C$, $x < X_C^{3-}(p_n) < X_C^{2-}(p_n) < X_C^{1-}(p_n) < X_C^{+}(p_n)$

for all p_n such as $g_n > C$, $X_C^{3-}(p_n) < X_C^{2-}(p_n) < X_C^{1-}(p_n) < x < X_C^{+}(p_n)$

If C is real different from an even number, and p_n enough large :

for all p_n such as $g_n < C$, $x < X_C^{3-}(p_n) < X_C^{2-}(p_n) < X_C^{1-}(p_n) < X_C^{+}(p_n)$

for all p_n such as $g_n > C$, $X_C^{3-}(p_n) < X_C^{2-}(p_n) < X_C^{1-}(p_n) < x < X_C^{+}(p_n)$

We give below some numerical results for C=2, C=6, C=Π for the primes numbers up to 300 (tables I.4, I.6, I.8) and for all primes numbers corresponding to the first occurrence of maximal gaps (tables I.5, I.7, I.9). We have shown in paragraphe I.2 that the local minimal values of lower bound functions of $x(p_n)$ was necessary reached for the prime numbers corresponding to the first occurrence of a maximal gap. Thus, we can limit our research of the lower bound of $x(p_n)$ to the set of these prime numbers.

for C=2, we can see that the minimum value of x is obtain for $p_n = 113$

Primes	Gaps	$X^{2-}(P_n)$	$F(X^{2-}(P_n))$	$X^-(P_n)$	$F(X^-(P_n))$	$X^+(P_n)$	$F(X^+(P_n))$
2	**1**	1,409383891	**2,047543071**	1,53457703	2,500299469	1,504935371	2,386283217
3	**2**	1	**2**	1	2	1	2
5	**2**	1	**2**	1	2	1	2
7	**4**	0,764708648	**1,82846929**	0,77894565	1,921360642	0,781651372	1,939467764
11	**2**	1	**2**	1	2	1	2
13	**4**	0,805566051	**1,904538249**	0,812937487	1,960658639	0,814055938	1,969302797
17	**2**	1	**2**	1	2	1	2
19	**4**	0,824272302	**1,931595547**	0,829516913	1,975152393	0,830224758	1,98109976
23	**6**	0,734345587	**1,855595552**	0,745602614	1,954348804	0,748098163	1,976881952
29	**2**	1	**2**	1	2	1	2
31	**6**	0,75222926	**1,884605669**	0,761169648	1,968050494	0,762987424	1,985429737
37	**4**	0,849672626	**1,959584518**	0,852801988	1,989476527	0,853155612	1,992881241
41	**2**	1	**2**	1	2	1	2
43	**4**	0,854417549	**1,963700645**	0,857235182	1,991419411	0,857542302	1,994462982
47	**6**	0,773488896	**1,913560699**	0,780152716	1,98101106	0,781367259	1,993542509
53	**6**	0,778964232	**1,920051171**	0,78511767	1,983743732	0,786206617	1,995220373
59	**2**	1	**2**	1	2	1	2
61	**6**	0,78504414	**1,926805701**	0,790664616	1,986490392	0,791626554	1,996878923
67	**4**	0,866822648	**1,972905887**	0,868931813	1,995446505	0,869139947	1,997684108
71	**2**	1	**2**	1	2	1	2
73	**6**	0,792340854	**1,93429832**	0,797366598	1,989395506	0,798192084	1,99858656
79	**4**	0,87090904	**1,97547282**	0,872816023	1,996458623	0,872997811	1,998470257
83	**6**	0,797260638	**1,938983339**	0,801911848	1,991120615	0,802654421	1,99956698
89	**8**	0,74742463	**1,903309751**	0,754531516	1,984302176	0,756012191	2,001583035
97	**4**	0,875662221	**1,978189478**	0,87735197	1,997455854	0,877506507	1,999226843
101	**2**	1	**2**	1	2	1	2
103	**4**	0,876986598	**1,978897069**	0,878619122	1,997700996	0,878766674	1,999408977
107	**2**	1	**2**	1	2	1	2
109	**4**	0,878210365	**1,979532434**	0,879791255	1,997915371	0,879932575	1,999566641
113	**14**	**0,660244738**	1,817742928	**0,672455603**	**1,962793959**	**0,676192115**	**2,009317475**
127	**4**	0,88139545	**1,981105304**	0,882847425	1,998420629	0,882973494	1,999930906
131	**6**	0,813008444	**1,952107477**	0,816594689	1,995461529	0,817115401	2,001832747
137	**2**	1	**2**	1	2	1	2
139	**10**	0,728798554	**1,893592701**	0,736251878	1,985141461	0,737949777	2,006579199
149	**2**	1	**2**	1	2	1	2
151	**6**	0,817423766	**1,955307871**	0,820745816	1,9963813	0,821214944	2,002248005
157	**6**	0,818598464	**1,956126249**	0,821852626	1,996605911	0,822308738	2,002343674
163	**4**	0,886252774	**1,983290352**	0,887522821	1,999053945	0,887628155	2,00036675
167	**6**	0,820429323	**1,957374732**	0,82357965	1,996939732	0,824016048	2,00248084
173	**6**	0,821459414	**1,9580629**	0,824552367	1,997119019	0,824977973	2,002551759
179	**2**	1	**2**	1	2	1	2
181	**10**	0,740350297	**1,906233094**	0,746835103	1,989208521	0,748233754	2,007553049
191	**2**	1	**2**	1	2	1	2
193	**4**	0,889321177	**1,984546253**	0,890485025	1,999375905	0,89057871	2,000574287
197	**2**	1	**2**	1	2	1	2
199	**12**	0,715291167	**1,885862617**	0,722948513	1,98534376	0,72480446	2,010201154
211	**12**	0,717915698	**1,888844041**	0,725350252	1,986279277	0,72713097	2,010321934
223	**4**	0,891816982	**1,985501459**	0,892899219	1,999597239	0,892984196	2,000708134
227	**2**	1	**2**	1	2	1	2
229	**4**	0,892263424	**1,98566631**	0,893331503	1,999633221	0,893414991	2,000728979
233	**6**	0,829699932	**1,963213628**	0,832360236	1,99834495	0,832706901	2,002967465
239	**2**	1	**2**	1	2	1	2
241	**10**	0,751813676	**1,91744136**	0,75742917	1,992415578	0,758574261	2,008046611
251	**6**	0,831641987	**1,964339812**	0,834206944	1,998583378	0,834536813	2,003029037
257	**6**	0,832249271	**1,964685416**	0,834784915	1,998654245	0,835109666	2,003045661
263	**6**	0,832838323	**1,965017703**	0,835345761	1,998721341	0,835665608	2,003060619
269	**2**	1	**2**	1	2	1	2
271	**6**	0,833597014	**1,96544145**	0,836068451	1,998805406	0,836382068	2,00307822
277	**4**	0,895358492	**1,986761185**	0,89633199	1,999854469	0,896405712	2,000849419
281	**2**	1	**2**	1	2	1	2
283	**10**	0,757813442	**1,922813073**	0,763008516	1,99379594	0,764036418	2,008137117
293	**14**	0,708703612	**1,883783703**	0,716173381	1,986495674	0,718038098	2,012963731

Table I.4

Primes	Gaps	$X^{2-}(P_n)$	$F(X^{2-}(P_n))$	$X^{-}(P_n)$	$F(X^{-}(P_n))$	$X^{+}(P_n)$	$F(X^{+}(P_n))$
2	**1**	1,409383891	**2,047543**	1,530278251	**2,483489**	1,504935371	**2,386283**
3	**2**	1	**2,000000**	1	**2,000000**	1	**2,000000**
7	**4**	0,764708648	**1,828469**	0,778933874	**1,921282**	0,781651372	**1,939468**
23	**6**	1	**2**	1	**2**	1	**2**
89	**8**	0,74742463	**1,903310**	0,754530833	**1,984294**	0,756012191	**2,001583**
113	**14**	0,660244738	**1,817743**	0,672453768	**1,962771**	0,676192115	**2,009317**
523	**18**	0,697334538	**1,877944**	0,704651831	**1,986836**	0,706572305	**2,016411**
887	**20**	0,704336039	**1,887031**	0,710775283	**1,989505**	0,71241456	**2,016443**
1 129	**22**	0,70134897	**1,885435**	0,707693494	**1,989378**	0,709329902	**2,017078**
1 327	**34**	0,654092857	**1,840721**	0,662412781	**1,979271**	0,665023514	**2,024794**
9 551	**36**	0,715637772	**1,901025**	0,72004951	**1,991679**	0,721117553	**2,014257**
15 683	**44**	0,710042588	**1,896989**	0,714394162	**1,990571**	0,71547438	**2,014497**
19 609	**52**	0,700645609	**1,889575**	0,705178964	**1,988964**	0,706351815	**2,015504**
31 397	**72**	0,684395688	**1,876054**	0,6892055	**1,985722**	0,690539387	**2,017238**
155 921	**86**	0,709719093	**1,897357**	0,713242647	**1,988831**	0,71411858	**2,012239**
360 653	**96**	0,719390173	**1,904783**	0,722467068	**1,989747**	0,723199397	**2,010516**
370 261	**112**	0,708771355	**1,896689**	0,712078731	**1,988086**	0,712904392	**2,011575**
492 113	**114**	0,713390161	**1,900272**	0,716523928	**1,988644**	0,717290295	**2,010868**
1 349 533	**118**	0,730237193	**1,912693**	0,732814968	**1,990572**	0,733398526	**2,008635**
1 357 201	**132**	0,722923567	**1,907433**	0,72564192	**1,989528**	0,72627854	**2,009255**
2 010 733	**148**	0,722569141	**1,907180**	0,725220644	**1,989280**	0,72584262	**2,009040**
4 652 353	**154**	0,734370677	**1,915600**	0,736668582	**1,990597**	0,737178752	**2,007639**
17 051 707	**180**	0,745078709	**1,922837**	0,74703	**1,991543**	0,74744147	**2,006340**
20 831 323	**210**	0,739302677	**1,918983**	0,741319151	**1,990729**	0,741756429	**2,006635**
47 326 693	**220**	0,748268365	**1,924920**	0,750061221	**1,991648**	0,75043341	**2,005786**
122 164 747	**222**	0,759973562	**1,932282**	0,761520553	**1,992802**	0,761823377	**2,004868**
189 695 659	**234**	0,762614517	**1,933883**	0,76409272	**1,993003**	0,764378182	**2,004625**
191 912 783	**248**	0,759857059	**1,932211**	0,761368882	**1,992693**	0,761664998	**2,004758**
387 096 133	**250**	0,76758106	**1,936834**	0,768946946	**1,993401**	0,769203997	**2,004229**
436 273 009	**282**	0,763146727	**1,934203**	0,764556702	**1,992900**	0,764828242	**2,004407**
1 294 268 491	**288**	0,773906306	**1,940482**	0,7751245	**1,993852**	0,775346203	**2,003721**
1 453 168 141	**292**	0,774466944	**1,940799**	0,775672451	**1,993891**	0,775891186	**2,003678**
2 300 942 549	**320**	0,77500256	**1,941102**	0,776176768	**1,993878**	0,776389213	**2,003578**
3 842 610 773	**336**	0,777889325	**1,942716**	0,779007006	**1,994093**	0,779206095	**2,003385**
4 302 407 359	**354**	0,776721104	**1,942066**	0,777844819	**1,993964**	0,778046253	**2,003412**
10 726 904 659	**382**	0,782027354	**1,944985**	0,783055931	**1,994363**	0,783235047	**2,003089**
20 678 048 297	**384**	0,787596014	**1,947956**	0,788545719	**1,994800**	0,788706052	**2,002818**
22 367 084 959	**394**	0,787232318	**1,947766**	0,788182139	**1,994760**	0,78834282	**2,002821**
25 056 082 087	**456**	0,78234195	**1,945155**	0,783331223	**1,994300**	0,783503196	**2,002969**
42 652 618 343	**464**	0,786204209	**1,947223**	0,787137939	**1,994603**	0,787296811	**2,002779**
127 976 334 671	**468**	0,794720666	**1,951623**	0,795544507	**1,995246**	0,795678053	**2,002410**
182 226 896 239	**474**	0,796941664	**1,952735**	0,797736786	**1,995400**	0,797864025	**2,002313**
241 160 624 143	**486**	0,798114001	**1,953317**	0,798891566	**1,995474**	0,799015145	**2,002261**
297 501 075 799	**490**	0,799360966	**1,953931**	0,800122851	**1,995557**	0,800243055	**2,002207**
303 371 455 241	**500**	0,798767553	**1,953638**	0,799533385	**1,995512**	0,799654634	**2,002222**
304 599 508 537	**514**	0,797790876	**1,953156**	0,798564043	**1,995429**	0,798687156	**2,002247**
416 608 695 821	**516**	0,799932376	**1,954211**	0,800680394	**1,995581**	0,800798012	**2,002165**
461 690 510 011	**532**	0,799574122	**1,954037**	0,800321949	**1,995549**	0,800439786	**2,002168**
614 487 453 523	**534**	0,801476767	**1,954968**	0,802202736	**1,995675**	0,802315848	**2,002097**
738 832 927 927	**540**	0,802373792	**1,955390**	0,803088363	**1,995727**	0,803199107	**2,002054**
1 346 294 310 749	**582**	0,803883803	**1,956132**	0,804572378	**1,995807**	0,804678133	**2,001975**
1 408 695 493 609	**588**	0,803836491	**1,956086**	0,804524282	**1,995816**	0,804629947	**2,001979**
1 968 188 556 461	**602**	0,805272143	**1,956792**	0,805941896	**1,995885**	0,806043905	**2,001916**
2 614 941 710 599	**652**	0,804445595	**1,956404**	0,805114334	**1,995796**	0,805216696	**2,001928**
7 177 162 611 713	**674**	0,809813177	**1,958916**	0,810424135	**1,996159**	0,810514648	**2,001701**
13 829 048 559 701	**716**	0,811869856	**1,959724**	0,812454713	**1,996231**	0,812540265	**2,001572**
19 581 334 192 423	**766**	0,811804383	**1,959641**	0,812382996	**1,996216**	0,812467669	**2,001617**
42 842 283 925 351	**778**	0,815873821	**1,961945**	0,816413866	**1,996460**	0,816490906	**2,001755**
90 874 329 411 493	**804**	0,81905969	**1,962341**	0,819569007	**1,996704**	0,819640204	**2,001221**
171 231 342 420 521	**806**	0,822380232	**1,963440**	0,822861539	**1,996399**	0,822927392	**1,998901**
218 209 405 436 543	**906**	0,820207947	**1,962402**	0,820697478	**1,996460**	0,820765405	**2,000854**

Table I.5

for C=6, we can see that the minimum value of x is obtain for $p_n = 113$

Primes	Gaps	$X^{2-}(P_n)$	$F(X^{2-}(P_n))$	$X^{-}(P_n)$	$F(X^{-}(P_n))$	$X^{+}(P_n)$	$F(X^{+}(P_n))$
2	1	2,058242006	5,429906392	3,10659915	21,74110049	2,543549878	10,5221896
3	2	1,523494642	6,279179845	1,650509009	8,1143397	1,616783094	7,584732868
5	2	1,421014918	6,035992942	1,476323569	6,924027768	1,464286901	6,721071099
7	4	1,137636617	6,151373262	1,142505599	6,243963571	1,142099131	6,23618524
11	2	1,323321409	5,908912743	1,345138467	6,340889005	1,341327456	6,263337329
13	4	1,113736569	6,060056777	1,116258489	6,115366452	1,116082006	6,11148054
17	2	1,286603481	5,89107396	1,301106116	6,212492739	1,29881495	6,16062283
19	4	1,102794113	6,03128825	1,104588543	6,074183009	1,104474247	6,071442248
23	6	1	6	1	6	1	6
29	2	1,251554356	5,888990462	1,260952561	6,125926799	1,259622578	6,091847141
31	6	1	6	1	6	1	6
37	4	1,087935876	6,00564521	1,089006641	6,035182245	1,088947715	6,033553165
41	2	1,233074253	5,893177274	1,240389471	6,092458931	1,239419575	6,065666496
43	4	1,085160274	6,002480583	1,086124382	6,029901259	1,086072903	6,02843409
47	6	1	6	1	6	1	6
53	6	1	6	1	6	1	6
59	2	1,216367147	5,899630067	1,222108243	6,068456519	1,221394318	6,047209406
61	6	1	6	1	6	1	6
67	4	1,077903757	5,996376034	1,078625458	6,018754134	1,07859003	6,017653723
71	2	1,208755295	5,903289359	1,213867347	6,059288768	1,213251092	6,040272447
73	6	1	6	1	6	1	6
79	4	1,075513371	5,994995501	1,076165893	6,015857986	1,076134793	6,014862043
83	6	1	6	1	6	1	6
89	8	0,947585875	5,977776104	0,947891898	5,988006088	0,947903029	5,988378494
97	4	1,072732938	5,993741732	1,07331113	6,012926397	1,073284535	6,012042673
101	2	1,195652486	5,910462811	1,199799933	6,045829524	1,199327458	6,030257081
103	4	1,071958227	5,993455674	1,072516838	6,012188683	1,072491404	6,011334522
107	2	1,193662161	5,911636083	1,19767546	6,044023433	1,197222345	6,028932603
109	4	1,071242369	5,993214656	1,071783313	6,011536331	1,071758916	6,010708848
113	14	0,852062077	5,875588741	0,854376912	5,957195907	0,854633893	5,966322392
127	4	1,069379214	5,992687926	1,069876047	6,009965669	1,069854198	6,009204811
131	6	1	6	1	6	1	6
137	2	1,185576998	5,916590652	1,189077032	6,037280999	1,188696405	6,024040102
139	10	0,913922341	5,958413851	0,914673123	5,985599727	0,914719187	5,987271634
149	2	1,182981698	5,918237827	1,186327412	6,035309206	1,185968058	6,022626979
151	6	1	6	1	6	1	6
157	6	1	6	1	6	1	6
163	4	1,066537862	5,992144091	1,066972444	6,007902132	1,066954078	6,007235363
167	6	1	6	1	6	1	6
173	6	1	6	1	6	1	6
179	2	1,177556778	5,921755142	1,180595631	6,031470359	1,180277804	6,019902831
181	10	0,917588793	5,965123858	0,918242025	5,98978191	0,918280285	5,991229191
191	2	1,17571405	5,922969918	1,178653382	6,030249782	1,178348792	6,019044938
193	4	1,064742961	5,991946542	1,065141205	6,00678832	1,065124808	6,00617653
197	2	1,174849064	5,923543232	1,177742486	6,029690898	1,177443963	6,018653565
199	12	0,889859589	5,941454746	0,891005475	5,985502296	0,891097225	5,989042782
211	12	0,890874896	5,943445086	0,891987446	5,986591551	0,892075631	5,990024499
223	4	1,063283009	5,991860772	1,063653328	6,005981607	1,063638409	6,005412106
227	2	1,17099157	5,926121447	1,173686421	6,027304619	1,173413838	6,016993372
229	4	1,063021857	5,991852051	1,063387331	6,005846195	1,063372666	6,005284034
233	6	1	6	1	6	1	6
239	2	1,16963151	5,927037897	1,172258712	6,026503441	1,171994852	6,016440156
241	10	0,921227198	5,970746538	0,92179287	5,993047127	0,921824446	5,994294363
251	6	1	6	1	6	1	6
257	6	1	6	1	6	1	6
263	6	1	6	1	6	1	6
269	2	1,166589897	5,929099076	1,169070169	6,024784511	1,168825047	6,0152609
271	6	1	6	1	6	1	6
277	4	1,061211358	5,991842954	1,061544469	6,004977441	1,06153147	6,004464349
281	2	1,165494666	5,929844679	1,167923484	6,024189531	1,167684855	6,014855284
283	10	0,923131487	5,973316491	0,923654811	5,994441196	0,923683274	5,995592238
293	14	0,87316228	5,929747823	0,874578435	5,987533272	0,874710848	5,992964216

Table I.6

Primes	Gaps	$X^{2-}(P_n)$	$F(X^{2-}(P_n))$	$X^{-}(P_n)$	$F(X^{-}(P_n))$	$X^{+}(P_n)$	$F(X^{+}(P_n))$
2	**1**	2,058242006	**5,429906**	2,866061886	**16,014810**	2,543549878	**10,522190**
3	**2**	1,523494642	**6,279180**	1,648218643	**8,077333**	1,616783094	**7,584733**
7	**4**	1,137636617	**6,151373**	1,142504221	**6,243937**	1,142099131	**6,236185**
23	**6**	1	**6**	1	**6**	1	**6**
89	**8**	0,947585875	**5,977776**	0,947891897	**5,988006**	0,947903029	**5,988378**
113	**14**	0,852062077	**5,875589**	0,854376846	**5,957194**	0,854633893	**5,966322**
523	**18**	0,848667269	**5,908727**	0,850496592	**5,989845**	0,850705001	**5,999155**
887	**20**	0,845403599	**5,910233**	0,847164103	**5,993858**	0,847369574	**6,003693**
1 129	**22**	0,838178003	**5,903176**	0,840040717	**5,994376**	0,84026971	**6,005682**
1 327	**34**	0,78822248	**5,829161**	0,791341083	**5,985179**	0,791865722	**6,011824**
9 551	**36**	0,823722082	**5,893108**	0,825417441	**5,997722**	0,825647376	**6,012051**
15 683	**44**	0,813098687	**5,879643**	0,814906704	**5,996559**	0,815169162	**6,013720**
19 609	**52**	0,801586268	**5,863506**	0,803577822	**5,994940**	0,803887921	**6,015665**
31 397	**72**	0,781151637	**5,832065**	0,783464388	**5,991114**	0,783869076	**6,019382**
155 921	**86**	0,794507502	**5,854218**	0,796273279	**5,992427**	0,796559867	**6,015162**
360 653	**96**	0,799024737	**5,861143**	0,800603046	**5,992530**	0,800852551	**6,013566**
370 261	**112**	0,788254439	**5,844720**	0,790002852	**5,990487**	0,790296933	**6,015356**
492 113	**114**	0,791270225	**5,849460**	0,79293231	**5,990842**	0,79320713	**6,014543**
1 349 533	**118**	0,802919477	**5,866924**	0,804295318	**5,992226**	0,804507851	**6,011818**
1 357 201	**132**	0,795578617	**5,856108**	0,797058268	**5,990875**	0,797296933	**6,012900**
2 010 733	**148**	0,793383347	**5,852798**	0,79485401	**5,990153**	0,795094253	**6,012893**
4 652 353	**154**	0,80155224	**5,864978**	0,802834785	**5,991180**	0,803034526	**6,011076**
17 051 707	**180**	0,807316822	**5,873217**	0,808431623	**5,991524**	0,808599324	**6,009526**
20 831 323	**210**	0,800842798	**5,863960**	0,802019622	**5,990203**	0,802203672	**6,010190**
47 326 693	**220**	0,80710394	**5,872922**	0,808156669	**5,991011**	0,808315238	**6,009002**
122 164 747	**222**	0,815965557	**5,885023**	0,816874985	**5,992201**	0,817004643	**6,007640**
189 695 659	**234**	0,817378289	**5,886890**	0,818253133	**5,992285**	0,818376748	**6,007329**
191 912 783	**248**	0,814589129	**5,883188**	0,81549035	**5,991808**	0,815619959	**6,007592**
387 096 133	**250**	0,820464596	**5,890909**	0,821279622	**5,992566**	0,821392529	**6,006786**
436 273 009	**282**	0,81572756	**5,884707**	0,816581001	**5,991734**	0,81670286	**6,007173**
1 294 268 491	**288**	0,823885912	**5,895270**	0,824625054	**5,992778**	0,824725193	**6,006112**
1 453 168 141	**292**	0,82418463	**5,895646**	0,824917223	**5,992796**	0,82501628	**6,006054**
2 300 942 549	**320**	0,823707286	**5,895045**	0,824428159	**5,992606**	0,824525938	**6,005963**
3 842 610 773	**336**	0,825511345	**5,897307**	0,82620113	**5,992773**	0,826293587	**6,005685**
4 302 407 359	**354**	0,82411095	**5,895553**	0,82480828	**5,992524**	0,824902615	**6,005764**
10 726 904 659	**382**	0,827620454	**5,899918**	0,82826374	**5,992884**	0,828348764	**6,005280**
20 678 048 297	**384**	0,831980195	**5,905194**	0,832574466	**5,993418**	0,832650733	**6,004834**
22 367 084 959	**394**	0,831476152	**5,904594**	0,832072025	**5,993330**	0,83214876	**6,004853**
25 056 082 087	**456**	0,826384423	**5,898393**	0,827013848	**5,992522**	0,827097728	**6,005179**
42 652 618 343	**464**	0,82932703	**5,902001**	0,829922079	**5,992887**	0,829999832	**6,004866**
127 976 334 671	**468**	0,836060561	**5,909994**	0,836585996	**5,993708**	0,836651557	**6,004240**
182 226 896 239	**474**	0,837739018	**5,911922**	0,838246733	**5,993896**	0,838309343	**6,004074**
241 160 624 143	**486**	0,838491201	**5,912788**	0,838988842	**5,993975**	0,839049885	**6,004001**
297 501 075 799	**490**	0,839428983	**5,913854**	0,839916953	**5,994065**	0,839976413	**6,003903**
303 371 455 241	**500**	0,838807036	**5,913157**	0,83929843	**5,993990**	0,839358572	**6,003960**
304 599 508 537	**514**	0,837824465	**5,912019**	0,838321792	**5,993830**	0,838383083	**6,004004**
416 608 695 821	**516**	0,839514287	**5,913938**	0,839995603	**5,994028**	0,840054217	**6,003845**
461 690 510 011	**532**	0,839010045	**5,913364**	0,839492538	**5,993950**	0,839551506	**6,003869**
614 487 453 523	**534**	0,840512091	**5,915078**	0,840980636	**5,994139**	0,84103729	**6,003773**
738 832 927 927	**540**	0,841155181	**5,915800**	0,84161682	**5,994158**	0,841672384	**6,003685**
1 346 294 310 749	**582**	0,84186078	**5,916552**	0,842308497	**5,994213**	0,842362112	**6,003605**
1 408 695 493 609	**588**	0,84175408	**5,916491**	0,842201675	**5,994184**	0,842255317	**6,003611**
1 968 188 556 461	**602**	0,842757036	**5,917507**	0,843193754	**5,994255**	0,843245716	**6,003494**
2 614 941 710 599	**652**	0,841570579	**5,916248**	0,842009506	**5,994110**	0,842062179	**6,003448**
7 177 162 611 713	**674**	0,845713281	**5,920593**	0,846115356	**5,994431**	0,846162176	**6,003296**
13 829 048 559 701	**716**	0,847016685	**5,921768**	0,847403427	**5,994385**	0,847448031	**6,002960**
19 581 334 192 423	**766**	0,846564437	**5,922028**	0,846949048	**5,994690**	0,846993555	**6,002899**
42 842 283 925 351	**778**	0,849793598	**5,925537**	0,850152996	**5,995544**	0,850193597	**6,003174**
90 874 329 411 493	**804**	0,852209834	**5,926880**	0,852549622	**5,994751**	0,852587312	**6,002075**
171 231 342 420 521	**806**	0,85490854	**5,926758**	0,855229702	**5,989502**	0,855264594	**5,996338**
218 209 405 436 543	**906**	0,852504422	**5,926270**	0,852833878	**5,990967**	0,852870339	**5,998291**

Table I.7

for C=Pi, we can see that the minimum value of x is obtain for $p_n = 113$.
We see on the table below that if $g_n = 2 < \text{Pi}$, $F(X^-) > \text{Pi}$ and if $g_n > \text{Pi}$, $F(X^-) < \text{Pi}$

Primes	Gaps	$X^{2-}(P_n)$	$F(X^{2-}(P_n))$	$X^-(P_n)$	$F(X^-(P_n))$	$X^+(P_n)$	$F(X^+(P_n))$
2	1	1,676095911	3,109610634	2,039899064	**5,291108782**	1,906391423	4,371742995
3	2	1,215181579	3,269237792	1,236319435	**3,424594516**	1,233648572	3,40461796
5	2	1,173057463	3,196864583	1,182368483	**3,276355629**	1,181410632	3,268097053
7	4	0,918000053	3,06885969	0,919727951	**3,086279455**	0,919828444	3,087295278
11	2	1,132900713	3,152011303	1,136584206	**3,191327206**	1,136286752	3,18813544
13	4	0,932239016	3,104472295	0,933134102	**3,114985026**	0,933176679	3,115485905
17	2	1,117807872	3,142352513	1,120257341	**3,171610684**	1,120080331	3,169487759
19	4	0,938758217	3,116602868	0,939395117	**3,124767172**	0,939422364	3,125116892
23	6	0,84354238	3,041234565	0,847446301	**3,094383925**	0,847907725	3,100722641
29	2	1,103400988	3,136835205	1,104988635	**3,158356801**	1,104886979	3,156974576
31	6	0,854075	3,063801823	0,857175703	**3,108715059**	0,85751452	3,113660026
37	4	0,947610328	3,128563169	0,947990384	**3,134178215**	0,948004199	3,134382508
41	2	1,095804774	3,135267926	1,097040615	**3,153335756**	1,096966924	3,152255598
43	4	0,949263952	3,13023641	0,949606152	**3,135446133**	0,949618184	3,135629452
47	6	0,866595898	3,085728826	0,868907159	**3,12205467**	0,869135645	3,125667713
53	6	0,869820607	3,090533658	0,871954869	**3,124843884**	0,872160207	3,128163912
59	2	1,088937346	3,134550798	1,089907291	**3,149826777**	1,089853349	3,148975342
61	6	0,873401379	3,09547948	0,875350807	**3,127640459**	0,875532663	3,13065684
67	4	0,953587177	3,13384254	0,953843339	**3,138086025**	0,95385155	3,138222148
71	2	1,085808507	3,134419133	1,086672189	**3,148520428**	1,086625747	3,14776061
73	6	0,877698791	3,100890737	0,879441958	**3,130592003**	0,879598494	3,133272457
79	4	0,955011301	3,134803465	0,955242908	**3,138756585**	0,955250096	3,138879362
83	6	0,880596306	3,104228109	0,882209576	**3,132342382**	0,882350678	3,134812921
89	8	0,829700568	3,06765524	0,8329313	**3,125024359**	0,833352228	3,132573997
97	4	0,956667803	3,13579185	0,956873027	**3,13942367**	0,956879155	3,139532171
101	2	1,08042262	3,134440559	1,081123345	**3,14665469**	1,081087899	3,146035732
103	4	0,957129353	3,136043721	0,957327628	**3,139589085**	0,957333483	3,139693836
107	2	1,0796045	3,134468498	1,080282562	**3,14641042**	1,080248592	3,145811095
109	4	0,957555841	3,136267711	0,957747844	**3,139734353**	0,957753456	3,139835725
113	14	**0,739090927**	**2,96776685**	**0,746291477**	3,101765403	**0,747853748**	**3,131589337**
127	4	0,958665855	3,136812637	0,958842202	**3,140079567**	0,958847217	3,140172519
131	6	0,889871004	3,113341969	0,89111491	**3,13674593**	0,891214492	3,13862689
137	2	1,076281108	3,13464088	1,076872457	**3,145514977**	1,076844002	3,144990896
139	10	0,80489337	3,052445198	0,808750749	**3,126434796**	0,809339405	3,137876353
149	2	1,075214314	3,134714708	1,075779591	**3,145258735**	1,075752752	3,144757319
151	6	0,892471416	3,115500538	0,893623688	**3,137680859**	0,893713564	3,139417321
157	6	0,893163256	3,11604776	0,894291982	**3,137909445**	0,894379406	3,139608895
163	4	0,960358652	3,137544029	0,960512903	**3,140520467**	0,960517105	3,140601576
167	6	0,894241543	3,116878641	0,895334255	**3,138249433**	0,895417962	3,13989237
173	6	0,894848217	3,117334532	0,895921028	**3,138432182**	0,896002699	3,140043942
179	2	1,07298441	3,13489494	1,073497844	**3,144768539**	1,073474152	3,144312258
181	10	0,813203878	3,063901666	0,816560041	**3,130978742**	0,817046779	3,140823564
191	2	1,07222696	3,134963542	1,07272358	**3,144615342**	1,072700889	3,14417372
193	4	0,961428002	3,137948792	0,961569355	**3,14075025**	0,961573099	3,140824468
197	2	1,071871409	3,13499695	1,072360272	**3,144545663**	1,07233804	3,144110806
199	12	0,787047219	3,04021041	0,791331011	**3,127258147**	0,792056236	3,142230523
211	12	0,789010281	3,043049855	0,793169442	**3,128309868**	0,79386583	3,142808566
223	4	0,962297799	3,138247927	0,962429241	**3,140911807**	0,962432641	3,140980741
227	2	1,070285793	3,135154771	1,070741107	**3,144251679**	1,070720835	3,143846113
229	4	0,962453386	3,138298732	0,962583109	**3,140938445**	0,96258645	3,141006464
233	6	0,899701484	3,120695722	0,900624233	**3,139685926**	0,900690972	3,14106377
239	2	1,069726742	3,135213663	1,070170627	**3,144154338**	1,070151014	3,143758761
241	10	0,821450815	3,07389015	0,824357092	**3,134517937**	0,824757051	3,142951476
251	6	0,900845261	3,121417706	0,901734939	**3,139930998**	0,901798493	3,141257579
257	6	0,901202922	3,12163827	0,902082432	**3,140003953**	0,902145015	3,141314795
263	6	0,901549845	3,121849885	0,902419572	**3,140073078**	0,902481225	3,141368786
269	2	1,068476493	3,135351002	1,068895553	**3,143948001**	1,068877353	3,14357414
271	6	0,901996677	3,122119098	0,902853917	**3,140159764**	0,902914387	3,141436169
277	4	0,963532029	3,138629597	0,963650265	**3,141105412**	0,96365322	3,141167322
281	2	1,0680263	3,135402251	1,068436667	**3,143877421**	1,068418956	3,143511172
283	10	0,825767142	3,078612859	0,828455841	**3,136023673**	0,828815545	3,143782552
293	14	0,776304067	3,034231667	0,780709022	**3,129085497**	0,781500223	3,146423629

Table I.8

Primes	Gaps	$X^{2-}(P_n)$	$F(X^{2-}(P_n))$	$X^{-}(P_n)$	$F(X^{-}(P_n))$	$X^{+}(P_n)$	$F(X^{+}(P_n))$
2	**1**	1,676095911	**3,109611**	2,005827961	**5,041618**	1,906391423	**4,371743**
3	**2**	1,215181579	**3,269238**	1,236255033	**3,424112**	1,233648572	**3,404618**
7	**4**	0,918000053	**3,068860**	0,919727777	**3,086278**	0,919828444	**3,087295**
23	**6**	0,84354238	**3,041235**	0,84744591	**3,094379**	0,84790773	**3,100723**
89	**8**	0,829700568	**3,067655**	0,832931159	**3,125022**	0,833352228	**3,132574**
113	**14**	0,739090927	**2,967767**	0,746290839	**3,101753**	0,747853748	**3,131589**
523	**18**	0,759539599	**3,022423**	0,764158215	**3,130242**	0,765064316	**3,151828**
887	**20**	0,762321619	**3,029395**	0,766482808	**3,133338**	0,767287607	**3,153838**
1 129	**22**	0,75759231	**3,025820**	0,76177219	**3,133358**	0,762600385	**3,155104**
1 327	**34**	0,70922661	**2,969019**	0,715105693	**3,123131**	0,716570397	**3,162722**
9 551	**36**	0,760065643	**3,033940**	0,763206519	**3,135461**	0,763821057	**3,155712**
15 683	**44**	0,752403621	**3,026773**	0,755576595	**3,134161**	0,756221915	**3,156457**
19 609	**52**	0,742137094	**3,016297**	0,745500864	**3,132412**	0,746220404	**3,157816**
31 397	**72**	0,724167056	**2,996706**	0,727841017	**3,128708**	0,728696524	**3,160260**
155 921	**86**	0,744571224	**3,019652**	0,74729947	**3,131217**	0,747876203	**3,155318**
360 653	**96**	0,752123824	**3,027444**	0,754524738	**3,131861**	0,755013718	**3,153561**
370 261	**112**	0,741442741	**3,016480**	0,744049668	**3,129981**	0,744609189	**3,154887**
492 113	**114**	0,745402629	**3,020635**	0,747875448	**3,130489**	0,748396067	**3,154117**
1 349 533	**118**	0,760113121	**3,035324**	0,762151542	**3,132306**	0,762550279	**3,151633**
1 357 201	**132**	0,752788301	**3,028170**	0,754952238	**3,131104**	0,755391495	**3,152419**
2 010 733	**148**	0,751677199	**3,027070**	0,753801499	**3,130678**	0,754235107	**3,152255**
4 652 353	**154**	0,761985544	**3,037129**	0,763830503	**3,131952**	0,764187947	**3,150661**
17 051 707	**180**	0,770661579	**3,045183**	0,772240875	**3,132735**	0,772533304	**3,149219**
20 831 323	**210**	0,764598639	**3,039600**	0,766242773	**3,131715**	0,766557038	**3,149636**
47 326 693	**220**	0,772452628	**3,046801**	0,773917546	**3,132639**	0,774186229	**3,148642**
122 164 747	**222**	0,782988976	**3,055991**	0,784253518	**3,133857**	0,78447256	**3,147545**
189 695 659	**234**	0,785125072	**3,057790**	0,78633622	**3,134029**	0,78654354	**3,147268**
191 912 783	**248**	0,782354583	**3,055453**	0,783596407	**3,133651**	0,783812269	**3,147446**
387 096 133	**250**	0,789318747	**3,061258**	0,790441084	**3,134408**	0,790628728	**3,146806**
436 273 009	**282**	0,784759988	**3,057484**	0,785924379	**3,133778**	0,7861241	**3,147053**
1 294 268 491	**288**	0,794450338	**3,065390**	0,795457207	**3,134804**	0,795620677	**3,146221**
1 453 168 141	**292**	0,794903314	**3,065749**	0,795900248	**3,134838**	0,796061682	**3,146171**
2 300 942 549	**320**	0,795022554	**3,065843**	0,7959971	**3,134769**	0,7961548	**3,146067**
3 842 610 773	**336**	0,797464275	**3,067759**	0,798393632	**3,134979**	0,798541898	**3,145838**
4 302 407 359	**354**	0,796200619	**3,066771**	0,797136815	**3,134806**	0,797287277	**3,145880**
10 726 904 659	**382**	0,80076832	**3,070308**	0,801627629	**3,135213**	0,801762081	**3,145491**
20 678 048 297	**384**	0,805840056	**3,074123**	0,806633621	**3,135702**	0,806754073	**3,145156**
22 367 084 959	**394**	0,805418671	**3,073811**	0,806213059	**3,135645**	0,806333943	**3,145162**
25 056 082 087	**456**	0,800445534	**3,070061**	0,801277085	**3,135056**	0,801407443	**3,145368**
42 652 618 343	**464**	0,803929771	**3,072699**	0,804715091	**3,135393**	0,804835671	**3,145132**
127 976 334 671	**468**	0,811713359	**3,078394**	0,812406452	**3,136125**	0,812507935	**3,144666**
182 226 896 239	**474**	0,813711347	**3,079811**	0,814380561	**3,136295**	0,814477335	**3,144549**
241 160 624 143	**486**	0,81471098	**3,080518**	0,815365953	**3,136371**	0,815460075	**3,144484**
297 501 075 799	**490**	0,815830856	**3,081301**	0,816472793	**3,136464**	0,816564393	**3,144420**
303 371 455 241	**500**	0,815225714	**3,080879**	0,815871399	**3,136410**	0,815963887	**3,144437**
304 599 508 537	**514**	0,814246615	**3,080186**	0,814899061	**3,136300**	0,814993094	**3,144473**
416 608 695 821	**516**	0,816202453	**3,081563**	0,816833755	**3,136467**	0,816923626	**3,144372**
461 690 510 011	**532**	0,815784191	**3,081260**	0,816415944	**3,136428**	0,816506117	**3,144366**
614 487 453 523	**534**	0,81752217	**3,082464**	0,81813553	**3,136564**	0,818222118	**3,144291**
738 832 927 927	**540**	0,818314815	**3,082995**	0,818918758	**3,136628**	0,819003587	**3,144244**
1 346 294 310 749	**582**	0,819494175	**3,083820**	0,820077495	**3,136702**	0,820158809	**3,144117**
1 408 695 493 609	**588**	0,819422451	**3,083776**	0,820005289	**3,136711**	0,820086574	**3,144146**
1 968 188 556 461	**602**	0,820680244	**3,084614**	0,821248201	**3,136782**	0,821326771	**3,144033**
2 614 941 710 599	**652**	0,819705756	**3,083931**	0,820274197	**3,136642**	0,820353329	**3,144075**
7 177 162 611 713	**674**	0,824569854	**3,087227**	0,82508968	**3,137115**	0,825159791	**3,143913**
13 829 048 559 701	**716**	0,8263169	**3,088287**	0,826815381	**3,137039**	0,82688184	**3,143890**
19 581 334 192 423	**766**	0,826092444	**3,088165**	0,826586534	**3,136948**	0,826652506	**3,143631**
42 842 283 925 351	**778**	0,829816487	**3,090912**	0,830277841	**3,138184**	0,830337926	**3,144104**
90 874 329 411 493	**804**	0,832686	**3,092896**	0,833121494	**3,138000**	0,833177115	**3,143677**
171 231 342 420 521	**806**	0,835750937	**3,093506**	0,836162509	**3,135498**	0,836213974	**3,143555**
218 209 405 436 543	**906**	0,833483358	**3,092773**	0,833903266	**3,135864**	0,833956603	**3,141357**

Table I.9

We can conclude that we have reached to build a function $X^{2-}(p_n)$ which give a good approximation and a good lower bound of $x(p_n)$ which verify $F_n(x) = (p_{n+1})^x - (p_n)^x = 1$, for any prime number p_n. We have proved that $x(p_n)$ couldn't be lower than 0.475 and under Cramer's conjecture or the weaker hypothesis $g_n < \ln^k(p_n)$, we have proposed, $x(p_n)$ reached a lower bound for $p_n = 113$. We have also proved that $F_n(1/k) = (p_{n+1})^{1/k} - (p_n)^{1/k} < 1/k$ for all integer k>2 and for k=2 if we take the weaker hypothesis $g_n < \ln^k(p_n)$. And at last, we have also build a lower bound function $X_C^{2-}(p_n)$ of $x(p_n)$ verifying $F_n(x) = (p_{n+1})^x - (p_n)^x = C$. We are going, in the following part, to study the behavior of the functions $F_x(p_n) = (p_{n+1})^x - (p_n)^x$ and show that $F_x(p_n)$ are positive upper bounded functions which tends toward 0 for all $0 \leq x < 1$.

I.5 – Study of $F_x(p_n) = (p_{n+1})^x - (p_n)^x$ for $x < 1$

In this paragraph, our goal is to study the function $F_x(p_n) = (p_{n+1})^x - (p_n)^x$ for $0 \leq x < 1$. We'll show that for $x < 0{,}475$, $F_x(p_n)$ is an upper bounded function which tends towards 0 when $p_n \rightarrow \infty$ and for $0{,}475 \leq x < 1$, under Cramer's conjecture or the weaker condition proposed above, $F_x(p_n)$ is a bounded function which tends toward 0 when $p_n \rightarrow \infty$. We'll build an upper bound function $M_x(p_n)$ of $F_x(p_n)$ and we'll analyse the behavior of $M_x(p_n)$. We'll also build a lower bound function $m_x(p_n)$ of $F_x(p_n)$. We'll analyse the behavior of $m_x(p_n)$ and gives an estimation of $p'_{n,x}$ giving the maximal value of $m_x(p_n)$. Finally, we 'll try to estimate the value of $p'_{n,x}$ such as for all $p_n > p'_{n,x}$, $F_x(p_n) < M_x(p_n) < m_x(p'_{n,x})$ and will gives the maximal values of $F_x(p_n)$ for different $0 \leq x < 1$.

I.5.1 -Upper bound function of $F_x(p_n)$

We are going to show that for any real x such as $0 < x < 1$, the function $F_x(p_n) = (p_{n+1})^x - (p_n)^x$ has an upper bound and tends toward 0 when $n \rightarrow \infty$, considering two different situations.:

1- Using the Baker, Harman and Pintz result, $g_n < p_n^{0{,}525}$, we'll prove that for any real x such as $0 < x < 0.475$, $F_x(p_n) = (p_{n+1})^x - (p_n)^x$ has an upper bound and tends toward 0 when $p_n \rightarrow \infty$.

2- Based on Cramer's conjecture or the weaker condition on prime gaps proposed in I.2.3, we'll state that for any real x such as $0{,}475 \leq x < 1$, $F_x(p_n) = (p_{n+1})^x - (p_n)^x$ has an upper bound and tends toward 0 when $p_n \rightarrow \infty$.

We start from the Taylor serie expansion of (I.5)

$$F_x(p_n) = x \,.\, {p_n}^x . \frac{g_n}{p_n} - {p_n}^x . O\left(\frac{g_n}{p_n}\right)^2 < x \,.\, {p_n}^x . \frac{g_n}{p_n} \tag{I.22}$$

For the same reasons than the one exposed in paragraph I.2, the research of an upper bound of (I.22) can be reduced to all primes corresponding to the first occurrence of maximal gaps.

Replacing in (I.22), g_n by G_n , we can establish a new upper bound function of $F_x(p_n)$:

$$F_x(p_n) < x \,.\, {p_n}^x . \frac{g_n}{p_n} < x \,.\, {p_n}^x . \frac{G_n}{p_n} = M_x(p_n)$$

Due to BHP result : $G_n < p_n^a$ with a =0,525 then :

$$|F_x(p_n) - M_x(p_n)| < p_n^x . O\left(\frac{g_n}{p_n}\right)^2 < O\left(\frac{G_n}{p_n^{1-x/2}}\right)^2 < O\left(\frac{p_n^a}{p_n^{1-x/2}}\right)^2 = O\left(\frac{1}{p_n^{0,95-x}}\right) \quad (I.23)$$

For all x < 0,95, $|F_x(p_n) - M_x(p_n)| \rightarrow 0$ when $n \rightarrow \infty$

Thanks to that, we can focus our study to the behavior of $M_x(p_n)$ which appears as a good estimation of $F_x(p_n)$.

If x < 0,475, we can write :

$$F_x(p_n) < M_x(p_n) < x . p_n^x . \frac{p_n^a}{p_n} = \frac{x}{p_n^{1-a-x}}$$

$$\lim_{n\to\infty} F_x(p_n) < \lim_{n\to\infty} M_x(p_n) < \lim_{n\to\infty}\left(\frac{x}{p_n^{1-a-x}}\right) = 0$$

If 0,475≤ x<1, based on Cramer's conjecture on the weaker condition proposed in I.2.3, where there is a real r such as $G_n < \ln^r(p_n)$ then :

$$F_x(p_n) < M_x(p_n) < x . p_n^x . \frac{\ln^r(p_n)}{p_n} = x . \frac{\ln^r(p_n)}{p_n^{1-x}}$$

$$\lim_{n\to\infty} F_x(p_n) < \lim_{n\to\infty} M_x(p_n) < \lim_{n\to\infty}\left(x . \frac{\ln^r(p_n)}{p_n^{1-x}}\right) = 0$$

The table I.10 below presents the digital values of $F_x(p_n)$ for different values of x<1 and for all prime numbers corresponding to the first occurrences of the maximal prime gaps G_n up to $P_n \leq 218\ 209\ 405\ 436\ 543$. Then for all 0≤x<1, $F_x(p_n)$ is a bounded positive function which tends forward to 0 when $n \rightarrow \infty$, Thus, there is one $p_{n,x}$ such as $F_x(p_{n,x}) = \mathrm{Max}_{n\in\mathbb{N}}(F_x(p_n))$ and $F_x(p_{n,k}) < M_x(p_{n,k}) < \mathrm{Max}_n M_x(p_n)$

We are now going to study more in detail the first and second derivatives functions of $F_x(p_n)$ and $M_x(p_n)$ to try to better estimate the values of $p_{n,k}$ where $F'_x(p_n) \approx M'_x(p_n) = 0$ and $F''_x(p_n) \approx M''_x(p_n) < 0$.

$$M'_x(p_n) = x^2 . p_n^{x-1} . \frac{G_n}{p_n} + x . p_n^x . \frac{G'_n}{p_n} - x . p_n^x . \frac{G_n}{p_n^2}$$

$$M'_x(p_n) = x . p_n^x . \frac{G_n}{p_n} . \left(\frac{G'_n}{G_n} - \frac{(1-x)}{p_n}\right) = M_x(p_n) . \left(\frac{G'_n}{G_n} - \frac{(1-x)}{p_n}\right)$$

As specified above, G'_n can be estimate by $\Delta G_n / \Delta p_n$:

$$M'_x(p_n) = \frac{M_x(p_n)}{p_n} . \left(\frac{p_n . G'_n}{G_n} - (1-x)\right) = \frac{M_x(p_n)}{p_n} . \left(\frac{\Delta G_n}{\Delta p_n} . \frac{p_n}{G_n} - (1-x)\right) \quad (I.24)$$

Regarding the second derivative $M''_x(p_n)$, we have :

$$M''_x(p_n) = M'_x(p_n) . \left(\frac{G'_n}{G_n} - \frac{(1-x)}{p_n}\right) + M_x(p_n) . \left(\frac{G''_n}{G_n} - \left(\frac{G'_n}{G_n}\right)^2 + \frac{(1-x)}{p_n^2}\right)$$

$$M''_x(p_n) = M_x(p_n) . \left(\frac{G'_n}{G_n} - \frac{(1-x)}{p_n}\right)^2 + M_x(p_n) . \left(\frac{G''_n}{G_n} - \left(\frac{G'_n}{G_n}\right)^2 + \frac{(1-x)}{p_n^2}\right)$$

$$M''_x(p_n) = M_x(p_n) . \left(\frac{G''_n}{G_n} - \frac{2(1-x)G'_n}{p_n . G_n} + \frac{(1-x)(2-x)}{p_n^2}\right) \quad (I.25)$$

We are looking p_n such as $M'_x(p_n) = 0$ and $M''_x(p_n) < 0$

If $M'_x(p_n) = 0$ is equivalent to $\frac{G'_n}{G_n} - \frac{(1-x)}{p_n} = 0$. Then we can replace in (25), $\frac{G'_n}{G_n} = \frac{(1-x)}{p_n}$ (I.26)

$$M''_x(p_n) = M_x(p_n).\left(\frac{G''_n}{G_n} - \frac{x(1-x)}{{p_n}^2}\right) \quad (I.27)$$

We can write (I.26) as follow $\frac{G'_n}{(1-x)} = \frac{G_n}{p_n}$ and using the approximation of (24), $G'_n = \frac{\Delta G_n}{\Delta p_n}$, we have to find all p_n such as :

$$\frac{\Delta G_n}{\Delta p_n}.\frac{p_n}{G_n} \approx (1-x) \quad (I.28)$$

To simplify the notation, we'll write G_n and G_{n+} two consecutive gaps of this integer sequence and p_n *and* p_{n+} the two primes numbers associated to these gaps.

In table I.7, we can see that the highest value of $F_x(p_n)$ is reached for the first p_n satisfying to the following conditions : $\frac{\Delta G_n}{\Delta p_n}.\frac{p_n}{G_n} > 1 - x$ and $\frac{\Delta G_{n+}}{\Delta p_{n+}}.\frac{p_{n+}}{G_{n+}} < 1 - x$

For x = 0.1, 1-x = 0.9 then the first p_n for which $\frac{\Delta G_n}{\Delta p_n}.\frac{p_n}{G_n} < 0.9$ is $p_n = 7$

For x = 0.2 up to x = 0.5, the first p_n for which $\frac{\Delta G_n}{\Delta p_n}.\frac{p_n}{G_n} < 1 - x$ is $p_n = 23$

Primes	Gaps	$\Delta G_n.P_n/G_n.\Delta P_n$	$Ln(P_n)$	0,950	0,900	0,895	0,800	0,700	0,600	0,500	0,400	0,300	0,200	0,100
2	1		0,69											
3	2	1,5000	1,10	1,774	1,569	1,5494	1,2157	0,9275	0,6933	0,5040	0,3518	0,2303	0,1340	**0,0585**
7	4	**0,8750**	1,95	3,406	2,893	2,8452	2,0662	1,4531	1,0013	**0,6709**	**0,4316**	**0,2603**	**0,1396**	0,0562
23	6	0,4792	3,14	4,844	3,899	3,8152	2,5031	1,5817	0,9792	0,5893	0,3405	0,1845	0,0888	0,0321
89	8	0,3371	4,49	6,059	4,576	4,4487	2,5853	1,4378	0,7833	0,4149	0,2110	0,1006	0,0426	0,0135
113	14	**2,0179**	4,73	10,469	7,807	7,5799	4,2997	2,3311	**1,2381**	0,6393	0,3169	0,1473	0,0608	0,0188
523	18	0,2835	6,26	12,494	8,648	8,3345	4,1038	1,9169	0,8771	0,3902	0,1667	0,0667	0,0237	0,0063
887	20	0,2437	6,79	13,524	9,120	8,7662	4,1074	1,8209	0,7908	0,3339	0,1353	0,0514	0,0174	0,0044
1 129	22	0,4241	7,03	14,699	9,794	9,4031	4,3066	1,8640	0,7903	0,3258	0,1289	0,0478	0,0158	0,0039
1 327	34	**2,3654**	7,19	22,531	14,890	14,2831	6,4401	**2,7420**	1,1436	0,4637	0,1805	0,0659	0,0214	0,0052
9 551	36	0,0645	9,16	21,626	12,956	12,3065	4,6049	1,6112	0,5522	0,1840	0,0589	0,0177	0,0047	0,0009
15 683	44	0,4650	9,66	25,785	15,069	14,2788	5,0972	1,6972	0,5536	0,1756	0,0534	0,0153	0,0039	0,0007
19 609	52	**0,7684**	9,88	30,135	17,416	16,4838	5,7609	1,8758	0,5983	0,1855	0,0552	0,0154	0,0038	0,0007
31 397	72	0,7399	10,35	40,756	23,006	21,7235	**7,2601**	2,2554	0,6863	0,2031	0,0577	0,0154	0,0036	0,0006
155 921	86	0,2038	11,96	44,934	23,412	21,9308	6,2948	1,6661	0,4320	0,1089	0,0263	0,0060	0,0012	0,0002
360 653	96	0,1835	12,80	48,099	24,033	22,4178	5,9420	1,4462	0,3448	0,0799	0,0178	0,0037	0,0007	0,0001
370 261	112	**5,5052**	12,82	56,042	27,964	26,0820	6,8959	1,6740	0,3981	0,0920	0,0204	0,0042	0,0008	0,0001
492 113	114	0,0709	13,11	56,237	27,665	25,7665	6,6309	1,5645	0,3616	0,0812	0,0175	0,0035	0,0006	0,0001
1 349 533	118	0,0534	14,12	55,347	25,888	23,9902	5,6096	1,1965	0,2500	0,0508	0,0099	0,0018	0,0003	0,0000
1 357 201	132	**18,7722**	14,12	61,896	28,943	26,8205	6,2680	1,3362	0,2790	0,0567	0,0110	0,0020	0,0003	0,0000
2 010 733	148	0,3326	14,51	68,048	31,201	28,8556	6,4965	1,3315	0,2673	0,0522	0,0098	0,0017	0,0003	0,0000
4 652 353	154	0,0686	15,35	67,899	29,853	27,4940	5,7158	1,0772	0,1989	0,0357	0,0062	0,0010	0,0001	0,0000
17 051 707	180	0,1986	16,65	74,372	30,643	28,0388	5,1524	0,8528	0,1383	0,0218	0,0033	0,0005	0,0001	0,0000
20 831 323	210	**0,7874**	16,85	85,903	35,042	**32,0315**	5,7752	0,9369	0,1489	0,0230	0,0034	0,0005	0,0001	0,0000
47 326 693	220	0,0812	17,67	86,376	33,819	30,7864	5,1344	0,7673	0,1123	0,0160	0,0022	0,0003	0,0000	0,0000
122 164 747	222	0,0147	18,62	83,125	31,039	28,1220	4,2860	0,5826	0,0776	0,0100	0,0012	0,0001	0,0000	0,0000
189 695 659	234	0,1441	19,06	85,711	31,308	28,3037	4,1371	0,5381	0,0686	0,0085	0,0010	0,0001	0,0000	0,0000
191 912 783	248	**4,8864**	19,07	90,786	33,142	29,9605	4,3744	0,5684	0,0723	0,0090	0,0011	0,0001	0,0000	0,0000
387 096 133	250	0,0159	19,77	88,363	31,146	28,0571	3,8324	0,4642	0,0551	0,0064	0,0007	0,0001	0,0000	0,0000
436 273 009	282	**1,0067**	19,89	99,080	34,715	31,2534	4,2207	0,5051	0,0592	0,0068	0,0007	0,0001	0,0000	0,0000
1 294 268 491	288	0,0314	20,98	95,833	31,800	28,4743	3,4680	0,3723	0,0392	0,0040	0,0004	0,0000	0,0000	0,0000
1 453 168 141	292	0,1253	21,10	96,603	31,871	28,5209	3,4357	0,3646	0,0379	0,0038	0,0004	0,0000	0,0000	0,0000
2 300 942 549	320	0,2375	21,56	103,461	33,358	29,7833	3,4345	0,3481	0,0346	0,0033	0,0003	0,0000	0,0000	0,0000
3 842 610 773	336	0,1187	22,07	105,884	33,275	29,6331	3,2546	0,3134	0,0296	0,0027	0,0002	0,0000	0,0000	0,0000
4 302 407 359	354	0,4758	22,18	110,928	34,664	30,8522	3,3524	0,3191	0,0298	0,0027	0,0002	0,0000	0,0000	0,0000
10 726 904 659	382	0,1224	23,10	114,357	34,140	30,2473	3,0134	0,2618	0,0223	0,0018	0,0001	0,0000	0,0000	0,0000
20 678 048 297	384	0,0108	23,75	111,245	32,138	28,3809	2,6566	0,2162	0,0172	0,0013	0,0001	0,0000	0,0000	0,0000
22 367 084 959	394	0,3361	23,83	113,694	32,717	28,8809	2,6833	0,2166	0,0171	0,0013	0,0001	0,0000	0,0000	0,0000
25 056 082 087	456	**1,2669**	23,94	130,841	**37,438**	33,0295	3,0358	0,2423	0,0189	0,0014	0,0001	0,0000	0,0000	0,0000
42 652 618 343	464	0,0418	24,48	129,642	36,122	31,7831	2,7773	0,2102	0,0156	0,0011	0,0001	0,0000	0,0000	0,0000
127 976 334 671	468	0,0128	25,58	123,769	32,642	28,5641	2,2486	0,1525	0,0101	0,0007	0,0000	0,0000	0,0000	0,0000
182 226 896 239	474	0,0425	25,93	123,160	31,912	27,8764	2,1220	0,1389	0,0089	0,0006	0,0000	0,0000	0,0000	0,0000
241 160 624 143	486	0,1010	26,21	124,522	31,816	27,7535	2,0571	0,1309	0,0082	0,0005	0,0000	0,0000	0,0000	0,0000
297 501 075 799	490	0,0431	26,42	124,235	31,412	27,3717	1,9888	0,1240	0,0076	0,0004	0,0000	0,0000	0,0000	0,0000
303 371 455 241	500	1,0336	26,44	126,647	31,990	27,8731	2,0215	0,1257	0,0077	0,0005	0,0000	0,0000	0,0000	0,0000
304 599 508 537	514	6,7558	26,44	130,167	32,872	28,6414	2,0764	0,1291	0,0079	0,0005	0,0000	0,0000	0,0000	0,0000
416 608 695 821	516	0,0144	26,76	128,643	31,983	27,8228	1,9579	0,1180	0,0070	0,0004	0,0000	0,0000	0,0000	0,0000
461 690 510 011	532	0,3080	26,86	131,953	32,638	28,3778	1,9776	0,1180	0,0069	0,0004	0,0000	0,0000	0,0000	0,0000
614 487 453 523	534	0,0151	27,14	130,569	31,837	27,6421	1,8747	0,1087	0,0062	0,0003	0,0000	0,0000	0,0000	0,0000
738 832 927 927	540	0,0660	27,33	130,825	31,607	27,4170	1,8272	0,1040	0,0058	0,0003	0,0000	0,0000	0,0000	0,0000
1 346 294 310 749	582	0,1599	27,93	136,833	32,081	27,7452	1,7466	0,0936	0,0049	0,0003	0,0000	0,0000	0,0000	0,0000
1 408 695 493 609	588	0,2304	27,97	137,931	32,266	27,8983	1,7487	0,0933	0,0049	0,0002	0,0000	0,0000	0,0000	0,0000
1 968 188 556 461	602	0,0818	28,31	138,872	31,947	27,5765	1,6744	0,0864	0,0044	0,0002	0,0000	0,0000	0,0000	0,0000
2 614 941 710 599	652	0,3101	28,59	148,286	33,631	28,9893	1,7134	0,0859	0,0042	0,0002	0,0000	0,0000	0,0000	0,0000
7 177 162 611 713	674	0,0513	29,60	145,746	31,426	26,9536	1,4473	0,0656	0,0029	0,0001	0,0000	0,0000	0,0000	0,0000
13 829 048 559 701	716	0,1220	30,26	149,821	31,266	26,7268	1,3485	0,0572	0,0024	0,0001	0,0000	0,0000	0,0000	0,0000
19 581 334 192 423	766	0,2222	30,61	157,522	32,307	27,5659	1,3457	0,0552	0,0022	0,0001	0,0000	0,0000	0,0000	0,0000
42 842 283 925 351	778	0,0284	31,39	153,857	30,339	25,7971	1,1689	0,0443	0,0016	0,0001	0,0000	0,0000	0,0000	0,0000
90 874 329 411 493	804	0,0612	32,14	153,141	29,080	24,6304	1,0390	0,0365	0,0013	0,0000	0,0000	0,0000	0,0000	0,0000
171 231 342 420 521	806	0,0053	32,77	148,645	27,356	23,0850	0,9169	0,0303	0,0010	0,0000	0,0000	0,0000	0,0000	0,0000
218 209 405 436 543	906	0,5127	33,02	165,125	30,002	25,2910	0,9818	0,0317	0,0010	0,0000	0,0000	0,0000	0,0000	0,0000

Table I.10

I.5.2 -Lower bound function of $F_x(p_n)$:

Our purpose is to build a lower bound function of $F_x(p_n)$ and to estimate the lowest prime $P_{n,max}$ where the upper bound function becomes lower than the maximal value of the lower bound function. It means that the maximal value of $F_x(p_n)$ will necessarily occur for a $p_n<P_{n,max}$. And if $P_{n,max}$ <218 209 405 436 543 we'll be able to compute and find the highest value of $F_x(p_n)$ in this interval and to conclude that this value is a global maximal value for $F_x(p_n)$.

Starting from the Taylor serie expansion of (5)

$$F_x(p_n) = x.p_n^x.\frac{g_n}{p_n}\left(1-\frac{(1-x)}{2}.\frac{g_n}{p_n}\right) + p_n^x.O\left(\frac{g_n}{p_n}\right)^2 \text{ with } O\left(\frac{g_n}{p_n}\right)^2 > 0$$

$$F_x(p_n) > x.p_n^x.\frac{g_n}{p_n}(1-\frac{(1-x)}{2}.\frac{g_n}{p_n})$$

As specified in the previous paragraph, we can limit our study to the prime numbers corresponding to the first occurrence of maximal gaps. As previously, P_n is the prime number associated to the first occurrence of the maximal gap G_n.

$$F_x(P_n) > x.P_n^x.\frac{G_n}{P_n}(1-\frac{(1-x)}{2}.\frac{G_n}{P_n}) \tag{I.29}$$

In 1931, Erik Westzynthius proved that maximal gaps grows more than logarithmically.

$$\text{Limsup } \frac{G_n}{\ln(P_n)} = \infty \quad \text{which imply that for n enough large } G_n > \ln(P_n) \tag{I.30}$$

Thanks to BHP result $G_n < P_n^a$ with a=0,525 and (I.30), we can define a lower bound function of $F_x(p_n)$:

$$F_x(P_n) > x.P_n^x.\frac{G_n}{P_n}\left(1-\frac{(1-x)}{2}.\frac{G_n}{P_n}\right) > x.p_n^x.\frac{\ln(P_n)}{P_n}\left(1-\frac{(1-x)}{2}.\frac{P_n^a}{P_n}\right) = m_x(P_n)$$

$$m_x(P_n) = x.\frac{\ln(P_n)}{P_n^{1-x}}.\left(1-\frac{(1-x)}{2P_n^{1-a}}\right)$$

We are going to estimate the first derivative of $m_x(P_n)$:

$$m'_x(P_n) = x\left(\frac{1}{P_n^{2-x}} - (1-x)\frac{\ln(P_n)}{P_n^{2-x}}\right).\left(1-\frac{(1-x)}{2P_n^{1-a}}\right) + x.\frac{\ln(P_n)}{P_n^{1-x}}.\frac{(1-x).(1-a)}{2P_n^{2-a}}$$

$$m'_x(P_n) = \frac{x}{2P_n^{3-a-x}}((1-(1-x)\ln(P_n)).\left(2P_n^{1-a} - (1-x)\right) + (1-x).(1-a).\ln(P_n))$$

$$m'_x(P_n) = \frac{x}{2P_n^{3-a-x}}((1-(1-x)\ln(P_n)).\left(2P_n^{1-a} - (1-x)\right) + (1-x).(1-a).\ln(P_n))$$

$$m'_x(P_n) = \frac{x}{2P_n^{3-a-x}}.\left(\left(2P_n^{1-a} - (1-x)\right) - (1-x)\left(2P_n^{1-a} - (2-a)\right).\ln(P_n)\right) \tag{I.31}$$

The maximal value of $m_x(P_n)$ is reached for $m'_x(P_{n,x}) = 0$, and $P_{n,x}$ satisfying to :

$$\ln(P_{n,x}) = \frac{2P_n^{1-a} - (1-x)}{(1-x)\left(2P_n^{1-a} - (2-a)\right)} = \frac{1}{(1-x)}.\frac{1-\frac{(1-x)}{2P_n^{1-a}}}{1-\frac{(2-a)}{2P_n^{1-a}}} \approx \frac{1}{(1-x)}.\frac{1-\frac{(1-x)}{2e^{(1-a)/(1-x)}}}{1-\frac{(2-a)}{2e^{(1-a)/(1-x)}}} > \frac{1}{(1-x)} \tag{I.32}$$

Thus $P_{n,x} = e^{\frac{1}{(1-x)} \cdot \frac{1-\frac{(1-x)}{2e^{(1-a)/(1-x)}}}{1-\frac{(2-a)}{2e^{(1-a)/(1-x)}}}} \approx e^{\frac{1}{(1-x)}} = P'_{n,x}$ In the table 8 below, we give for different x, the numerical values of $P_{n,x}$, $P'_{n,x}$, $m_x(P_{n,x})$, $m_x(P'_{n,x})$,. We can see that $\frac{|m_x(p_n)-m_x(p'_n)|}{m_x(p_n)} < 1.5\%$ for $0.1 \leq x \leq 0{,}9$

x	$P'_{n,x}\approx$	$m_x(P'_{n,x})$	$P_{n,x}\approx$	$m_x(P_{n,x})$	$\|m_x(P_{n,x})-m_x(P'_{n,x})\|/m_x(P_{n,x})$
0,1	3	0,030024624	4	0,030445391	1,38%
0,2	3	0,071653628	5	0,072093963	0,61%
0,3	4	0,129666555	7	0,129622113	0,03%
0,4	5	0,211916534	9	0,210941533	0,46%
0,5	7	0,332310923	13	0,330249272	0,62%
0,6	12	0,518160095	21	0,515412608	0,53%
0,7	28	0,831952717	45	0,829582868	0,29%
0,8	148	1,457830517	204	1,456915542	0,06%
0,9	22026	3,309482719	23385	3,309463809	0,00%

We can see in table I.11 above that the upper bound of the lower bound function $m_x(p_n)$ of $F_x(P_n)$ is correctly estimated if we compare with the upper bounds of $m_x(p_n)$ appearing in the table I.13 and I.15.

I.5.3 -Estimation of P_{no} such as for $P_n > P_{no}$ $M_x(P_n) < Max_{pn}\, m_x(P_n)$:

As specified above, the local maximal values of $F_x(P_n)$ are reached only for the prime numbers (P_n), where P_n is the first occurrence of a maximal gaps G_n. If we consider a maximal gap G_n, we define $\mathbb{P}_{G_n} = \{p_i \in \mathbb{P} \ / \ g_i = G_n\}$ and $P_n = Min\{p_i \in \mathbb{P}_{G_n}\}$ and $\mathbb{P}_{(G_n)} = \{P_n \in \mathbb{P}$ such as $P_n = Min\{p_i \in \mathbb{P}_{G_n}\}$ }. We consider $P_{n,k}$ such as $F_x(P_{n,k}) = Max_{P_n \in \mathbb{P}_{(G_n)}} F_x(P_n)$. Our approach is to show that there is one P_{no} enough large such as for all $P_n > P_{no}$, $F_x(P_n) < M_x(P_n) < Max_{pn}\, m_x(P_n) = m_x(P'_{n,x})$.

In the previous paragraph, we have shown that $F_x(P_{n,k})$ was correctly estimated by $F_x(P'_{n,k}) = F_x\left(e^{\frac{1}{(1-x)}}\right)$.

We are going to search the best value of $\alpha = \alpha(x) > 1$, such as :

$$M_x\left(e^{\frac{\alpha}{(1-x)}}\right) < m_x\left(e^{\frac{1}{(1-x)}}\right) \quad \text{and} \quad P_{no} = e^{\frac{\alpha}{(1-x)}} \tag{I.33}$$

We can consider the two different cases define at the beginning of this study :

For $0<x<0{,}475$, then $G_n < P_n^{\,a}$ *with* $a = 0{,}525$

$$x \,.\, P_n^{\,x} . \frac{G_n}{P_n} < x \,.\, P_n^{\,x} . \frac{P_n^{\,a}}{P_n} < x . \frac{\ln(P'_{n,x})}{P'^{\,1-x}_{n,x}} . \left(1 - \frac{(1-x)}{2P'^{\,1-a}_{n,x}}\right)$$

$$\frac{1}{P_n^{\,1-x-a}} < \frac{\ln(P'_{n,x})}{P'^{\,1-x}_{n,x}} . \left(1 - \frac{(1-x)}{2P'^{\,1-a}_{n,x}}\right)$$

And we can replace P_n and $P'_{n,x}$ by their expression defined in (33) :

$$\frac{1}{e^{\frac{k(1-x-a)}{1-x}}} < \frac{\ln\left(e^{\frac{1}{(1-x)}}\right)}{e} . \left(1 - \frac{(1-x)}{2.e^{\frac{(1-a)}{(1-x)}}}\right)$$

And finally :

$$\alpha > \frac{1-x}{1-a-x} . \ln\left(\frac{e(1-x)}{1 - \frac{(1-x)}{2.} . e^{\frac{(a-1)}{(1-x)}}}\right) \tag{I.34}$$

Table I.12 below, gives the for value of $\alpha(x)$ for different values of x<0,475

x	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45
α	2,88	2,89	2,92	2,99	3,11	3,36	3,84	5,08	11,56

Table I.12

Thanks to that, we can now estimate the values of $P_{no} = e^{\frac{\alpha}{(1-x)}}$ and we 'll compare with the numerical values of table I.13

For x=0.1, α =2.89 and $p_{no,x} = e^{\frac{2,89}{(1-x)}} \approx 25$. So we can see in the table below that $F_x(p_n) \leq 0,059$ for all $P_n < 25$ and for $P_n > 25$, we have proved here that $F_x(p_n) < M_x(p_n) < Max_{pn}\, m_x(p_n) = m_x(e^{\frac{1}{(1-x)}})$. The nearest and lowest prime satisfying to this condition is $P_n = 89$

For x=0.2, α =2,99 and $p_{no,x} = e^{\frac{2,99}{0,8}} \approx 42$. So we can see in the table below that $F_x(P_n) < 0,14$ for all $P_n \leq 42$ and for $P_n > 42$, we have proved here that $F_x(p_n) < M_x(p_n) < Max_{pn}\, m_x(p_n) = m_x(e^{\frac{1}{(1-x)}})$. The nearest and lowest prime satisfying to this condition is $P_n = 89$

For x=0.3, α =3,36 and $p_{no,x} = e^{\frac{3,36}{0,7}} \approx 121$. So we can see in the table below that $F_x(p_n) < 0,27$ for all $P_n \leq 121$ and for $P_n > 121$, we have proved here that $F_x(p_n) < M_x(p_n) < Max_{pn}\, m_x(p_n) = m_x(e^{\frac{1}{(1-x)}})$. The closest and lowest prime satisfying to this condition is $P_n = 523$.

For x=0.4, α =5,08 and $p_{no,x} = e^{\frac{5,08}{0,6}} \approx 4753$. So we can see in the table below that $F_x(p_n) < 0,44$ for all $P_n \leq 4753$ and for $P_n > 4753$, we have proved here that $F_x(p_n) < M_x(p_n) < Max_{pn}\, m_x(p_n) = m_x(e^{\frac{1}{(1-x)}})$. The closest and lowest prime satisfying to this condition is $P_n = 9551$.

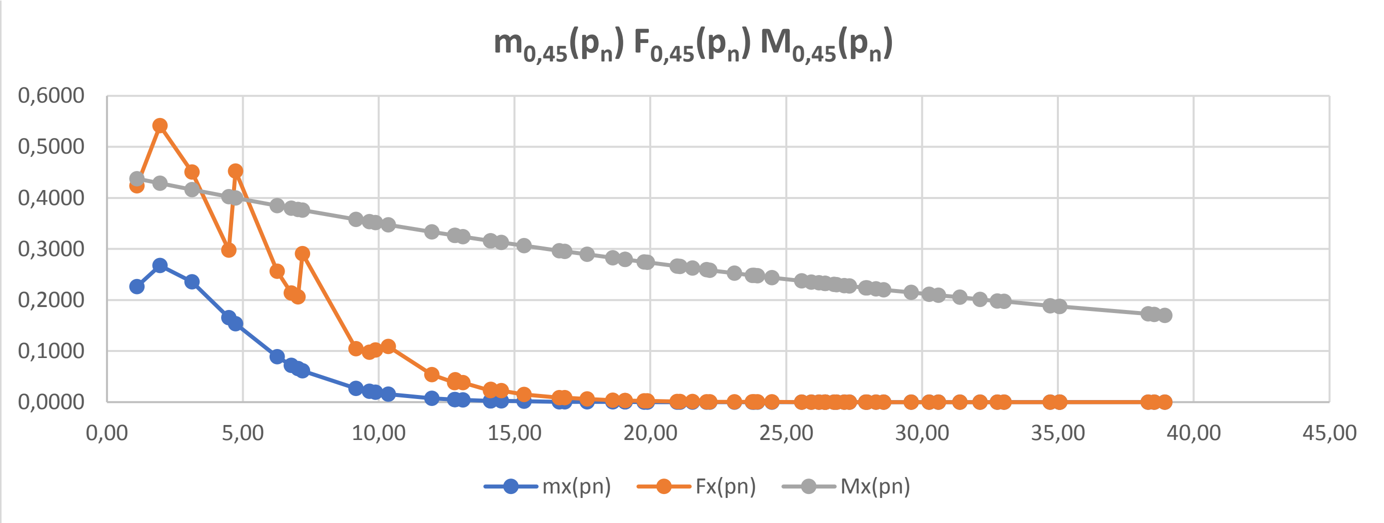
For x=0.45, α =11,56 and $p_{no,x} = e^{\frac{11,56}{0,55}} \approx 143\,397\,866\,765$. So we can see in the table below that $F_x(p_n) < 0,55$ for all $P_n \leq 143\,397\,866\,765$ and for $P_n > 143\,397\,866\,765$, we have proved here that $F_x(p_n) < M_x(p_n) < Max_{pn}$ $m_x(p_n) = m_x(e^{\frac{1}{(1-x)}})$. The closest and lowest prime satisfying to this condition is $P_n = 1\,294\,268\,491$.

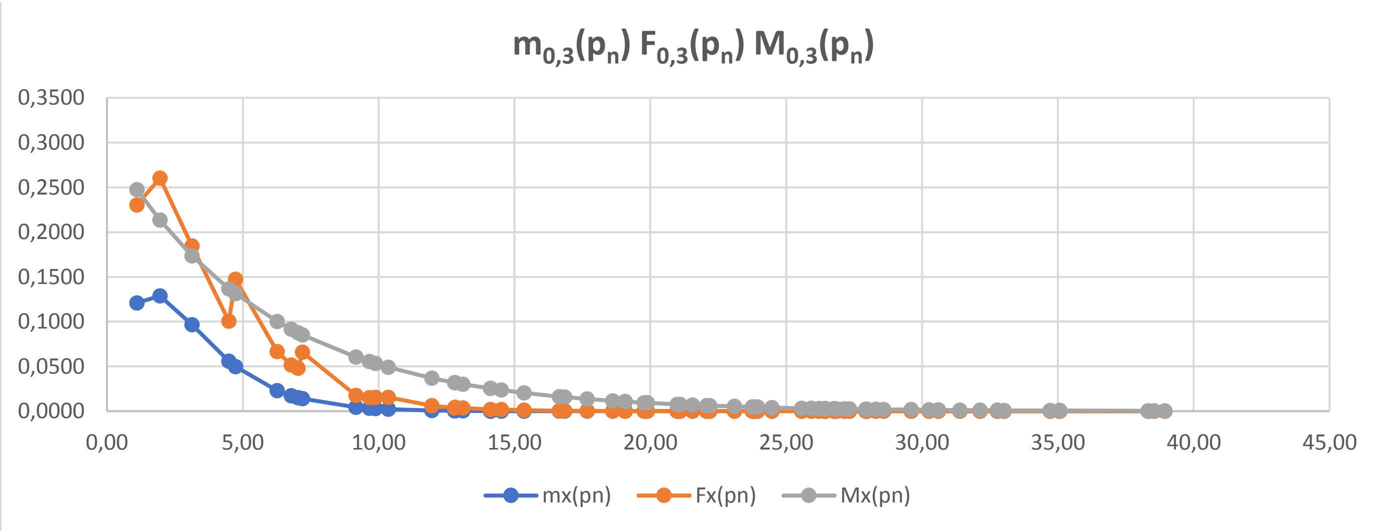
Remark: The value of $P_{no,x}$ obtained for x=0,45 seems to be over-estimated but it's due to the fact that the formulation of α contains an undefined for x = 1-a. This is only due to the fact we use the BHP result as upper bound for (G_n). For all x values not to close from 1-a, the estimation of $P_{no,x}$ is quite good and totally coherent with the numerical results of table I.13 below.

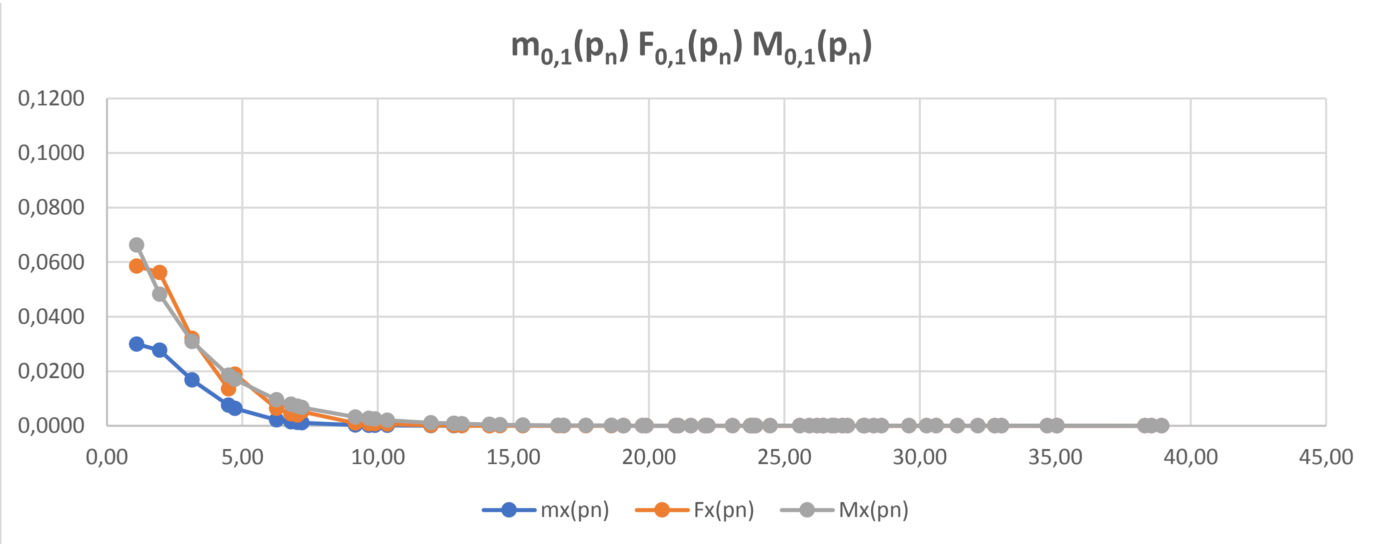
We can conclude that for all x<0.475, $F_x(p_n)$ is a bounded function who tends toward 0 when $p_n \to \infty$ and the upper bound of $F_x(p_n)$ for every x is reached for $p_n < 218\,209\,405\,436\,543$ and thus can be estimated numerically for all x<0.45.

Primes	Gaps	Ln(Pn)	$m_x(p_n)$	$F_x(p_n)$	$M_x(p_n)$	$m_x(p_n)$	$F_x(p_n)$	$M_x(p_n)$	$m_x(p_n)$	$F_x(p_n)$	$M_x(p_n)$	$m_x(p_n)$	$F_x(p_n)$	$M_x(p_n)$	$m_x(p_n)$	$F_x(p_n)$	$M_x(p_n)$
				X= 0,450			X= 0,400			X= 0,300			X= 0,200			X= 0,100	
2	1	0,69															
3	2	1,10	0,2261	0,424	0,4378	0,1868	0,3518	0,3684	0,1210	0,2303	0,2475	0,0696	0,1340	0,1479	0,0300	0,0585	0,0662
7	4	1,95	0,2675	0,541	0,4286	0,2133	0,4316	0,3457	0,1287	0,2603	0,2134	0,0690	0,1396	0,1171	0,0277	0,0562	0,0482
23	6	3,14	0,2359	0,451	0,4161	0,1782	0,3405	0,3162	0,0965	0,1845	0,1733	0,0464	0,0888	0,0844	0,0168	0,0321	0,0309
89	8	4,49	0,1655	0,298	0,4022	0,1172	0,2110	0,2857	0,0558	0,1006	0,1368	0,0236	0,0426	0,0582	0,0075	0,0135	0,0186
113	14	4,73	0,1534	0,453	0,3998	0,1074	0,3169	0,2806	0,0499	0,1473	0,1312	0,0206	0,0608	0,0545	0,0064	0,0188	0,0170
523	18	6,26	0,0888	0,257	0,3848	0,0576	0,1667	0,2501	0,0231	0,0667	0,1003	0,0082	0,0237	0,0358	0,0022	0,0063	0,0096
887	20	6,79	0,0722	0,214	0,3798	0,0457	0,1353	0,2404	0,0173	0,0514	0,0915	0,0059	0,0174	0,0309	0,0015	0,0044	0,0078
1 129	22	7,03	0,0656	0,206	0,3775	0,0410	0,1289	0,2361	0,0152	0,0478	0,0877	0,0050	0,0158	0,0289	0,0012	0,0039	0,0072
1 327	34	7,19	0,0614	0,291	0,3760	0,0381	0,1805	0,2333	0,0139	0,0659	0,0852	0,0045	0,0214	0,0277	0,0011	0,0052	0,0067
9 551	36	9,16	0,0266	0,105	0,3579	0,0149	0,0589	0,2012	0,0045	0,0177	0,0603	0,0012	0,0047	0,0161	0,0002	0,0009	0,0032
15 683	44	9,66	0,0214	0,097	0,3534	0,0117	0,0534	0,1938	0,0033	0,0153	0,0553	0,0008	0,0039	0,0140	0,0002	0,0007	0,0027
19 609	52	9,88	0,0193	0,102	0,3515	0,0105	0,0552	0,1906	0,0029	0,0154	0,0532	0,0007	0,0038	0,0132	0,0001	0,0007	0,0025
31 397	72	10,35	0,0156	0,109	0,3474	0,0083	0,0577	0,1840	0,0022	0,0154	0,0490	0,0005	0,0036	0,0116	0,0001	0,0006	0,0021
155 921	86	11,96	0,0075	0,054	0,3337	0,0037	0,0263	0,1632	0,0008	0,0060	0,0370	0,0002	0,0012	0,0075	0,0000	0,0002	0,0011
360 653	96	12,80	0,0051	0,038	0,3268	0,0024	0,0178	0,1532	0,0005	0,0037	0,0320	0,0001	0,0007	0,0059	0,0000	0,0001	0,0008
370 261	112	12,82	0,0050	0,044	0,3266	0,0023	0,0204	0,1529	0,0005	0,0042	0,0318	0,0001	0,0008	0,0059	0,0000	0,0001	0,0008
492 113	114	13,11	0,0044	0,038	0,3243	0,0020	0,0175	0,1497	0,0004	0,0035	0,0303	0,0001	0,0006	0,0054	0,0000	0,0001	0,0007
1 349 533	118	14,12	0,0027	0,023	0,3162	0,0012	0,0099	0,1388	0,0002	0,0018	0,0254	0,0000	0,0003	0,0041	0,0000	0,0000	0,0005
1 357 201	132	14,12	0,0027	0,025	0,3162	0,0012	0,0110	0,1387	0,0002	0,0020	0,0253	0,0000	0,0003	0,0041	0,0000	0,0000	0,0005
2 010 733	148	14,51	0,0022	0,023	0,3131	0,0010	0,0098	0,1347	0,0002	0,0017	0,0237	0,0000	0,0003	0,0037	0,0000	0,0000	0,0004
4 652 353	154	15,35	0,0015	0,015	0,3066	0,0006	0,0062	0,1265	0,0001	0,0010	0,0204	0,0000	0,0001	0,0029	0,0000	0,0000	0,0003
17 051 707	180	16,65	0,0008	0,009	0,2968	0,0003	0,0033	0,1147	0,0000	0,0005	0,0163	0,0000	0,0001	0,0021	0,0000	0,0000	0,0002
20 831 323	210	16,85	0,0007	0,009	0,2953	0,0003	0,0034	0,1130	0,0000	0,0005	0,0157	0,0000	0,0001	0,0019	0,0000	0,0000	0,0002
47 326 693	220	17,67	0,0005	0,006	0,2893	0,0002	0,0022	0,1063	0,0000	0,0003	0,0136	0,0000	0,0000	0,0016	0,0000	0,0000	0,0001
122 164 747	222	18,62	0,0003	0,004	0,2825	0,0001	0,0012	0,0990	0,0000	0,0001	0,0115	0,0000	0,0000	0,0012	0,0000	0,0000	0,0001
189 695 659	234	19,06	0,0002	0,003	0,2794	0,0001	0,0010	0,0958	0,0000	0,0001	0,0107	0,0000	0,0000	0,0011	0,0000	0,0000	0,0001
191 912 783	248	19,07	0,0002	0,003	0,2793	0,0001	0,0011	0,0957	0,0000	0,0001	0,0107	0,0000	0,0000	0,0011	0,0000	0,0000	0,0001
387 096 133	250	19,77	0,0002	0,002	0,2745	0,0001	0,0007	0,0908	0,0000	0,0001	0,0094	0,0000	0,0000	0,0009	0,0000	0,0000	0,0001
436 273 009	282	19,89	0,0002	0,002	0,2737	0,0001	0,0007	0,0900	0,0000	0,0001	0,0092	0,0000	0,0000	0,0008	0,0000	0,0000	0,0001
1 294 268 491	288	20,98	0,0001	0,001	0,2663	0,0000	0,0004	0,0829	0,0000	0,0000	0,0076	0,0000	0,0000	0,0006	0,0000	0,0000	0,0000
1 453 168 141	292	21,10	0,0001	0,001	0,2656	0,0000	0,0004	0,0822	0,0000	0,0000	0,0075	0,0000	0,0000	0,0006	0,0000	0,0000	0,0000
2 300 942 549	320	21,56	0,0001	0,001	0,2625	0,0000	0,0003	0,0794	0,0000	0,0000	0,0069	0,0000	0,0000	0,0005	0,0000	0,0000	0,0000
3 842 610 773	336	22,07	0,0001	0,001	0,2592	0,0000	0,0002	0,0764	0,0000	0,0000	0,0063	0,0000	0,0000	0,0005	0,0000	0,0000	0,0000
4 302 407 359	354	22,18	0,0001	0,001	0,2584	0,0000	0,0002	0,0758	0,0000	0,0000	0,0062	0,0000	0,0000	0,0004	0,0000	0,0000	0,0000
10 726 904 659	382	23,10	0,0000	0,001	0,2526	0,0000	0,0001	0,0708	0,0000	0,0000	0,0053	0,0000	0,0000	0,0003	0,0000	0,0000	0,0000
20 678 048 297	384	23,75	0,0000	0,000	0,2485	0,0000	0,0001	0,0674	0,0000	0,0000	0,0047	0,0000	0,0000	0,0003	0,0000	0,0000	0,0000
22 367 084 959	394	23,83	0,0000	0,000	0,2480	0,0000	0,0001	0,0670	0,0000	0,0000	0,0046	0,0000	0,0000	0,0003	0,0000	0,0000	0,0000
25 056 082 087	456	23,94	0,0000	0,000	0,2473	0,0000	0,0001	0,0664	0,0000	0,0000	0,0045	0,0000	0,0000	0,0003	0,0000	0,0000	0,0000
42 652 618 343	464	24,48	0,0000	0,000	0,2440	0,0000	0,0001	0,0638	0,0000	0,0000	0,0041	0,0000	0,0000	0,0002	0,0000	0,0000	0,0000
127 976 334 671	468	25,58	0,0000	0,000	0,2374	0,0000	0,0000	0,0588	0,0000	0,0000	0,0034	0,0000	0,0000	0,0002	0,0000	0,0000	0,0000
182 226 896 239	474	25,93	0,0000	0,000	0,2353	0,0000	0,0000	0,0572	0,0000	0,0000	0,0032	0,0000	0,0000	0,0002	0,0000	0,0000	0,0000
241 160 624 143	486	26,21	0,0000	0,000	0,2337	0,0000	0,0000	0,0560	0,0000	0,0000	0,0031	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000
297 501 075 799	490	26,42	0,0000	0,000	0,2325	0,0000	0,0000	0,0552	0,0000	0,0000	0,0029	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000
303 371 455 241	500	26,44	0,0000	0,000	0,2324	0,0000	0,0000	0,0551	0,0000	0,0000	0,0029	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000
304 599 508 537	514	26,44	0,0000	0,000	0,2323	0,0000	0,0000	0,0551	0,0000	0,0000	0,0029	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000
416 608 695 821	516	26,76	0,0000	0,000	0,2305	0,0000	0,0000	0,0538	0,0000	0,0000	0,0028	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000
461 690 510 011	532	26,86	0,0000	0,000	0,2299	0,0000	0,0000	0,0534	0,0000	0,0000	0,0027	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000
614 487 453 523	534	27,14	0,0000	0,000	0,2283	0,0000	0,0000	0,0522	0,0000	0,0000	0,0026	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000
738 832 927 927	540	27,33	0,0000	0,000	0,2272	0,0000	0,0000	0,0515	0,0000	0,0000	0,0025	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000
1 346 294 310 749	582	27,93	0,0000	0,000	0,2239	0,0000	0,0000	0,0492	0,0000	0,0000	0,0023	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000
1 408 695 493 609	588	27,97	0,0000	0,000	0,2236	0,0000	0,0000	0,0491	0,0000	0,0000	0,0022	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000
1 968 188 556 461	602	28,31	0,0000	0,000	0,2217	0,0000	0,0000	0,0479	0,0000	0,0000	0,0021	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000
2 614 941 710 599	652	28,59	0,0000	0,000	0,2202	0,0000	0,0000	0,0469	0,0000	0,0000	0,0020	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000
7 177 162 611 713	674	29,60	0,0000	0,000	0,2147	0,0000	0,0000	0,0434	0,0000	0,0000	0,0017	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000
13 829 048 559 701	716	30,26	0,0000	0,000	0,2112	0,0000	0,0000	0,0414	0,0000	0,0000	0,0015	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
19 581 334 192 423	766	30,61	0,0000	0,000	0,2094	0,0000	0,0000	0,0403	0,0000	0,0000	0,0014	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
42 842 283 925 351	778	31,39	0,0000	0,000	0,2053	0,0000	0,0000	0,0380	0,0000	0,0000	0,0012	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
90 874 329 411 493	804	32,14	0,0000	0,000	0,2015	0,0000	0,0000	0,0359	0,0000	0,0000	0,0011	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
171 231 342 420 521	806	32,77	0,0000	0,000	0,1983	0,0000	0,0000	0,0342	0,0000	0,0000	0,0010	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
218 209 405 436 543	906	33,02	0,0000	0,000	0,1971	0,0000	0,0000	0,0336	0,0000	0,0000	0,0009	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
1 189 459 969 825 480	916	34,71	0,0000	0,000	0,1889	0,0000	0,0000	0,0296	0,0000	0,0000	0,0007	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
1 686 994 940 955 800	924	35,06	0,0000	0,000	0,1873	0,0000	0,0000	0,0288	0,0000	0,0000	0,0006	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
1 693 182 318 746 370	1132	35,07	0,0000	0,000	0,1873	0,0000	0,0000	0,0288	0,0000	0,0000	0,0006	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
43 841 547 845 541 000	1184	38,32	0,0000	0,000	0,1726	0,0000	0,0000	0,0226	0,0000	0,0000	0,0004	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
55 350 776 431 903 200	1198	38,55	0,0000	0,000	0,1716	0,0000	0,0000	0,0222	0,0000	0,0000	0,0004	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000

Table I.13







We can see on the graphics and table above that $F_x(p_n)$ is under $M_x(p_n)$ for all $p_n > 113$ and independently of the x. We can also see in the three graphics above, that the upper bound functions $M_x(p_n)$ (in grey) becomes always lower than the maximal value of $m_x(p_n)$ (in blue).

For 0,475≤x<1, then from Cramer's conjecture, $G_n < \ln^2(P_n)$
(we could also use the weaker condition where we suppose there is r∊ℝ such as $G_n < \ln^r(P_n)$)

$$x.P_n^{\ x}.\frac{G_n}{P_n} < x.P_n^{\ x}.\frac{\ln^2(P_n)}{P_n} < x.\frac{\ln(P_{n,x})}{P_{n,x}^{\ 1-x}}.\left(1-\frac{(1-x)}{2P_{n,x}^{\ 1-a}}\right)$$

$$\frac{\ln^2(P_n)}{P_n^{\ 1-x}} < \frac{\ln(P'_{n,x})}{P'_{n,x}{}^{1-x}}.\left(1-\frac{(1-x)}{2P'_{n,x}{}^{1-a}}\right)$$

And we can replace p_n and $p'_{n,x}$ by their literal expression :

$$\frac{\ln^2\left(e^{\frac{\alpha}{(1-x)}}\right)}{e^{\alpha}} < \frac{\ln\left(e^{\frac{1}{(1-x)}}\right)}{e}.\left(1-\frac{(1-x)}{2.e^{\frac{(1-a)}{(1-x)}}}\right)$$

$$1 < (1-x)\frac{e^{\alpha-1}}{\alpha^2}.\left(1-\frac{(1-x)}{2.e^{\frac{(1-a)}{(1-x)}}}\right) \qquad (I.35)$$

As (I.35) is an implicite function of α, we are going to estimate α, in the table below

α\x	0,5	0,55	0,6	0,65	0,7	0,75	0,8	0,85	0,9	0,95
5,2000	0,9170	0,8406	0,7629	0,6832	0,6009	0,5151	0,4250	0,3296	0,2278	0,1184
5,3396	**1,0000**	0,9167	0,8319	0,7450	0,6552	0,5617	0,4634	0,3594	0,2484	0,1291
5,4776	1,0909	**1,0000**	0,9075	0,8127	0,7148	0,6127	0,5055	0,3920	0,2709	0,1408
5,6293	1,2020	1,1019	**1,0000**	0,8956	0,7876	0,6752	0,5571	0,4320	0,2986	0,1551
5,7990	1,3422	1,2304	1,1167	**1,0000**	0,8795	0,7539	0,6220	0,4824	0,3334	0,1732
5,9934	1,5262	1,3991	1,2697	1,1371	**1,0000**	0,8573	0,7073	0,5485	0,3791	0,1970
6,2224	1,7803	1,6320	1,4811	1,3264	1,1665	**1,0000**	0,8250	0,6398	0,4422	0,2298
6,5029	2,1578	1,9781	1,7952	1,6077	1,4139	1,2120	**1,0000**	0,7755	0,5359	0,2785
6,8657	2,7824	2,5507	2,3148	2,0730	1,8232	1,5629	1,2895	**1,0000**	0,6911	0,3591
7,3796	4,0263	3,6910	3,3497	2,9998	2,6383	2,2616	1,8659	1,4470	**1,0000**	0,5196
8,2595	7,7483	7,1029	6,4462	5,7728	5,0771	4,3522	3,5908	2,7846	1,9244	**1,0000**

Table I.14

For x=0.5, α≈5.3396 and $p_{no,x} = e^{\frac{5.3396}{0,5}} \approx 43442$. So we can see in the table below that $F_x(p_n)$<0,671 for all $p_n \leq 43442$ and for $p_n > 43442$, we have proved here that $F_x(p_n) < M_x(p_n) < Max_{pn}\, m_x(p_n) = m_x(e^{\frac{1}{(1-x)}})$. The closest prime value satisfying this condition is $P_n = 31397$

For x=0.6, α≈5.6293 and $p_{no,x} = e^{\frac{5.6293}{0,4}} \approx 1\,294\,001$. So we can see in the table below that $F_x(p_n)$<1,24 for all $p_n \leq 1\,294\,001$ and for $p_n > 1\,294\,001$, we have proved here that $F_x(p_n) < M_x(p_n) < Max_{pn}\, m_x(p_n) = m_x(e^{\frac{1}{(1-x)}})$. The closest and lowest prime value satisfying this condition is $P_n = 1349\,533$

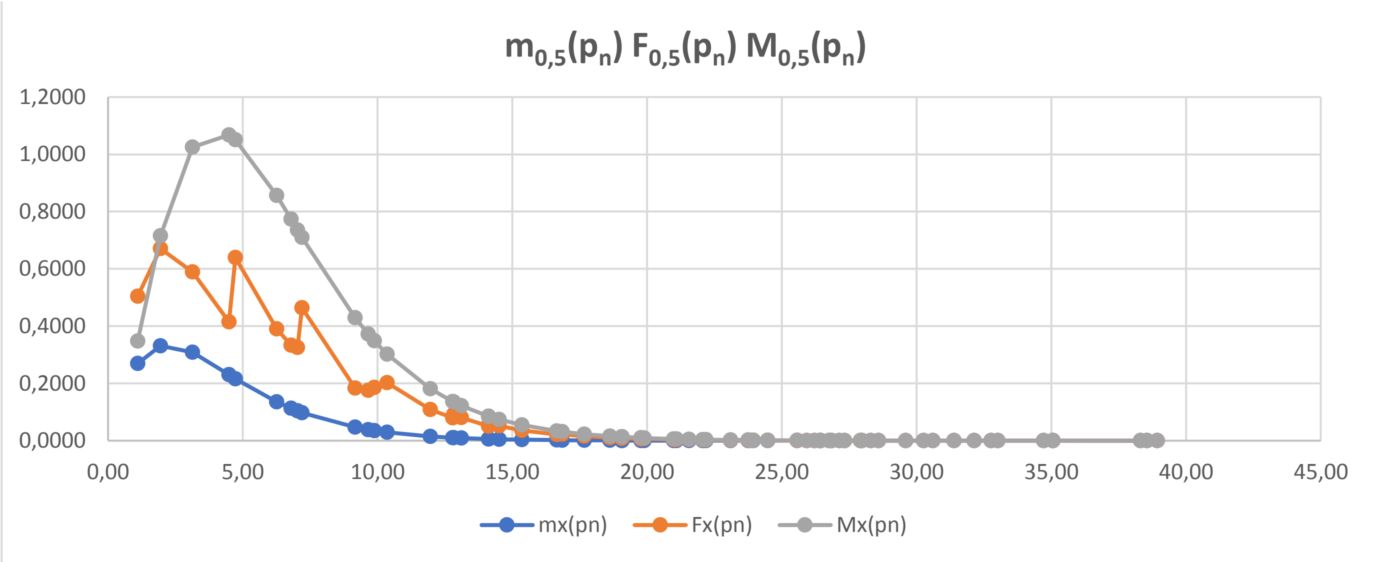
For x=0.7, α≈5.9934 and $p_{no,x} = e^{\frac{5.9934}{0,3}} \approx 474\,608\,114$. So we can see in the table below that $F_x(p_n)$<2,75 for all $p_n \leq 474\,608\,114$ and for $p_n > 474\,608\,114$, we have proved here that $F_x(p_n) < M_x(p_n) < Max_{pn}\, m_x(p_n) = m_x(e^{\frac{1}{(1-x)}})$. The closest and lowest prime value satisfying this condition is $P_n = 387\,096\,133$

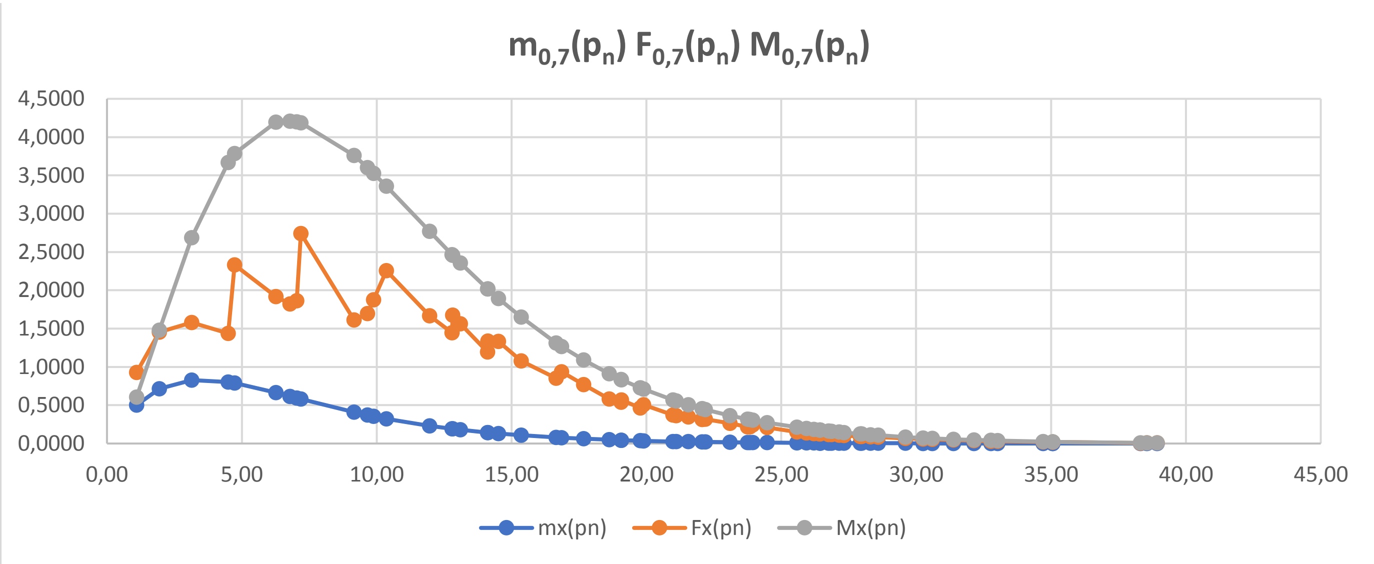
For x=0.8, α≈6.5029 and $p_{no,x} = e^{\frac{6.5029}{0,2}} \approx 132\,089\,389\,169\,128$. So we can see in the table below that $F_x(p_n)$<2,75 for all $p_n \leq 132\,089\,389\,169\,128$. and for $p_n > 132\,089\,389\,169\,128$., we have proved here that $F_x(p_n) < M_x(p_n) < Max_{pn}\, m_x(p_n) = m_x(e^{\frac{1}{(1-x)}})$. The closest and lowest prime value satisfying this condition is P_n= 90 874 329 411 493

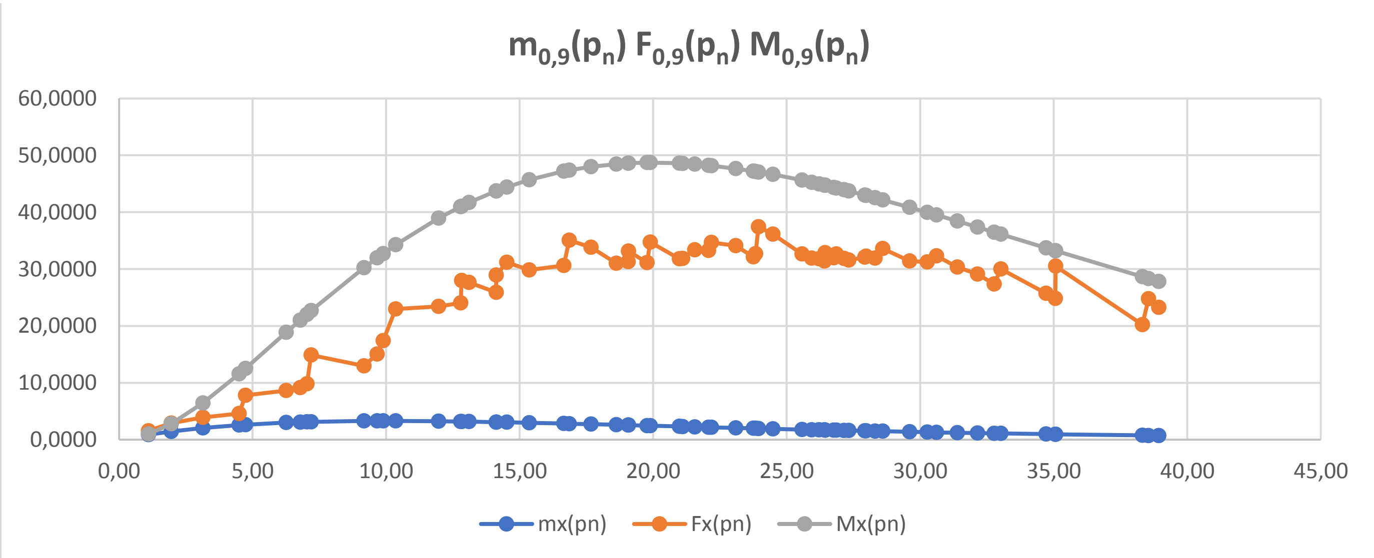
We can conclude that for all 0,475<x<1, under Cramer's conjecture , $F_x(p_n)$ is a bounded function who tends toward 0 when $p_n \to \infty$ and the upper bound of $F_x(p_n)$ for every x is reached for $p_n < 218\,209\,405\,436\,543$ and thus can be estimated numerically for all 0,475<x<0,8.

Primes	Gaps	ln((Pn)	$m_x(p_n)$	$F_x(p_n)$	$M_x(p_n)$	$m_x(p_n)$	$F_x(p_n)$	$M_x(p_n)$	$m_x(p_n)$	$F_x(p_n)$	$M_x(p_n)$	$m_x(p_n)$	$F_x(p_n)$	$M_x(p_n)$	$m_x(p_n)$	$F_x(p_n)$	$M_x(p_n)$
				X= 0,900			X= 0,800			X= 0,700			X= 0,600			X= 0,500	
2	1	0,69															
3	2	1,10	0,8596	1,569	0,9732	0,6637	1,2157	0,7751	0,5039	0,9275	0,6076	0,3744	0,6933	0,4667	0,2701	0,5040	0,3484
7	4	1,95	1,4130	2,893	2,8053	1,0130	2,0662	2,0527	0,7146	1,4531	1,4785	0,4935	1,0013	1,0432	**0,3313**	**0,6709**	0,7156
23	6	3,14	2,0392	3,899	6,4667	1,3096	2,5031	4,2010	**0,8278**	1,5817	2,6865	**0,5125**	0,9792	1,6830	0,3085	0,5893	1,0250
89	8	4,49	2,5635	4,576	11,5753	1,4459	2,5853	6,5681	0,8028	1,4378	3,6687	0,4366	0,7833	2,0074	0,2308	0,4149	**1,0678**
113	14	4,73	2,6379	7,807	12,5365	**1,4537**	4,2997	6,9457	0,7886	**2,3311**	3,7881	0,4190	**1,2381**	**2,0238**	0,2165	0,6393	1,0512
523	18	6,26	3,0049	8,648	18,8574	1,4247	4,1038	8,9636	0,6649	1,9169	4,1941	0,3040	0,8771	1,9224	0,1351	0,3902	0,8567
887	20	6,79	3,0926	9,120	21,0336	1,3916	4,1074	**9,4835**	0,6164	1,8209	**4,2091**	0,2675	0,7908	1,8300	0,1128	0,3339	0,7735
1 129	22	7,03	3,1268	9,794	22,0177	1,3737	4,3066	9,6906	0,5941	1,8640	4,1984	0,2517	0,7903	1,7819	0,1037	0,3258	0,7352
1 327	34	7,19	3,1478	14,890	22,6723	1,3610	6,4401	9,8187	0,5792	**2,7420**	4,1858	0,2415	1,1436	1,7480	0,0979	0,4637	0,7097
9 551	36	9,16	3,2966	12,956	30,2305	1,1712	4,6049	10,7470	0,4096	1,6112	3,7609	0,1403	0,5522	1,2893	0,0467	0,1840	0,4297
15 683	44	9,66	3,3073	15,069	31,9657	1,1183	5,0972	10,8140	0,3722	1,6972	3,6013	0,1214	0,5536	1,1748	0,0385	0,1756	0,3726
19 609	52	9,88	**3,3092**	17,416	32,7220	1,0943	5,7609	10,8253	0,3562	1,8758	3,5254	0,1136	0,5983	1,1246	0,0352	0,1855	0,3488
31 397	72	10,35	3,3077	23,006	34,2617	1,0436	**7,2601**	10,8135	0,3241	2,2554	3,3596	0,0986	0,6863	1,0225	0,0292	0,2031	0,3025
155 921	86	11,96	3,2546	23,412	38,9228	0,8750	6,2948	10,4655	0,2315	1,6661	2,7700	0,0600	0,4320	0,7182	0,0151	0,1089	0,1810
360 653	96	12,80	3,2029	24,033	40,9883	0,7918	5,9420	10,1344	0,1927	1,4462	2,4666	0,0459	0,3448	0,5881	0,0106	0,0799	0,1363
370 261	112	12,82	3,2011	27,964	41,0488	0,7893	6,8959	10,1227	0,1916	1,6740	2,4573	0,0456	0,3981	0,5843	0,0105	0,0920	0,1351
492 113	114	13,11	3,1804	27,665	41,6876	0,7622	6,6309	9,9919	0,1798	1,5645	2,3575	0,0416	0,3616	0,5449	0,0093	0,0812	0,1224
1 349 533	118	14,12	3,0966	25,888	43,7122	0,6709	5,6096	9,4718	0,1431	1,1965	2,0203	0,0299	0,2500	0,4221	0,0061	0,0508	0,0858
1 357 201	132	14,12	3,0961	28,943	43,7225	0,6705	6,2680	9,4687	0,1429	1,3362	2,0185	0,0298	0,2790	0,4215	0,0061	0,0567	0,0856
2 010 733	148	14,51	3,0597	31,201	44,4101	0,6370	6,4965	9,2469	0,1306	1,3315	1,8953	0,0262	0,2673	0,3805	0,0051	0,0522	0,0743
4 652 353	154	15,35	2,9761	29,853	45,6936	0,5698	5,7158	8,7485	0,1074	1,0772	1,6488	0,0198	0,1989	0,3044	0,0036	0,0357	0,0546
17 051 707	180	16,65	2,8348	30,643	47,2048	0,4766	5,1524	7,9370	0,0789	0,8528	1,3137	0,0128	0,1383	0,2130	0,0020	0,0218	0,0336
20 831 323	210	16,85	2,8120	35,042	47,3884	0,4634	5,7752	7,8099	0,0752	0,9369	1,2670	0,0119	0,1489	0,2014	0,0018	0,0230	0,0311
47 326 693	220	17,67	2,7166	33,819	48,0100	0,4124	5,1344	7,2890	0,0616	0,7673	1,0893	0,0090	0,1123	0,1595	0,0013	0,0160	0,0227
122 164 747	222	18,62	2,6034	31,039	48,4784	0,3595	4,2860	6,6942	0,0489	0,5826	0,9099	0,0065	0,0776	0,1212	0,0008	0,0100	0,0157
189 695 659	234	19,06	2,5502	31,308	48,6099	0,3370	4,1371	6,4234	0,0438	0,5381	0,8355	0,0056	0,0686	0,1065	0,0007	0,0085	0,0132
191 912 783	248	19,07	2,5488	33,142	48,6127	0,3364	4,3744	6,4163	0,0437	0,5684	0,8337	0,0056	0,0723	0,1061	0,0007	0,0090	0,0131
387 096 133	250	19,77	2,4635	31,146	48,7144	0,3031	3,8324	5,9941	0,0367	0,4642	0,7260	0,0044	0,0551	0,0861	0,0005	0,0064	0,0099
436 273 009	282	19,89	2,4490	34,715	**48,7193**	0,2978	4,2207	5,9234	0,0356	0,5051	0,7089	0,0042	0,0592	0,0831	0,0005	0,0068	0,0095
1 294 268 491	288	20,98	2,3167	31,800	48,6073	0,2526	3,4680	5,3009	0,0271	0,3723	0,5691	0,0029	0,0392	0,0598	0,0003	0,0040	0,0061
1 453 168 141	292	21,10	2,3027	31,871	48,5795	0,2482	3,4357	5,2368	0,0263	0,3646	0,5557	0,0027	0,0379	0,0578	0,0003	0,0038	0,0058
2 300 942 549	320	21,56	2,2471	33,358	48,4409	0,2314	3,4345	4,9873	0,0234	0,3481	0,5055	0,0023	0,0346	0,0502	0,0002	0,0033	0,0048
3 842 610 773	336	22,07	2,1856	33,275	48,2349	0,2138	3,2546	4,7179	0,0206	0,3134	0,4542	0,0019	0,0296	0,0428	0,0002	0,0027	0,0039
4 302 407 359	354	22,18	2,1721	34,664	48,1826	0,2101	3,3524	4,6598	0,0200	0,3191	0,4436	0,0019	0,0298	0,0414	0,0002	0,0027	0,0038
10 726 904 659	382	23,10	2,0641	34,140	47,6727	0,1822	3,0134	4,2079	0,0158	0,2618	0,3656	0,0013	0,0223	0,0311	0,0001	0,0018	0,0026
20 678 048 297	384	23,75	1,9879	32,138	47,2176	0,1643	2,6566	3,9030	0,0134	0,2162	0,3176	0,0011	0,0172	0,0253	0,0001	0,0013	0,0020
22 367 084 959	394	23,83	1,9789	32,717	47,1586	0,1623	2,6833	3,8677	0,0131	0,2166	0,3122	0,0010	0,0171	0,0247	0,0001	0,0013	0,0019
25 056 082 087	456	23,94	1,9659	**37,438**	47,0715	0,1594	3,0358	3,8169	0,0127	0,2423	0,3047	0,0010	0,0189	0,0238	0,0001	0,0014	0,0018
42 652 618 343	464	24,48	1,9054	36,122	46,6382	0,1465	2,7773	3,5859	0,0111	0,2102	0,2714	0,0008	0,0156	0,0201	0,0001	0,0011	0,0015
127 976 334 671	468	25,58	1,7838	32,642	45,6210	0,1229	2,2486	3,1427	0,0083	0,1525	0,2131	0,0006	0,0101	0,0142	0,0000	0,0007	0,0009
182 226 896 239	474	25,93	1,7457	31,912	45,2623	0,1161	2,1220	3,0097	0,0076	0,1389	0,1970	0,0005	0,0089	0,0126	0,0000	0,0006	0,0008
241 160 624 143	486	26,21	1,7158	31,816	44,9680	0,1109	2,0571	2,9075	0,0071	0,1309	0,1851	0,0004	0,0082	0,0115	0,0000	0,0005	0,0007
297 501 075 799	490	26,42	1,6936	31,412	44,7421	0,1072	1,9888	2,8328	0,0067	0,1240	0,1766	0,0004	0,0076	0,0108	0,0000	0,0004	0,0006
303 371 455 241	500	26,44	1,6915	31,990	44,7208	0,1069	2,0215	2,8259	0,0066	0,1257	0,1758	0,0004	0,0077	0,0107	0,0000	0,0005	0,0006
304 599 508 537	514	26,44	1,6911	32,872	44,7164	0,1068	2,0764	2,8245	0,0066	0,1291	0,1756	0,0004	0,0079	0,0107	0,0000	0,0005	0,0006
416 608 695 821	516	26,76	1,6584	31,983	44,3704	0,1015	1,9579	2,7162	0,0061	0,1180	0,1637	0,0004	0,0070	0,0097	0,0000	0,0004	0,0006
461 690 510 011	532	26,86	1,6477	32,638	44,2548	0,0998	1,9776	2,6815	0,0060	0,1180	0,1599	0,0003	0,0069	0,0093	0,0000	0,0004	0,0005
614 487 453 523	534	27,14	1,6183	31,837	43,9279	0,0953	1,8747	2,5866	0,0055	0,1087	0,1499	0,0003	0,0062	0,0085	0,0000	0,0003	0,0005
738 832 927 927	540	27,33	1,5996	31,607	43,7134	0,0925	1,8272	2,5270	0,0053	0,1040	0,1438	0,0003	0,0058	0,0080	0,0000	0,0003	0,0004
1 346 294 310 749	582	27,93	1,5395	32,081	42,9952	0,0838	1,7466	2,3407	0,0045	0,0936	0,1254	0,0002	0,0049	0,0066	0,0000	0,0003	0,0003
1 408 695 493 609	588	27,97	1,5350	32,266	42,9398	0,0832	1,7487	2,3272	0,0044	0,0933	0,1242	0,0002	0,0049	0,0065	0,0000	0,0002	0,0003
1 968 188 556 461	602	28,31	1,5023	31,947	42,5264	0,0787	1,6744	2,2289	0,0041	0,0864	0,1150	0,0002	0,0044	0,0058	0,0000	0,0002	0,0003
2 614 941 710 599	652	28,59	1,4748	33,631	42,1690	0,0751	1,7134	2,1483	0,0038	0,0859	0,1077	0,0002	0,0042	0,0053	0,0000	0,0002	0,0003
7 177 162 611 713	674	29,60	1,3803	31,426	40,8589	0,0636	1,4473	1,8817	0,0029	0,0656	0,0853	0,0001	0,0029	0,0038	0,0000	0,0001	0,0002
13 829 048 559 701	716	30,26	1,3213	31,266	39,9795	0,0570	1,3485	1,7243	0,0024	0,0572	0,0732	0,0001	0,0024	0,0030	0,0000	0,0001	0,0001
19 581 334 192 423	766	30,61	1,2908	32,307	39,5057	0,0538	1,3457	1,6456	0,0022	0,0552	0,0675	0,0001	0,0022	0,0027	0,0000	0,0001	0,0001
42 842 283 925 351	778	31,39	1,2241	30,339	38,4236	0,0472	1,1689	1,4800	0,0018	0,0443	0,0561	0,0001	0,0016	0,0021	0,0000	0,0001	0,0001
90 874 329 411 493	804	32,14	1,1627	29,080	37,3683	0,0415	1,0390	1,3351	0,0015	0,0365	0,0470	0,0001	0,0013	0,0016	0,0000	0,0000	0,0001
171 231 342 420 521	806	32,77	1,1128	27,356	36,4707	0,0373	0,9169	1,2230	0,0012	0,0303	0,0404	0,0000	0,0010	0,0013	0,0000	0,0000	0,0000
218 209 405 436 543	906	33,02	1,0942	30,002	36,1257	0,0358	0,9818	1,1824	0,0012	0,0317	0,0381	0,0000	0,0010	0,0012	0,0000	0,0000	0,0000
1 189 459 969 825 480	916	34,71	0,9709	25,742	33,7034	0,0268	0,7144	0,9311	0,0007	0,0193	0,0253	0,0000	0,0005	0,0007	0,0000	0,0000	0,0000
1 686 994 940 955 800	924	35,06	0,9470	24,820	33,2046	0,0253	0,6641	0,8858	0,0007	0,0175	0,0233	0,0000	0,0005	0,0006	0,0000	0,0000	0,0000
1 693 182 318 746 370	1132	35,07	0,9468	30,500	33,1993	0,0252	0,8123	0,8853	0,0007	0,0214	0,0232	0,0000	0,0006	0,0006	0,0000	0,0000	0,0000
43 841 547 845 541 000	1184	38,32	0,7473	20,250	28,6346	0,0144	0,3672	0,5515	0,0003	0,0063	0,0105	0,0000	0,0001	0,0002	0,0000	0,0000	0,0000
55 350 776 431 903 200	1198	38,55	0,7345	24,750	28,3162	0,0138	0,4414	0,5328	0,0003	0,0074	0,0099	0,0000	0,0002	0,0002	0,0000	0,0000	0,0000

Table I.15







Under Cramer's conjecture (in grey), we see that for all prime up to the 66th maximal gap, $F_x(p_n)$ remains under $M_x(p_n)$. For x<0,85, the value of Pno will be lower than the prime number Pn,max corresponding to the highest known maximal gap Gn,max= 1550. In theses conditions we can see the grey curve be under the maximal value of the blue one, which insure that the upper bound of $F_x(p_n)$is reached for a prime number lower than Pn,max .

Part 2 : Study of $L_k(p_n) = Ln^k(p_{n+1}) - Ln^k(p_n)$ as a variant of Andrica's conjecture

In 2018 Matt Wisser [1] [2] from Victoria University of Wellington has proposed a variant of Andrica's conjecture. He developed an original approach to estimate some upper bounds of the function $L_k(p_n) = \ln^k(p_{n+1})-\ln^k(p_n)$. We propose a new approach to prove that for all k, $L_k(p_n)$ is a bounded function which tends toward 0 when $n \to \infty$. As we have processed in Part 1, we intend to build a lower and an upper bound functions $L_k^-(p_n)$ and $L_k^+(p_n)$, to state that these functions are bounded and tends toward 0 when $n \to \infty$. We'll try to estimate the prime number $P_{n_o,k}$ such as for all prime $P_n > P_{n_o,k}$, $L_k^+(p_n) < Max_{pn}\ L_k^-(p_n)$. And we'll conclude, if $P_{no} < P_{n,max}$ which is the prime number associated to the highest maximal gap known $G_{n,max} = 1550$, that $L_k(p_n)$ has an upper bound and that this upper bound is reached for $P_n < P_{n,max}$. We'll also state a recurrence relation on k to the upper bound of $L_k(p_n)$ with the upper bound of $L_{k-1}(p_n)$.

In a second time, we'll prove that for all p_n, there is a real y such as $L_y(p_n) = \ln^y(p_{n+1})-\ln^y(p_n)=1$ and try to build an explicite formulation of $y(p_n)$.

I – Upper and Lower bound functions of $L_k(p_n)$:

We are first going to factorize $L_k(p_n)$:

$$L_k(p_n) = \ln^k(p_{n+1}) - \ln^k(p_n) = \left(\ln(p_{n+1}) - \ln(p_n)\right).\sum_{i=1}^{k} \ln^{k-i}(p_{n+1}).\ln^{i-1}(p_n)$$

As $p_{n+1} > p_n$ then $Ln(p_{n+1}) > Ln(p_n)$, Thus we can give a first lower and upper bounds estimation of $F_k(p_n)$:

$$k.\ln^{k-1}(p_n).\left(\ln(p_{n+1}) - \ln(p_n)\right) < L_k(p_n) < k.\ln^{k-1}(p_{n+1}).\left(\ln(p_{n+1}) - \ln(p_n)\right) \quad (II.1)$$

We are going to use a Talor serie expansion of $Ln(p_{n+1}) - Ln(p_n)$:

$$\ln(p_{n+1}) - \ln(p_n) = \ln\left(p_n.\left(1 + \frac{g_n}{p_n}\right)\right) - \ln(p_n) = \ln\left(1 + \frac{g_n}{p_n}\right) = \frac{g_n}{p_n}.\left(1 - \frac{g_n}{2p_n}\right) + O\left(\frac{g_n}{p_n}\right)^3$$

$$\frac{g_n}{p_n}.\left(1 - \frac{g_n}{2p_n}\right) < \ln(p_{n+1}) - \ln(p_n) < \frac{g_n}{p_n}$$

We can replace into (II.1) :

$$k.\ln^{k-1}(p_n).\frac{g_n}{p_n}.\left(1 - \frac{g_n}{2p_n}\right) < L_k(p_n) < k.\ln^{k-1}(p_{n+1}).\frac{g_n}{p_n}$$

$$k.\frac{\ln^{k-1}(p_n)}{p_n}.g_n.\left(1 - \frac{g_n}{2p_n}\right) < L_k(p_n) < k.\frac{\ln^{k-1}(p_{n+1})}{p_{n+1}}.\frac{p_{n+1}}{p_n}.g_n$$

$$k.\frac{\ln^{k-1}(p_n)}{p_n}.g_n.\left(1 - \frac{g_n}{2p_n}\right) < L_k(p_n) < k.\frac{\ln^{k-1}(p_{n+1})}{p_{n+1}}.g_n.\left(1 + \frac{g_n}{p_n}\right) \quad (II.2)$$

If we consider the function $M_k(p_n) = k.\frac{\ln^{k-1}(p_n)}{p_n}$, (II.2) can be written :

$$M_k(p_n).g_n.\left(1 - \frac{g_n}{2p_n}\right) < L_k(p_n) < M_k(p_{n+1}).g_n.\left(1 + \frac{g_n}{p_n}\right) \quad (II.3)$$

we can easily see that the function $\frac{\ln^{k-1}(p_n)}{p_n}$ is decreasing for all $p_n > e^{k-1}$, thus $M_k(p_{n+1}) < M_k(p_n)$

$$M_k(p_n).g_n.\left(1-\frac{g_n}{2p_n}\right) < L_k(p_n) < M_k(p_n).g_n.\left(1+\frac{g_n}{p_n}\right) \qquad \text{for } p_n > e^{k-1} \qquad (II.4)$$

$$\left(1-\frac{g_n}{2p_n}\right) < \frac{L_k(p_n)}{M_k(p_n).g_n} < \left(1+\frac{g_n}{p_n}\right)$$

thanks to BHP result, we know that $g_n < {p_n}^a$ with $a = 0.525$

$$\left(1-\frac{1}{2{p_n}^{1-a}}\right) < \frac{L_k(p_n)}{M_k(p_n).g_n} < \left(1+\frac{1}{{p_n}^{1-a}}\right)$$

$$-\frac{1}{2{p_n}^{1-a}} < \frac{L_k(p_n)}{M_k(p_n).g_n} - 1 < \frac{1}{{p_n}^{1-a}}$$

Thus , $\lim lim_{p_n\to\infty}\left(\frac{L_k(p_n)}{M_k(p_n).g_n}\right) = 1$ which imply that for n enough large, $M_k(p_n).g_n$ and $L_k(p_n)$ have the same behavior.

An interesting corollary of this result is the fact that the value of $p_{n,k}$ such as $L_k(p_{n,k}) = Max_{pi \in \mathbb{P}} L_k(p_i)$ must be higher than e^{k-1} . the proof of this point will be presented in the next paragraph.

Our purpose is now to build two function $L_k^-(p_n)$ and $L_k^+(p_n)$ independent from g_n:

As for all integer, $g_n \geq 2$ and t $g_n < {p_n}^a$ with $a = 0.525$:

$$2k.\frac{\ln^{k-1}(p_n)}{p_n}\left(1-\frac{1}{2{p_n}^{1-a}}\right) < L_k(p_n) < k.\frac{\ln^{k-1}(p_n)}{{p_n}^{1-a}}.\left(1+\frac{1}{{p_n}^{1-a}}\right) \qquad (II.5)$$

then we can define : $L_k^-(p_n) = 2k.\frac{\ln^{k-1}(p_n)}{p_n}\left(1-\frac{1}{2{p_n}^{1-a}}\right)$ and $L_k^+(p_n) = k.\frac{\ln^{k-1}(p_n)}{{p_n}^{1-a}}.\left(1+\frac{1}{{p_n}^{1-a}}\right)$ which satisfy

$$L_k^-(p_n) < L_k(p_n) < L_k^+(p_n) \qquad \text{for } p_n > e^{k-1} \qquad (II.6)$$

Thus we have built two functions $L_k^-(p_n)$and $L_k^+(p_n)$, independent from g_n, satisfying to (II.6), which are two positive bounded functions which tends toward 0 when n→∞. Thus we can state that for all integer k, $L_k(p_n)$ has an upper bound and tends toward 0 when n→∞.

II – An approach based on the maximal gaps integer sequence

For the same reason exposed in Part 1, we can limit our study to prime numbers corresponding to the first occurrence of maximal G_N . gaps show that we can concentrate our analysis on maximal gaps. We'll write (G_N) the integer sequence corresponding to all maximal gaps and (P_N), the integer sequence of prime numbers corresponding to the first occurrence of all element of (G_N).

For each $G_N \in (G_N)$, we define $\mathbb{P}_{G_N} = \{p_i \in \mathbb{P} / g_i = G_N\}$ and $P_N = Min_{\mathbb{P}_{G_N}}(p_i)$

We can rewrite (II.4) as following :

$$M_k(p_n).g_n.\left(1-\frac{G_n}{2p_n}\right) < M_k(p_n).g_n.\left(1-\frac{g_n}{2p_n}\right) < L_k(p_n) < M_k(p_n).g_n.\left(1+\frac{g_n}{p_n}\right) < M_k(p_n).G_n.\left(1+\frac{G_n}{p_n}\right) \qquad (II.7)$$

As our purpose is to estimate as well as possible $p_{n,k}$ such as $L_k(p_{n,k}) = Max_{pi \in \mathbb{P}} L_k(p_i)$, we can conjecture that $p_{n,k} \in (P_N)$. To try to state this hypothesis, we suppose that $p_{n,k} \notin (P_N)$, there is G_N such as $p_{n,k} \in \mathbb{P}_{G_N}$ and by definition $p_{n,k} > P_N$. It's easy to prove that $L_k(p_{n,k}) < L_k(P_N)$.

$$L_k(P_N) - L_k(p_{n,k}) = \ln^k(P_N + G_N) - \ln^k(P_N) - (\ln^k(p_{n,k} + G_N) - \ln^k(p_{n,k}))$$

$$L_k(P_N) - L_k(p_{n,k}) = \left(\ln^k(p_{n,k}) - \ln^k(P_N)\right) - \left(\ln^k(p_{n,k} + G_N) - \ln^k(P_N + G_N)\right)$$

We can remark that $\ln^k(x+d) - \ln^k(y+d)$ is decreasing when d grows, for $x>y>e^{k-1}$.

We can also insure that $p_{n,k}$ and P_N cannot be lower than e^{k-1}. Effectively $M_k(P_N)$ is a strictly growing function up to $p_n=e^{k-1}$ and G_N is also a growing function. So for all $p_n<e^{k-1}$, the product $M_k(P_N).G_N$ of two growing function is also a growing function. Which imply that the maximum value cannot be smaller than e^{k-1}.

Thus we can conclude that $p_{n,k}>P_N>e^{k-1}$ and $L_k(P_N) - L_k(p_{n,k}) > 0$ which contradict our hypothesis $L_k(p_{n,k}) = \mathrm{Max}_{p_i \in \mathbb{P}} L_k(p_i)$ and $p_{n,k} \notin (P_N)$. We can conclude that $p_{n,k} \in (P_N)$ and limit our study to prime numbers belonging to the integer sequence (P_N).

We can verify in tables II.1, II.2 and II.3 that for $k \leq 25$, $p_{n,k} > e^{k-1}$ and then belong to (P_N).

In these conditions, (II.7) can be written :

$$M_k(P_N).G_N.\left(1 - \frac{G_N}{2P_N}\right) < L_k(P_N) < M_k(P_N).G_N.\left(1 + \frac{G_N}{P_N}\right) \quad \text{(II.8)}$$

We have also: $$1 - \frac{1}{2{P_N}^{1-a}} < 1 - \frac{G_N}{2P_N} < \frac{L_k(P_N)}{M_k(P_N).G_N} < 1 + \frac{G_N}{P_N} < 1 + \frac{1}{{P_N}^{1-a}} \quad \text{(II.9)}$$

From (II.8) we can state that the behavior of $L_k(P_N)$ will be correctly represented by the behavior of $M_k(P_N).G_N$. Effectivelly, we have :

$$|L_k(P_N) - M_k(P_N).G_N| < M_k(P_N).\frac{{G_N}^2}{P_N} = k.\ln^{k-1}(P_N).\frac{{G_N}^2}{{P_N}^2} < k.\frac{\ln^{k-1}(P_N)}{{P_N}^{2-2a}} \qquad \text{with a=0,525}$$

Which tends towards to 0 when N→∞

Thanks to Erik Westzynthius (I.30) result regarding Maximal gaps and BHP result, we can write, $\ln(P_N) < G_N < {P_N}^a$ and replace into (II.8)

$$M_k(P_N).\ln(P_N).\left(1 - \frac{1}{2{P_N}^{1-a}}\right) < L_k(P_N) < M_k(P_N).{P_N}^a.\left(1 + \frac{1}{{P_N}^{1-a}}\right)$$

$$k.\frac{\ln^{k-1}(P_N)}{P_N}.\ln(P_N).\left(1 - \frac{1}{2{P_N}^{1-a}}\right) < L_k(P_N) < k.\frac{\ln^{k-1}(P_N)}{P_N}.{P_N}^a.\left(1 + \frac{1}{{P_N}^{1-a}}\right)$$

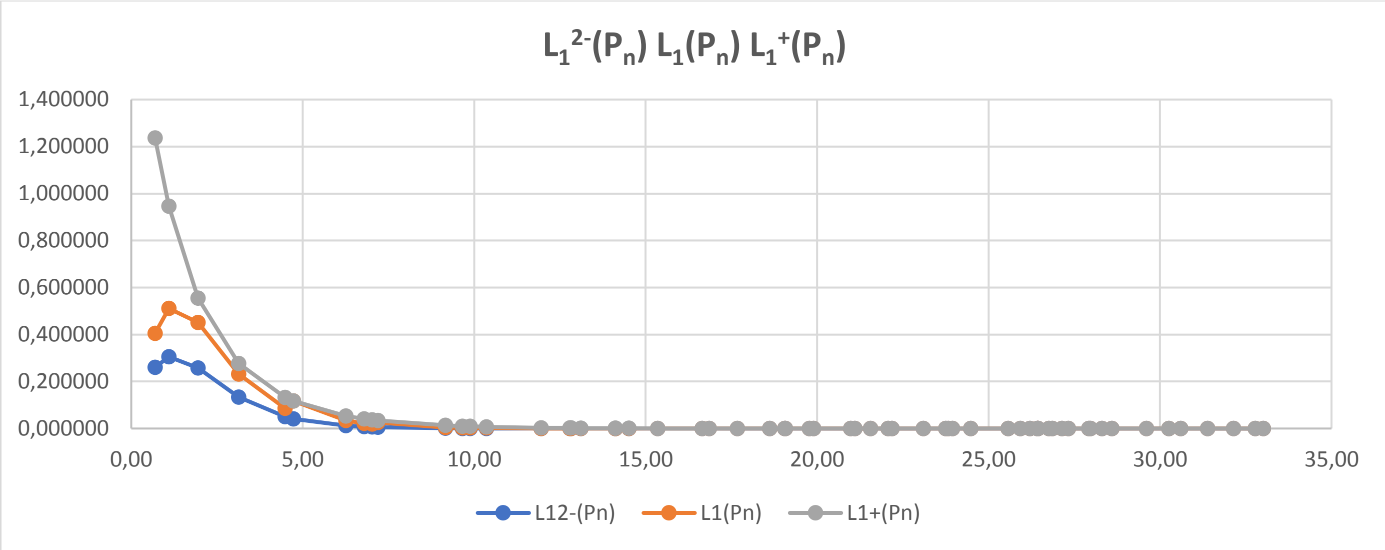
$$k.\frac{\ln^{k}(P_N)}{P_N}.\left(1 - \frac{1}{2{P_N}^{1-a}}\right) < L_k(P_N) < k.\frac{\ln^{k-1}(P_N)}{{P_N}^{1-a}}.\left(1 + \frac{1}{{P_N}^{1-a}}\right) = L_k^{+}(P_N) \quad \text{(II.10)}$$

We can define a new lower bound function which satisfy (II.10) only for prime numbers belonging to (P_N):

$$L_k^{2-}(P_N) = k.\frac{\ln^{k}(P_N)}{P_N}.\left(1 - \frac{1}{2{P_N}^{1-a}}\right) \quad \text{(II.11)}$$

We presents in table II.1 and II.2, the numerical values of $L_k^{2-}(P_N)$, $L_k(P_N)$ and $L_k^{+}(P_N)$ for k<10 and for all $P_{N(n)} \leq 218\,209\,405\,436\,543$. We have computed for all $P_N \in (P_N)$, and all k<30, the numerical values of $L_k^{2-}(P_N)$, $L_k(P_N)$ and $L_k^{+}(P_N)$ and the results are summarized in table II.3 which present the upper bound of these three functions.

Primes	Gaps	ln(Pn)	$L_1^{2-}(P_n)$	$L_1(P_n)$	$L_1^{+}(P_n)$	$L_2^{2-}(P_n)$	$L_2(P_n)$	$L_2^{+}(P_n)$	$L_3^{2-}(P_n)$	$L_3(P_n)$	$L_3^{+}(P_n)$	$L_4^{2-}(P_n)$	$L_4(P_n)$	$L_4^{+}(P_n)$	$L_5^{2-}(P_n)$	$L_5(P_n)$	$L_5^{+}(P_n)$
				k= 1,000			k= 2,000			k= 3,000			k= 4,000			k= 5,000	
2	**1**	**0,69**	**0,260**	0,405	**1,237**	0,360	0,726	1,715	0,375	0,993	1,783	0,346	1,226	1,648	0,300	1,440	1,428
3	**2**	**1,10**	0,305	**0,511**	0,946	0,671	1,383	2,078	1,105	2,843	3,424	1,619	5,253	5,015	2,223	9,198	6,887
7	**4**	**1,95**	0,258	0,452	0,554	**1,005**	**1,963**	**2,157**	2,932	6,419	6,296	7,608	18,723	16,336	18,506	51,377	39,735
23	**6**	**3,14**	0,133	0,232	0,276	0,836	1,507	1,733	**3,933**	**7,355**	**8,151**	16,444	31,911	34,078	64,451	129,858	133,565
89	**8**	**4,49**	0,050	0,086	0,133	0,450	0,780	1,191	3,031	5,303	8,018	**18,142**	32,044	47,986	101,790	181,534	269,239
113	**14**	**4,73**	0,042	0,117	0,117	0,394	1,118	1,107	2,792	**8,026**	7,850	17,601	**51,218**	49,479	**104,009**	306,445	292,380
523	**18**	**6,26**	0,012	0,034	0,054	0,150	0,425	0,673	1,406	3,999	6,318	11,731	33,467	**52,731**	91,787	262,572	412,592
887	**20**	**6,79**	0,008	0,022	0,041	0,104	0,303	0,562	1,057	3,092	5,718	9,568	28,032	51,753	81,182	238,235	439,115
1 129	**22**	**7,03**	0,006	0,019	0,037	0,087	0,272	0,516	0,922	2,868	5,445	8,645	26,920	51,035	75,959	236,855	448,412
1 327	**34**	**7,19**	0,005	0,025	0,034	0,078	0,364	0,488	0,840	3,938	5,264	8,056	37,824	50,471	72,408	**340,571**	453,655
9 551	**36**	**9,16**	0,001	0,004	0,013	0,018	0,069	0,239	0,242	0,948	3,284	2,954	11,590	40,124	33,839	132,794	**459,638**
15 683	**44**	**9,66**	0,001	0,003	0,010	0,012	0,054	0,198	0,172	0,785	2,875	2,221	10,107	37,034	26,822	122,069	447,201
19 609	**52**	**9,88**	0,001	0,003	0,009	0,010	0,052	0,182	0,148	0,776	2,704	1,947	10,232	35,634	24,050	126,433	440,243
31 397	**72**	**10,35**	0,000	0,002	0,007	0,007	0,047	0,153	0,106	0,737	2,369	1,464	10,175	32,703	18,955	131,710	423,275
155 921	**86**	**11,96**	0,000	0,001	0,003	0,002	0,013	0,082	0,033	0,237	1,470	0,524	3,771	23,431	7,838	56,362	350,207
360 653	**96**	**12,80**	0,000	0,000	0,002	0,001	0,007	0,059	0,017	0,131	1,129	0,297	2,230	19,259	4,755	35,675	308,035
370 261	**112**	**12,82**	0,000	0,000	0,002	0,001	0,008	0,058	0,017	0,149	1,119	0,292	2,550	19,137	4,680	40,875	306,711
492 113	**114**	**13,11**	0,000	0,000	0,002	0,001	0,006	0,052	0,014	0,119	1,021	0,240	2,086	17,850	3,929	34,176	292,442
1 349 533	**118**	**14,12**	0,000	0,000	0,001	0,000	0,002	0,035	0,006	0,052	0,733	0,118	0,984	13,798	2,076	17,354	243,456
1 357 201	**132**	**14,12**	0,000	0,000	0,001	0,000	0,003	0,035	0,006	0,058	0,732	0,117	1,095	13,778	2,068	19,335	243,191
2 010 733	**148**	**14,51**	0,000	0,000	0,001	0,000	0,002	0,029	0,005	0,047	0,641	0,088	0,900	12,410	1,602	16,331	225,147
4 652 353	**154**	**15,35**	0,000	0,000	0,001	0,000	0,001	0,021	0,002	0,023	0,482	0,048	0,479	9,858	0,917	9,195	189,182
17 051 707	**180**	**16,65**	0,000	0,000	0,000	0,000	0,000	0,012	0,001	0,009	0,306	0,018	0,195	6,784	0,375	4,058	141,215
20 831 323	**210**	**16,85**	0,000	0,000	0,000	0,000	0,000	0,011	0,001	0,009	0,285	0,015	0,193	6,394	0,326	4,065	134,687
47 326 693	**220**	**17,67**	0,000	0,000	0,000	0,000	0,000	0,008	0,000	0,004	0,212	0,008	0,103	4,993	0,182	2,267	110,304
122 164 747	**222**	**18,62**	0,000	0,000	0,000	0,000	0,000	0,005	0,000	0,002	0,150	0,004	0,047	3,722	0,092	1,092	86,643
189 695 659	**234**	**19,06**	0,000	0,000	0,000	0,000	0,000	0,004	0,000	0,001	0,127	0,003	0,034	3,239	0,066	0,814	77,183
191 912 783	**248**	**19,07**	0,000	0,000	0,000	0,000	0,000	0,004	0,000	0,001	0,127	0,003	0,036	3,227	0,066	0,855	76,945
387 096 133	**250**	**19,77**	0,000	0,000	0,000	0,000	0,000	0,003	0,000	0,001	0,098	0,002	0,020	2,577	0,039	0,494	63,707
436 273 009	**282**	**19,89**	0,000	0,000	0,000	0,000	0,000	0,003	0,000	0,001	0,093	0,001	0,020	2,479	0,036	0,506	61,658
1 294 268 491	**288**	**20,98**	0,000	0,000	0,000	0,000	0,000	0,002	0,000	0,000	0,062	0,001	0,008	1,735	0,016	0,216	45,509
1 453 168 141	**292**	**21,10**	0,000	0,000	0,000	0,000	0,000	0,002	0,000	0,000	0,059	0,001	0,008	1,670	0,014	0,199	44,032
2 300 942 549	**320**	**21,56**	0,000	0,000	0,000	0,000	0,000	0,002	0,000	0,000	0,050	0,000	0,006	1,432	0,010	0,150	38,583
3 842 610 773	**336**	**22,07**	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,041	0,000	0,004	1,204	0,007	0,104	33,224
4 302 407 359	**354**	**22,18**	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,039	0,000	0,004	1,159	0,006	0,100	32,137
10 726 904 659	**382**	**23,10**	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,028	0,000	0,002	0,848	0,003	0,051	24,471
20 678 048 297	**384**	**23,75**	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,021	0,000	0,001	0,675	0,002	0,030	20,042
22 367 084 959	**394**	**23,83**	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,021	0,000	0,001	0,657	0,002	0,028	19,565
25 056 082 087	**456**	**23,94**	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,020	0,000	0,001	0,631	0,002	0,030	18,893
42 652 618 343	**464**	**24,48**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,016	0,000	0,001	0,524	0,001	0,020	16,023
127 976 334 671	**468**	**25,58**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,010	0,000	0,000	0,355	0,000	0,008	11,333
182 226 896 239	**474**	**25,93**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,009	0,000	0,000	0,312	0,000	0,006	10,123
241 160 624 143	**486**	**26,21**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,008	0,000	0,000	0,282	0,000	0,005	9,250
297 501 075 799	**490**	**26,42**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,007	0,000	0,000	0,262	0,000	0,004	8,644
303 371 455 241	**500**	**26,44**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,007	0,000	0,000	0,260	0,000	0,004	8,589
304 599 508 537	**514**	**26,44**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,007	0,000	0,000	0,260	0,000	0,004	8,578
416 608 695 821	**516**	**26,76**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,006	0,000	0,000	0,232	0,000	0,003	7,749
461 690 510 011	**532**	**26,86**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,006	0,000	0,000	0,223	0,000	0,003	7,494
614 487 453 523	**534**	**27,14**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,006	0,000	0,000	0,201	0,000	0,002	6,825
738 832 927 927	**540**	**27,33**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,005	0,000	0,000	0,188	0,000	0,002	6,425
1 346 294 310 749	**582**	**27,93**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,004	0,000	0,000	0,151	0,000	0,001	5,270
1 408 695 493 609	**588**	**27,97**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,004	0,000	0,000	0,148	0,000	0,001	5,191
1 968 188 556 461	**602**	**28,31**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,003	0,000	0,000	0,131	0,000	0,001	4,644
2 614 941 710 599	**652**	**28,59**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,003	0,000	0,000	0,118	0,000	0,001	4,223
7 177 162 611 713	**674**	**29,60**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,002	0,000	0,000	0,081	0,000	0,000	3,004
13 829 048 559 701	**716**	**30,26**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,002	0,000	0,000	0,063	0,000	0,000	2,401
19 581 334 192 423	**766**	**30,61**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,056	0,000	0,000	2,131
42 842 283 925 351	**778**	**31,39**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,041	0,000	0,000	1,625
90 874 329 411 493	**804**	**32,14**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,031	0,000	0,000	1,250
171 231 342 420 521	**806**	**32,77**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,024	0,000	0,000	1,000
218 209 405 436 543	**906**	**33,02**	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,022	0,000	0,000	0,918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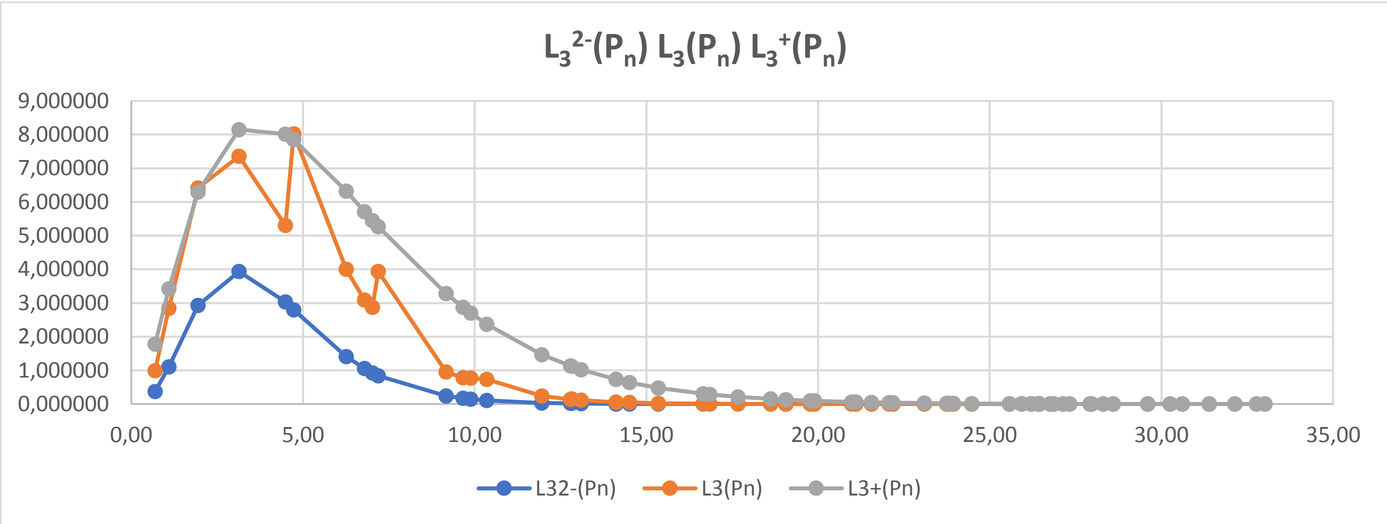
L12-(Pn)
L1(Pn)
L1+(Pn)


$L_3^{2-}(P_n)$ $L_3(P_n)$ $L_3^{+}(P_n)$
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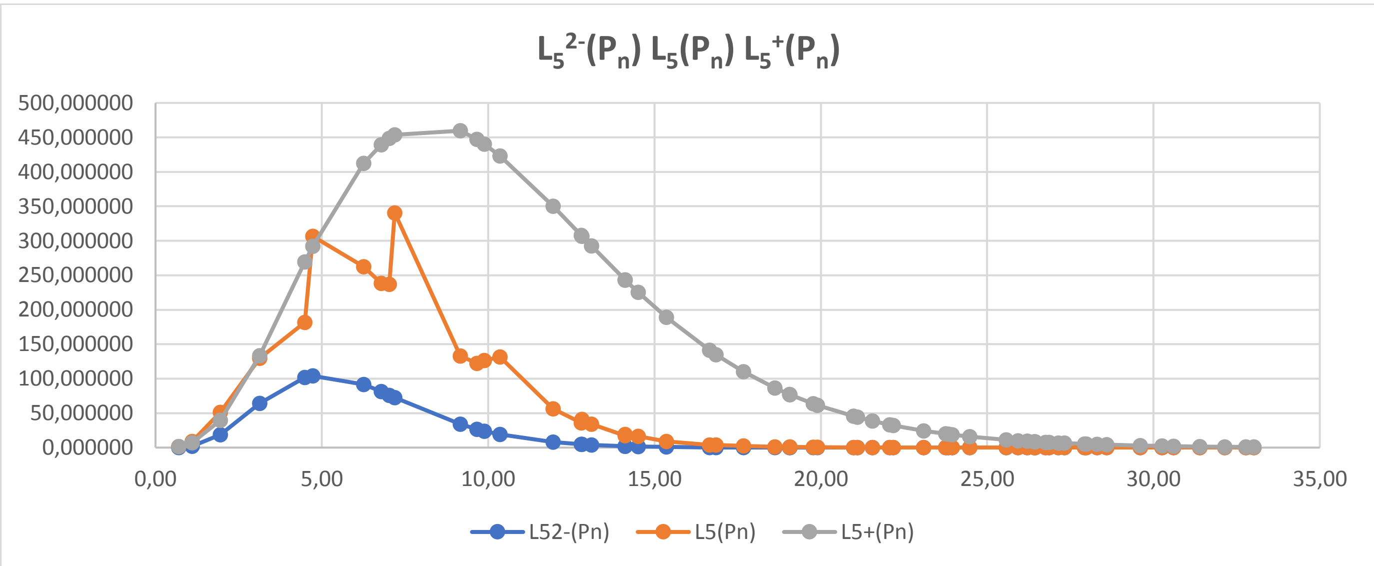
L32-(Pn)
L3(Pn)
L3+(Pn)


$L_5^{2-}(P_n)$ $L_5(P_n)$ $L_5^{+}(P_n)$
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10,00
15,00
20,00
25,00
30,00
35,00
L52-(Pn)
L5(Pn)
L5+(Pn)

Primes	Gaps	ln(Pn)	$L_6^{2-}(P_n)$	$L_6(P_n)$	$L_6^{+}(P_n)$	$L_7^{2-}(P_n)$	$L_7(P_n)$	$L_7^{+}(P_n)$	$L_8^{2-}(P_n)$	$L_8(P_n)$	$L_8^{+}(P_n)$	$L_9^{2-}(P_n)$	$L_9(P_n)$	$L_9^{+}(P_n)$	$L_{10}^{2-}(P_n)$	$L_{10}(P_n)$	$L_{10}^{+}(P_n)$
2	1		k= 6			k= 7			k= 8			k= 9			k= 10		
3	2	1,10	2,93	15,62	9,08	3,76	26,04	11,64	4,72	42,90	14,61	5,83	70,12	18,06	7,11	114,05	22,04
7	4	1,95	43,21	135,81	92,79	98,10	350,19	210,64	218,17	887,47	468,45	477,60	2 220,99	1 025,51	1 032,63	5 506,51	2 217,28
23	6	3,14	242,50	507,52	502,55	887,09	1 929,24	1 838,37	3 178,81	7 186,96	6 587,65	11 213,04	26 366,16	23 237,49	39 064,92	95 572,66	80 956,67
89	8	4,49	548,28	987,30	1 450,22	2 871,19	5 220,60	7 594,42	14 728,84	27 042,67	38 958,37	74 376,47	137 896,13	196 728,69	370 943,27	694 500,55	981 159,50
113	14	4,73	590,03	1 760,25	1 658,63	3 254,19	9 830,64	9 147,84	17 581,48	53 784,42	49 423,31	93 503,79	289 676,44	262 848,56	491 142,99	1 540 977,73	1 380 652,30
523	18	6,26	**689,46**	1 977,66	3 099,19	5 034,98	14 481,76	22 632,87	36 019,30	103 881,28	161 911,22	253 648,96	733 524,70	1 140 183,50	1 764 151,49	5 115 619,17	7 930 079,45
887	20	6,79	661,26	1 943,72	3 576,77	5 236,64	15 417,97	28 325,01	40 623,45	119 802,58	219 732,28	310 213,90	916 359,89	1 677 947,27	2 339 648,72	6 922 633,13	12 655 162,20
1 129	22	7,03	640,71	2 000,59	3 782,31	**5 254,17**	16 428,65	31 017,26	42 208,05	132 157,00	249 169,17	333 769,59	1 046 500,25	1 970 360,88	2 606 773,00	8 184 529,90	15 388 710,20
1 327	34	7,19	624,79	**2 943,90**	3 914,50	5 241,45	**24 740,35**	32 839,24	**43 073,80**	203 673,00	269 870,13	348 445,95	1 650 525,16	2 183 117,28	2 783 957,72	13 210 408,61	17 442 321,22
9 551	36	9,16	372,14	1 460,68	5 054,76	3 978,83	15 620,46	54 044,54	41 672,65	163 636,11	566 040,93	**429 643,00**	1 687 426,77	5 835 854,41	4 374 912,04	17 186 035,23	59 424 567,56
15 683	44	9,66	310,93	1 415,28	5 184,13	3 504,29	15 953,04	58 427,14	38 688,66	176 153,16	645 057,84	420 463,44	1 914 688,03	7 010 407,45	4 513 129,59	20 554 673,79	75 247 630,68
19 609	52	9,88	285,24	1 499,76	5 221,50	3 289,14	17 296,12	60 209,26	37 153,12	195 398,12	680 106,16	413 113,47	2 172 964,32	7 562 244,54	**4 536 786,35**	23 866 556,99	83 048 098,35
31 397	72	10,35	235,52	1 636,73	**5 259,35**	2 845,10	19 774,24	63 534,03	33 667,98	**234 027,85**	751 841,20	392 190,81	**2 726 439,61**	8 758 029,82	4 512 141,15	31 371 060,15	100 760 818,09
155 921	86	11,96	112,46	808,73	5 024,96	1 568,82	11 282,06	70 097,91	21 438,39	154 175,87	957 906,35	288 383,77	2 073 982,03	12 885 509,90	3 831 372,11	27 554 880,26	171 192 657,36
360 653	96	12,80	73,02	547,79	4 729,81	1 090,06	8 177,63	**70 607,96**	15 940,59	119 587,86	1 032 544,28	229 466,93	1 721 500,70	14 863 609,71	3 262 426,14	24 475 540,70	211 322 075,49
370 261	112	12,82	72,01	628,92	4 719,16	1 077,14	9 408,07	70 593,75	15 784,02	137 864,34	1 034 457,75	227 679,94	1 988 676,48	14 921 751,91	3 243 670,91	**28 332 264,26**	212 584 619,46
492 113	114	13,11	61,80	537,51	4 599,45	945,00	8 219,09	70 329,69	14 154,97	123 113,38	1 053 455,39	208 711,79	1 815 294,71	15 532 959,13	3 039 414,93	26 435 893,78	226 202 404,52
1 349 533	118	14,12	35,16	293,96	4 123,74	579,08	4 840,84	67 909,04	9 341,56	78 091,37	1 095 490,68	148 341,04	1 240 069,70	17 396 039,05	2 326 526,33	19 448 857,54	272 833 081,69
1 357 201	132	14,12	35,05	327,63	4 120,91	577,43	5 397,55	67 889,57	9 318,66	87 107,05	1 095 616,30	148 036,66	1 383 791,82	17 405 017,38	2 322 684,51	21 711 668,55	273 083 466,63
2 010 733	148	14,51	27,89	284,44	3 921,34	472,34	4 816,38	66 400,17	7 834,92	79 891,56	**1 101 408,86**	127 930,69	1 304 493,63	17 984 091,47	2 063 096,93	21 037 201,65	290 023 646,05
4 652 353	154	15,35	16,89	169,41	3 485,39	302,52	3 034,46	62 429,34	5 308,07	53 243,16	1 095 394,72	91 681,02	919 616,60	18 919 651,21	1 563 964,48	15 687 535,46	322 745 782,40
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20 831 323	210	16,85	6,60	82,21	2 723,70	129,70	1 616,23	53 549,61	2 497,92	31 127,60	1 031 332,98	47 356,69	590 131,61	**19 552 489,53**	886 726,07	11 049 869,00	366 108 815,85
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122 164 747	222	18,62	2,05	24,41	1 936,05	44,48	530,28	42 059,36	946,56	11 284,92	895 065,57	19 828,90	236 402,01	18 750 273,26	410 257,34	4 891 126,65	387 940 676,17
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387 096 133	250	19,77	0,93	11,72	1 511,70	21,38	270,28	34 874,79	483,13	6 108,03	788 137,82	10 747,58	135 878,93	17 532 879,70	236 138,50	2 985 438,74	385 220 425,13
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1 453 168 141	292	21,10	0,36	5,04	1 114,74	8,96	124,02	27 437,40	216,04	2 990,22	661 539,67	5 127,61	70 970,37	15 701 074,20	120 197,00	1 663 625,24	368 050 833,01
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1 346 294 310 749	582	27,93	0,00	0,04	176,62	0,07	1,44	5 754,82	2,20	45,83	183 683,07	69,11	1 440,09	5 771 216,16	2 144,46	44 688,25	179 089 667,12
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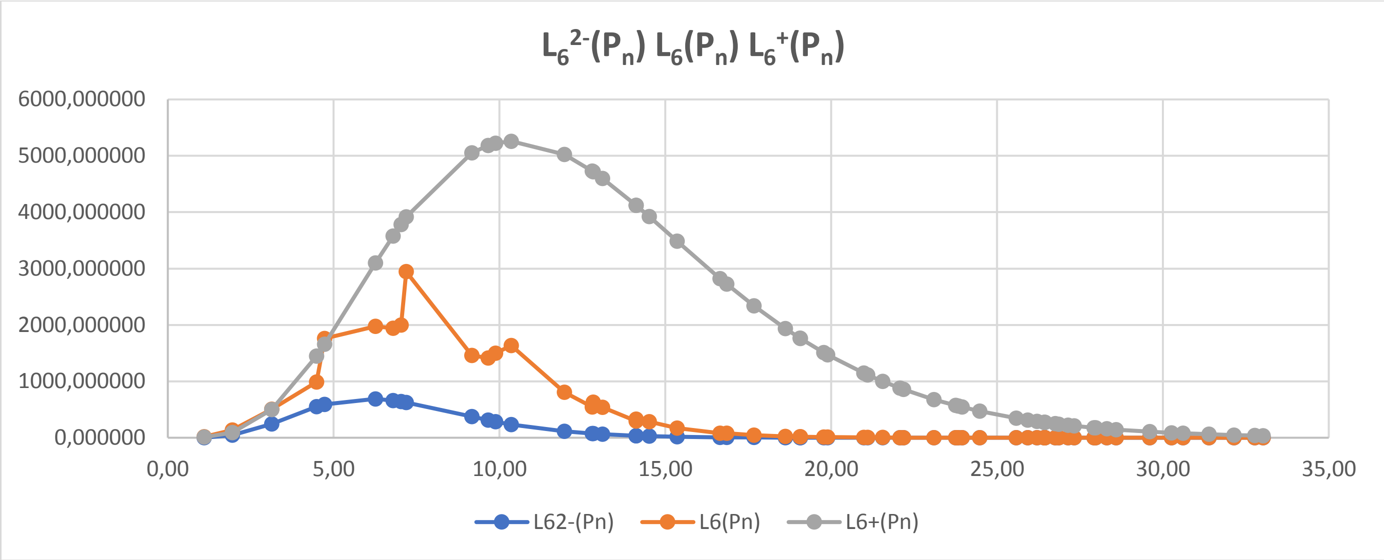
1 968 188 556 461	602	28,31	0,00	0,03	157,77	0,05	1,10	5 210,56	1,68	35,64	168 572,88	53,38	1 135,16	5 368 481,79	1 678,98	35 704,88	168 857 450,85
2 614 941 710 599	652	28,59	0,00	0,03	144,91	0,04	0,95	4 833,87	1,37	31,16	157 955,87	43,96	1 002,34	5 080 855,37	1 396,44	31 843,50	161 414 614,01
7 177 162 611 713	674	29,60	0,00	0,01	106,70	0,02	0,44	3 685,03	0,66	14,96	124 667,27	21,89	498,32	4 151 690,17	719,86	16 390,19	136 553 357,83
13 829 048 559 701	716	30,26	0,00	0,01	87,19	0,01	0,28	3 077,84	0,41	9,62	106 432,58	13,84	327,37	3 622 966,62	465,14	11 006,13	121 803 302,42
19 581 334 192 423	766	30,61	0,00	0,01	78,26	0,01	0,23	2 794,33	0,31	7,87	97 739,74	10,83	271,05	3 365 306,17	368,27	9 217,38	114 441 341,57
42 842 283 925 351	778	31,39	0,00	0,00	61,22	0,00	0,12	2 241,74	0,18	4,36	80 417,23	6,21	154,02	2 839 702,38	216,69	5 371,63	99 037 922,95
90 874 329 411 493	804	32,14	0,00	0,00	48,21	0,00	0,07	1 807,82	0,10	2,51	66 404,69	3,62	90,66	2 401 065,00	129,44	3 237,50	85 746 029,54
171 231 342 420 521	806	32,77	0,00	0,00	39,34	0,00	0,04	1 504,27	0,06	1,53	56 343,98	2,29	56,36	2 077 447,29	83,51	2 052,25	75 651 481,95
218 209 405 436 543	906	33,02	0,00	0,00	36,38	0,00	0,04	1 401,26	0,05	1,42	52 873,76	1,92	52,72	1 963 918,44	70,54	1 934,00	72 046 296,31



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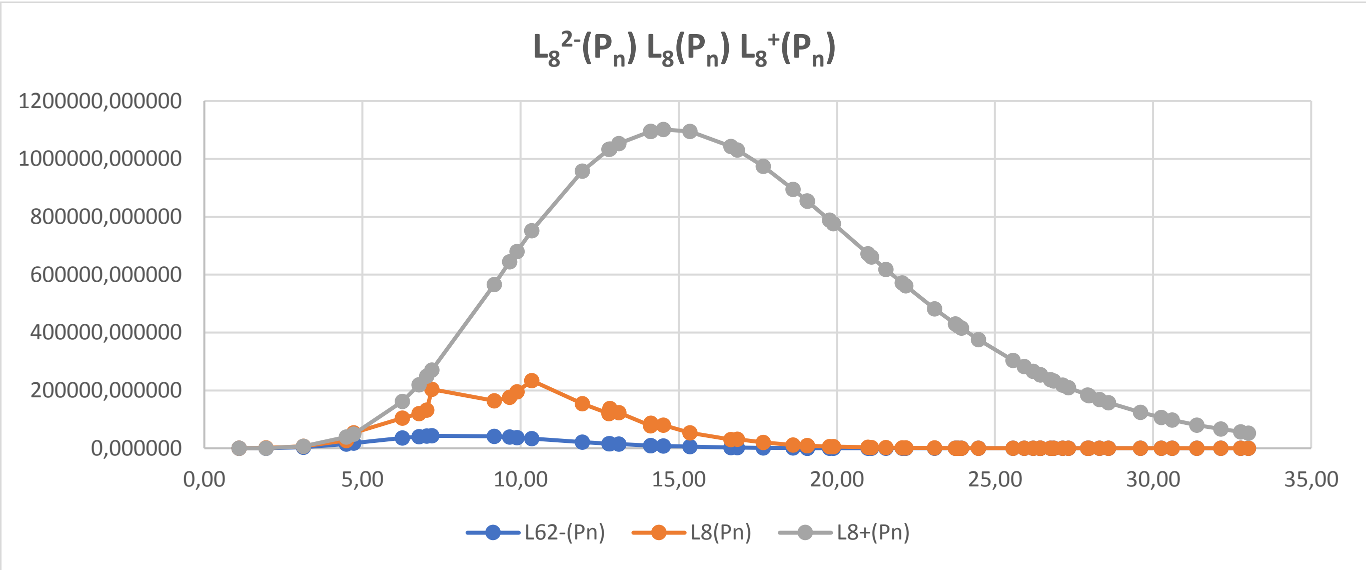

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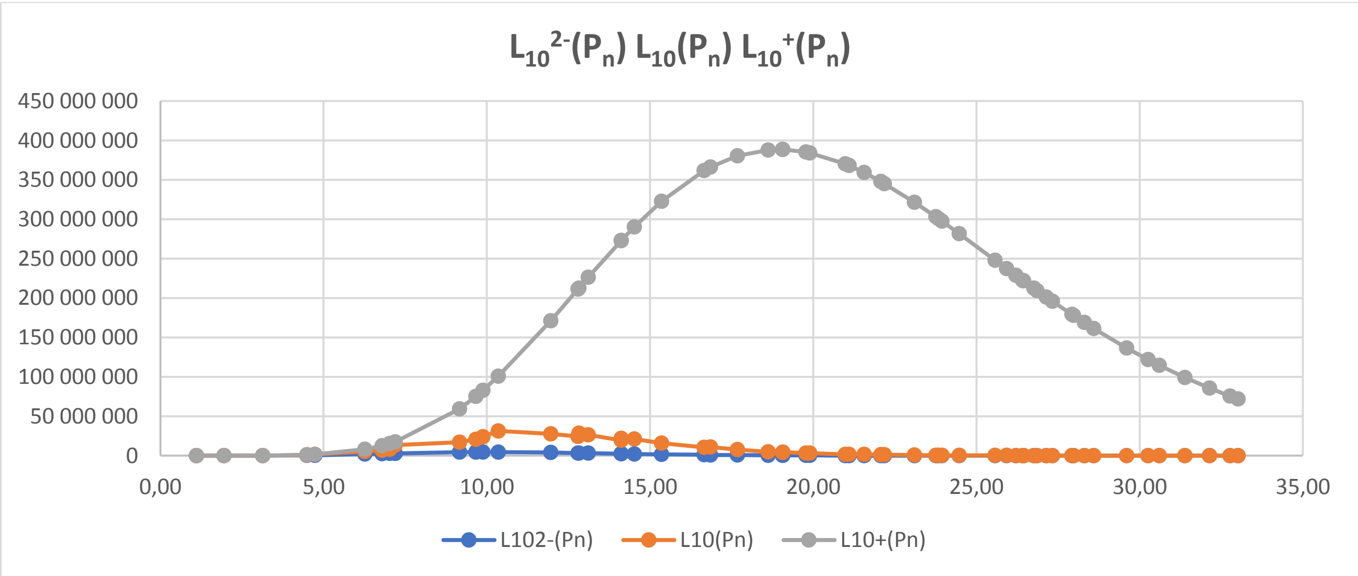

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25,00
30,00
35,00
L102-(Pn)
L10(Pn)
L10+(Pn)

II.3 – Behaviour of the lower and upper bound functions $L^{2-}_k(p_n)$ and $L^{+}_k(p_n)$

As our purpose is to estimate the value of P_{no} when $L^{+}_k(P_N)$ becomes lower than $\mathrm{Max}_{pi\,\epsilon\,\mathbb{P}}\, L_k^{2-}\left(P_N^{2-}\right) = L_k^{2-}\left(P_{N,k}^{2-}\right)$, we need to $P_{N,k}^{2-}$ and thus compute the first derivative of $L_k^{2-}(P_N)$

First derivative of $L^{2-}_k(p_n)$

$$\frac{dL_k^{2-}(P_N)}{dP_N} = 2k^2.\frac{\ln^{k-1}(P_N)}{2{P_N}^{3-a}}.\left({P_N}^{1-a} - \frac{1}{k}\left({P_N}^{1-a} - 1 + \frac{a}{2}\right).\ln(P_N) - \frac{1}{2}\right)$$

If we define $P_{N,k}^{2-}$ as the prime number for which $L_k^{2-}\left(P_{N,k}^{2-}\right) = \mathrm{Max}_{P_N \epsilon \mathbb{P}_{Max}}\, L_k^{2-}(P_N)$, $P_{N,k}$ must satisfy :

$$\frac{dL_k^{2-}(P_{N,k}^{2-})}{dP_N} = 0 \qquad \text{which imply} \qquad {P_N}^{1-a} - \frac{1}{k}\left({P_N}^{1-a} - 1 + \frac{a}{2}\right).\ln(P_N) - \frac{1}{2} = 0$$

If P_N is large, then $P_{N,k} = e^k$ gives a first estimation. But a better estimation can be given by :

$$P_{N,k}^{2-} \approx e^{k.\left(\frac{1-\frac{1}{2}.e^{k(a-1)}}{1-\frac{(2-a)}{2}.e^{k(a-1)}}\right)} \qquad \text{(II.12)}$$

First derivative of $L^{+}_k(p_n)$

$$\frac{dL_k^{+}(P_N)}{dP_N} = (k-1).\frac{\ln^{k-2}(P_N)}{2{P_N}^{3-2a}}.\left({P_N}^{1-a} - \frac{(1-a)}{k-1}\left({P_N}^{1-a} + 2\right).\ln(P_N) + 1\right)$$

If we define $P_{N,k}^{+}$ as the prime number for which $L_k^{+}\left(P_{N,k}^{+}\right) = \mathrm{Max}_{P_N \epsilon \mathbb{P}_{Max}}\, L_k^{+}(P_N)$, $P_{N,k}$ must satisfy :

$$\frac{dL_k^{+}(P_{N,k}^{+})}{dP_N} = 0 \quad \text{which imply} \quad {P_N}^{1-a} - \frac{(1-a)}{k-1}\left({P_N}^{1-a} + 2\right).\ln(P_N) + 1 = 0$$

If P_N is large, then $P_{N,k} = e^{\frac{k-1}{1-a}}$ gives a first estimation. But a better estimation can be given by :

$$P_{N,k}^{+} \approx e^{\frac{k-1}{1-a}.\left(\frac{1-e^{1-k}}{1-2.e^{1-k}}\right)} \qquad \text{(II.13)}$$

k	Gn	$P^{2-}_{N,k}$	$P_{N,k}$	$P^{+}_{N,k}$	$L^{2-}_{N,k}(P^{2-}_{N,k})$	$L_{N,k}(P_{N,k})$	$L^{+}_{N,k}(P^{+}_{N,k})$
1	2	4	3	1		0,51	
2	4	10	7	4	0,88	1,96	2,18
3	6	25	113	39	3,57	3,51	8,31
4	14	64	113	406	17,40	51,22	52,87
5	34	167	1 327	3 897	100,51	340,57	469,29
6	34	440	1 327	34 737	674,21	2 943,90	5 260,50
7	34	1 166	1 327	296 647	5 163,60	24 740,35	70 657,82
8	72	3 113	31 397	2 479 105	44 526,46	234 027,85	1 102 190,69
9	72	8 350	31 397	20 510 675	427 331,21	2 726 439,61	19 552 555,81
10	72	22 487	31 397	168 935 955	4 520 449,10	31 371 060,15	388 457 386,43
11	112	60 725	370 261	1 388 758 205	52 276 549,54	399 607 495,22	8 541 447 795,12
12	112	164 318	370 261	11 407 170 239	656 291 658,80	5 589 614 319,57	205 891 790 335,23
13	148	445 269	2 010 733	93 666 147 312	8 890 522 390,57	83 617 440 511,50	5 398 385 259 199,99
14	148	1 207 795	2 010 733	769 001 833 747	129 275 727 735,77	1 306 983 392 428,00	152 949 269 629 900,00
15	210	3 278 408	20 831 323	6 313 174 159 164	2 008 483 436 487,93	22 526 940 330 496,00	4 656 405 979 805 580,00
16	210	8 903 032	20 831 323	51 827 276 934 671	33 206 213 701 666,20	404 931 625 041 920,00	151 588 979 197 246 000,00
17	210	24 185 331	20 831 323	425 466 247 153 719	582 112 367 132 113,00	7 250 390 520 168 450,00	5 254 836 747 375 160 000,00
18	282	65 714 327	436 273 009	3 492 772 409 868 440	10 785 366 340 923 100,00	139 301 911 911 203 000,00	193 244 272 347 804 000 000,00
19	282	178 579 276	436 273 009	28 673 113 449 022 000	210 596 296 266 572 000,00	2 925 199 322 721 030 000,00	7 514 032 201 733 600 000 000,00
20	354	485 337 734	4 302 407 359	235 385 214 084 396 000	4 322 390 679 783 290 000,00	61 730 375 961 021 500 000,00	308 016 993 282 465 000 000 000,00
21	456	1 319 121 962	25 056 082 087	1 932 339 407 684 920 000	93 031 549 951 990 100 000,00	1 466 708 506 822 120 000 000,00	13 275 698 427 130 100 000 000 000,00
22	456	3 585 455 134	25 056 082 087	15 863 083 359 450 200 000	2 095 254 447 482 860 000 000,00	36 791 783 075 559 200 000 000,00	600 175 065 414 524 000 000 000 000,00
23	456	9 745 761 797	25 056 082 087	130 224 226 887 788 000 000	49 282 311 735 643 700 000 000,00	921 000 003 891 318 000 000 000,00	28 398 068 380 891 800 000 000 000 000,00
24	456	26 490 812 582	25 056 082 087	1 069 044 943 513 680 000 000	1 208 400 916 738 400 000 000 000,00	23 011 592 656 012 300 000 000 000,00	1 403 541 844 533 810 000 000 000 000 000,00
25	464	72 007 876 002	42 652 618 343	8 776 071 188 729 050 000 000	30 837 322 738 430 800 000 000 000,00	581 344 786 051 355 000 000 000 000,00	72 326 194 190 570 200 000 000 000 000 000,00

Table II.3

II.4 – Estimation of p_{no} such as for $P_N > p_{no}$, $L_k^{2-}(P_N) < L_k^{2-}(P_{N,k}^{2-})$

We are going to try to estimate for which value of p_n, the upper bound function becomes lower than the maximum of the lower bound function.

$$L_k(p_n) < L_k^{+}(p_n) < L_k^{2-}(p_{n,k}^{2-}) = Max_{p_n \in \mathbb{P}} L_k^{2-}(p_n) \qquad \text{(II.14)}$$

As $P_{N,k}^{2-} \approx e^{k.\left(\frac{1-\frac{1}{2}.e^{k(a-1)}}{1-\frac{(2-a)}{2}.e^{k(a-1)}}\right)}$ is a relatively complex formulation, we are going to use the following approximation $P_N^{2-} \approx e^k$. Effectively for instance, for k=4, $\frac{1-\frac{1}{2}.e^{k(a-1)}}{1-\frac{(2-a)}{2}.e^{k(a-1)}} \approx 1{,}04$ and $P_N^{2-} \approx e^k \approx 55$ *and* $P_{N,k}^{2-} \approx 64$. As in the integer sequence (P_N), $P_4 = 23$ and $P_5 = 89$ the approximation of $P_{N,k}^{2-}$ proposed above is coherent.

So we have to estimate $\quad L_k^{2-}(e^k) = k.\frac{\ln^k(e^k)}{e^k}.\left(1 - \frac{1}{2e^{k(1-a)}}\right) < L_k^{2-}(P_{n,k}^{2-})$

We are going to look for a real α>1 such as $p_{no} \approx e^{\alpha.\frac{(k-1)}{(1-a)}}$ which satisfy the following inequality

$$L_k^{+}\left(e^{\alpha.\frac{(k-1)}{(1-a)}}\right) = k.\frac{\ln^{k-1}\left(e^{\alpha.\frac{(k-1)}{(1-a)}}\right)}{e^{\alpha.(k-1)}}.\left(1 + \frac{1}{e^{\alpha.(k-1)}}\right) < k.\frac{\ln^k(e^k)}{e^k}.\left(1 - \frac{1}{2e^{k(1-a)}}\right) = L_k^{2-}(e^k)$$

$$\frac{\left(\alpha.\frac{(k-1)}{(1-a)}\right)^{k-1}}{e^{\alpha.(k-1)}} < \frac{k^k}{e^k}.\frac{\left(1-\frac{1}{2e^{k(1-a)}}\right)}{\left(1+\frac{1}{e^{\alpha.(k-1)}}\right)} \quad \Leftrightarrow \quad \frac{\alpha}{e^\alpha} < \left(\frac{1-a}{k-1}\right).\left(\frac{k}{e}\right)^{\frac{k}{k-1}}.\left(\frac{1-\frac{1}{2e^{k(1-a)}}}{1+\frac{1}{e^{\alpha.(k-1)}}}\right)^{\frac{1}{k-1}}$$

For k>1 we can state that $\quad \left(\frac{1-a}{k-1}\right).\left(\frac{k}{e}\right)^{\frac{k}{k-1}}.\left(\frac{1-\frac{1}{2e^{k(1-a)}}}{1+\frac{1}{e^{\alpha.(k-1)}}}\right)^{\frac{1}{k-1}} > \left(\frac{1-a}{k-1}\right).\frac{k}{e}.\left(\frac{1-\frac{1}{2e^{k(1-a)}}}{1+\frac{1}{e^{(k-1)}}}\right)^{\frac{1}{k-1}} > \frac{1-a}{e}$

As for all integer k, $\left(\frac{k}{k-1}\right).\left(\frac{1-\frac{1}{2e^{k(1-a)}}}{1+\frac{1}{e^{(k-1)}}}\right)^{\frac{1}{k-1}} > 1$. So we have to solve $\frac{\alpha}{e^\alpha} < \frac{1-a}{e}$ which gives $\alpha \approx \frac{11}{4}$

The table beside gives for k<7, the numerical value of $P_{no,k} \approx e^{\frac{11}{4}.\frac{(k-1)}{(1-a)}}$ satisfying (II.14) and compare with $P_{No,k}$ in table II.1 and table II.2. We see that our estimation over-estimate the value of $P_{No,k}$

k	$P_{no,k}$	$P_{No,k}$ in table II.1-II.2
2	327	523
3	106 825	9 551
4	34 914 789	492 113
5	11 411 583 088	122 164 747
6	3 729 772 743 500	10 726 904 659

We can conclude that for all $p_n > e^{\frac{11}{4}.\frac{(k-1)}{(1-a)}}$ and all integer k>1, $L_k(p_n) < L_k^{+}(p_n) < L_k^{2-}(p_{n,k}^{2-})$ which means that $P_{N,k} < e^{e^{\frac{11}{4}.\frac{(k-1)}{(1-a)}}}$ for all integer k.

We have tested numerically all prime of the integer sequence (P_N) corresponding to the prime numbers associated with the first occurrence of the maximal gaps, up to $P_n = 218\ 209\ 405\ 436\ 543$ and $G_n = 906$. We were limited by the accuracy of Excel. We intend in a future work to push the investigation up to $P_n =$

18 361 375 334 787 046 697 corresponding to the highest known maximal gap $G_n = 1550$ and $n =$ 423 731 791 997 205 041, to be able to gives upper bound of $L_k(p_n)$ for some values of k>7.

Thanks to the previous results we can validate the following upper bounds of $L_k(p_n) = \ln^k(p_{n+1}) - \ln^k(p_n)$ for k≤7 (see table II.3)

$$L_1(p_n) = \ln(p_{n+1}) - \ln(p_n) < 0{,}52 \qquad L_2(p_n) = \ln^2(p_{n+1}) - \ln^2(p_n) < 2$$

$$L_3(p_n) = \ln^3(p_{n+1}) - \ln^3(p_n) < 4 \qquad L_4(p_n) = \ln^4(p_{n+1}) - \ln^4(p_n) < 52$$

$$L_5(p_n) = \ln^5(p_{n+1}) - \ln^5(p_n) < 341 \qquad L_6(p_n) = \ln^6(p_{n+1}) - \ln^6(p_n) < 2944$$

$$L_6(p_n) = \ln^7(p_{n+1}) - \ln^7(p_n) < 24741$$

II.5 – Better bounds estimation from recurrence on k

We start from relation (II.4):

$$M_k(p_n).g_n.\left(1 - \frac{g_n}{2p_n}\right) < L_k(p_n) < M_k(p_n).g_n.\left(1 + \frac{g_n}{p_n}\right) \qquad \text{for } p_n > e^{k-1}$$

And according to BHP result, $g_n < (p_n)^a$ with a=0,525

$$M_k(p_n).g_n.\left(1 - \frac{1}{2{p_n}^{1-a}}\right) < L_k(p_n) < M_k(p_n).g_n.\left(1 + \frac{1}{{p_n}^{1-a}}\right) \qquad \text{(II.15)}$$

The advandage of this formulation is the fact that we can establish a relation between $L_{k-1}(p_n)$ and $L_k(p_n)$ for all p_n and without g_n. For $L_{k-1}(p_n)$, we can apply (II.15):

$$M_{k-1}(p_n).g_n.\left(1 - \frac{1}{2{p_n}^{1-a}}\right) < L_{k-1}(p_n) < M_{k-1}(p_n).g_n.\left(1 + \frac{1}{{p_n}^{1-a}}\right) \qquad \text{(II.16)}$$

From (II.15) and (II.16)we can get :

$$\frac{k}{k-1}.\ln(p_n)\,C(p_n) < \frac{L_k(p_n)}{L_{k-1}(p_n)} < \frac{k}{k-1}.\ \frac{\ln(p_n)}{C(p_n)} \qquad \text{with} \quad C(p_n) = \frac{\left(1-\frac{1}{2{p_n}^{1-a}}\right)}{\left(1+\frac{1}{{p_n}^{1-a}}\right)} < 1 \ \text{for all p} \qquad \text{(II.17)}$$

Consider for all integer k, $P_{n,k}$ the prime number for which $F_k(P_{n,k})$ is maximum. In this conditions, we have :

$$\frac{L_k(P_{n,k-1})}{L_{k-1}(P_{n,k-1})} \le \frac{L_k(P_{n,k})}{L_{k-1}(P_{n,k-1})} \le \frac{L_k(P_{n,k})}{L_{k-1}(P_{n,k})}$$

Thanks to (II.16) :

$$\frac{k}{k-1}.\ln(P_{n,k-1})\,C(P_{n,k-1}) < \frac{L_k(P_{n,k-1})}{L_{k-1}(P_{n,k-1})} \le \frac{L_k(P_{n,k})}{L_{k-1}(P_{n,k-1})} \le \frac{L_k(P_{n,k})}{L_{k-1}(P_{n,k})} < \frac{k}{k-1}.\ \frac{\ln(P_{n,k})}{C(P_{n,k})}$$

$$\frac{k}{k-1}.\ln(P_{n,k-1})\,C(P_{n,k-1}) < \frac{L_k(P_{n,k})}{L_{k-1}(P_{n,k-1})} < \frac{k}{k-1}.\ \frac{\ln(P_{n,k})}{C(P_{n,k})}$$

$$\frac{k}{k-1}.\ln(P_{n,k-1})\,C(P_{n,k-1}).L_{k-1}(P_{n,k-1}) < L_k(P_{n,k}) < \frac{k}{k-1}.\ \frac{\ln(P_{n,k})}{C(P_{n,k})}.L_{k-1}(P_{n,k-1}) \qquad \text{(II.18)}$$

We define two new lower and upper bound functions :

$$L_k^{3-}(P_{n,k}) = \frac{k}{k-1}.\ln(P_{n,k-1})\,C(P_{n,k-1}).L_{k-1}(P_{n,k-1}) \quad \text{and} \quad L_k^{2+}(P_{n,k}) = \frac{k}{k-1}.\ \frac{\ln(P_{n,k})}{C(P_{n,k})}\ .L_{k-1}(P_{n,k-1})$$

The table (II.4) below, will give for each integer k up to 30, the numerical values of the lower and upper bounds of $L_k(p_{n,k})$ presented in (II.18).

Thanks to (II.18), we can see that the lower bound of $L_k(p_{n,k})$ is only dependent of the level k-1and gives some numerical values very close from $L_k(p_{n,k})$. It means that we can estimate $L_k(p_{n,k})$ from $L_{k-1}(P_{n,k-1})$ and $P_{n,k-1}$ without any information about prime gaps.

K	Pn,k	Gaps	$L_k^{3-}(P_n)$	$L_k(P_n)/L_k^{3-}(P_n)$	$L_k(P_n)$	$L_k^{2+}(P_n)$	$L_k(P_n)/L_k^{2+}(P_n)$
1	3	2			0,67		
2	7	4	0,65	3,03676571	1,96	4,52	0,43426304
3	113	14	3,29	2,44040167	8,03	16,26	0,49369214
4	113	14	43,32	1,18222958	51,22	59,07	0,86705919
5	1 327	34	259,20	1,31394732	340,57	483,44	0,70447933
6	1 327	34	2 798,49	1,05196132	2 943,90	3 085,98	0,95396059
7	1 327	34	23 518,28	1,05196240	24 740,35	25 934,32	0,95396157
8	31 397	72	193 612,23	1,20874515	234 027,85	295 991,73	0,79065671
9	31 397	72	2 696 458,70	1,01111862	2 726 439,61	2 756 143,97	0,98922249
10	31 397	72	31 026 092,56	1,01111863	31 371 060,15	31 712 845,42	0,98922250
11	370 261	112	353 423 636,96	1,13067564	399 607 495,22	443 967 035,74	0,90008371
12	370 261	112	5 570 605 585,19	1,00341233	5 589 614 319,57	5 608 555 617,79	0,99662278
13	2 010 733	148	77 379 188 888,64	1,08061924	83 617 440 511,50	88 022 066 199,56	0,94995998
14	2 010 733	148	1 304 994 775 159,13	1,00152385	1 306 983 392 428,00	1 308 968 402 035,86	0,99848353
15	20 831 323	210	20 293 666 177 617,60	1,11004784	22 526 940 330 496,00	23 610 295 435 578,90	0,95411514
16	20 831 323	210	404 728 764 895 014,00	1,00050122	404 931 625 041 920,00	405 134 344 485 731,00	0,99949962
17	20 831 323	210	7 246 758 264 556 170,00	1,00050122	7 250 390 520 168 450,00	7 254 020 257 045 710,00	0,99949962
18	436 273 009	282	129 305 834 748 391 000,00	1,07730569	139 301 911 911 203 000,00	152 740 268 939 947 000,00	0,91201824
19	436 273 009	282	2 924 853 870 792 310 000,00	1,00011811	2 925 199 322 721 030 000,00	2 925 544 719 503 790 000,00	0,99988194
20	4 302 407 359	354	61 248 838 739 379 200 000,00	1,00786198	61 730 375 961 021 500 000,00	68 305 940 868 206 000 000,00	0,90373363
21	25 056 082 087	456	1 437 739 666 357 580 000 000,00	1,02014888	1 466 708 506 822 120 000 000,00	1 552 027 277 194 470 000 000,00	0,94502753
22	25 056 082 087	456	36 791 148 800 458 500 000 000,00	1,00001724	36 791 783 075 559 200 000 000,00	36 792 417 582 700 100 000 000,00	0,99998275
23	25 056 082 087	456	920 984 122 302 756 000 000 000,00	1,00001724	921 000 003 891 318 000 000 000,00	921 015 883 428 386 000 000 000,00	0,99998276
24	25 056 082 087	456	23 011 195 895 870 800 000 000 000,00	1,00001724	23 011 592 656 012 300 000 000 000,00	23 011 989 461 651 200 000 000 000,00	0,99998276
25	42 652 618 343	464	573 946 744 068 867 000 000 000 000,00	1,01288977	581 344 786 051 355 000 000 000 000,00	586 716 086 234 403 000 000 000 000,00	0,99084515
26	2 614 941 710 599	652	14 798 170 925 310 900 000 000 000 000,00	1,11625268	16 518 497 946 691 300 000 000 000 000,00	17 286 874 263 177 600 000 000 000 000,00	0,95555146
27	2 614 941 710 599	652	490 465 738 715 257 000 000 000 000 000,00	1,00000123	490 466 342 952 388 000 000 000 000 000,00	490 467 598 386 435 000 000 000 000 000,00	0,99999744
28	2 614 941 710 599	652	14 542 905 529 616 300 000 000 000 000 000,00	1,00000187	14 542 932 725 262 400 000 000 000 000 000,00	14 542 960 671 127 900 000 000 000 000 000,00	0,99999808
29	19 581 334 192 423	766	430 665 088 119 679 000 000 000 000 000 000,00	1,05459080	454 175 441 613 020 000 000 000 000 000 000,00	460 991 742 408 187 000 000 000 000 000 000,00	0,98521383
30	19 581 334 192 423	766	14 379 621 528 494 800 000 000 000 000 000 000,00	0,99999813	14 379 594 583 688 900 000 000 000 000 000 000,00	14 379 642 481 382 800 000 000 000 000 000 000,00	0,99999667

II.6 – Analysis of the integer sequence of k, $(P_{n,k})$:

When we look at the numerical results of the table (II.3) and (II.4), we can see that the values of the prime numbers corresponding to the maximal values of $L_k(p_n)$, is a growing integer sequence. We are going to prove this assertion below .

If we consider that the relation (II.17):

$$\frac{k}{k-1}.\ln(p_n).C(p_n) < \frac{L_k(p_n)}{L_{k-1}(p_n)} < \frac{k}{k-1}.\ \frac{\ln(p_n)}{C(p_n)}$$

$$\frac{k}{k-1}.\ln(p_n).C(p_n).L_{k-1}(p_n) < L_k(p_n) < \frac{k}{k-1}.\ \frac{\ln(p_n)}{C(p_n)}.L_{k-1}(p_n)$$

$$\frac{k}{k-1}.\ln(p_n).(1-C(p_n)).L_{k-1}(p_n) < \frac{k}{k-1}.\ln(p_n).L_{k-1}(p_n) - L_k(p_n) < \frac{k}{k-1}.\ \ln(p_n).\left(\frac{1-C(p_n)}{C(p_n)}\right).L_{k-1}(p_n) \quad \text{(II.19)}$$

As $C(p_n) = \frac{\left(1-\frac{1}{2p_n^{1-a}}\right)}{\left(1+\frac{1}{p_n^{1-a}}\right)} < 1$ we have $1-C(p_n) = 1-\left(\frac{1-\frac{1}{2p_n^{1-a}}}{1+\frac{1}{p_n^{1-a}}}\right) = \left(\frac{\frac{3}{2p_n^{1-a}}}{1+\frac{1}{p_n^{1-a}}}\right)$

then for $p_n > 3$ we have: $\frac{1}{p_n^{1-a}} < 1 - C(p_n) < \frac{3}{2p_n^{1-a}}$ and $\frac{1}{2} < C(p_n) < 1$

We can replace into (II.19):

$$0 < \frac{k}{k-1}.\frac{\ln(p_n)}{p_n^{1-a}}.L_{k-1}(p_n) < \frac{k}{k-1}.\ln(p_n).L_{k-1}(p_n) - L_k(p_n) < \frac{3k}{k-1}.\frac{\ln(p_n)}{p_n^{1-a}}.L_{k-1}(p_n)$$

$$0 < \frac{1}{p_n^{1-a}} < 1 - \frac{L_k(p_n)}{\frac{k}{k-1}.\ln(p_n).L_{k-1}(p_n)} < \frac{3}{p_n^{1-a}}$$

$$\frac{k}{k-1}.\ln(p_n).L_{k-1}(p_n).\left(1 - \frac{3}{p_n^{1-a}}\right) < L_k(p_n) < \frac{k}{k-1}.\ln(p_n).L_{k-1}(p_n).\left(1 - \frac{1}{p_n^{1-a}}\right) \quad \text{(II.20)}$$

(II.20) shows that $L_k(p_n) = \frac{k}{k-1}.\ln(p_n).L_{k-1}(p_n).\left(1 - O\left(\frac{1}{p_n^{1-a}}\right)\right)$ with $\frac{1}{p_n^{1-a}} < O\left(\frac{1}{p_n^{1-a}}\right) < \frac{3}{p_n^{1-a}}$ (II.21)

Which means that we can link the behavior of $L_k(p_n)$ to the behavior of $L_{k-1}(p_n)$. We are going now to compute the first derivative of the two parts of (II.21).

$$\frac{dL_k(p_n)}{dp_n} = \frac{k}{k-1}.\left(L_{k-1}(p_n).\left(\frac{1}{p_n} + \ln(p_n).O\left(\frac{1}{p_n^{2-a}}\right)\right) + \ln(p_n).\frac{dL_{k-1}(p_n)}{dp_n}\right) \quad \text{(II.22)}$$

Considering the prime number $p_{n,k-1}$ defined by $L_{k-1}(p_n) = \mathrm{Max}_{p_n \in \mathbb{P}} L_{k-1}(p_n)$ and which imply $\frac{dL_{k-1}(p_n)}{dp_n} = 0$

then :

$$\frac{dL_k(p_{n,k-1})}{dp_n} = \frac{k}{k-1}.\left(\frac{L_{k-1}(p_{n,k-1})}{p_{n,k-1}} + \ln(p_{n,k-1}).\frac{dL_{k-1}(p_{n,k-1})}{dp_n}\right) = \frac{k}{k-1}.\frac{L_{k-1}(p_{n,k-1})}{p_{n,k-1}} > 0 \quad \text{(II.23)}$$

Which means that for all integer k, $p_{n,k-1} \le p_{n,k}$

Then we can conclude that for $p_{n,k}$ enough large, $p_{n,k-1} \le p_{n,k}$ and we can verifie in tables 1 and 2 that it is true for all $p_{n,k}$.

II.6 – $G_{k,r}(p_n) = Ln^k(p_{n+1})^r - Ln^k(p_n)^r$ bounds estimation from recurrence on k

The function $G_{k,r}(p_n)$ is linked to $L_k(p_n)$ by : $G_{k,r}(p_n) = \ln^k(p_{n+1})^r - \ln^k(p_n)^r = r^k\, L_k(p_n)$

All the proofs, in the previous paragraphs, regarding $L_k(p_n)$ are systematically applicable to $G_{k,r}(p_n)$

$G_1(p_n) = \ln(p_{n+1})^r - \ln(p_n)^r < 0{,}52 * r$ $\quad$ $G_2(p_n) = \ln^2(p_{n+1})^r - \ln^2(p_n)^r < 2 * r^2$

$G_3(p_n) = \ln^3(p_{n+1})^r - \ln^3(p_n)^r < 4 * r^3$ $\quad$ $G_4(p_n) = \ln^4(p_{n+1})^r - \ln^4(p_n)^r < 52 * r^4$

$G_5(p_n) = \ln^5(p_{n+1})^r - \ln^5(p_n)^r < 341 * r^5$ $\quad$ $G_6(p_n) = \ln^6(p_{n+1})^r - \ln^6(p_n)^r < 2944 * r^6$

$$G_7(p_n) = \ln^7(p_{n+1})^r - \ln^7(p_n)^r < 24741 * r^7$$

II.7 – $L_y(p_n) = Ln^y(p_{n+1}) - Ln^y(p_n) = 1$ has a solution for all p_n

As proved in (II.4), we can write for all real $y \in \mathbb{R}^+$

$$M_y(p_n).g_n.\left(1-\frac{g_n}{2p_n}\right) < L_y(p_n) < M_y(p_n).g_n.\left(1+\frac{g_n}{p_n}\right) \qquad \text{for } p_n > e^{k-1}$$

With $M_y(p_n) = y.\frac{\ln^{y-1}(p_n)}{p_n}$. So we can focus our study on $M_y(p_n).g_n$ which represent a very good approximation of $L_y(p_n)$. We are going to show that $M_y(p_n).g_n = 1$ has a solution y=y(p_n) for all p_n.

$$M_y(p_n).g_n = y.\frac{\ln^{y-1}(p_n)}{p_n}.g_n = 1 \qquad \Leftrightarrow \qquad y.\ln^{y-1}(p_n) = \frac{p_n}{g_n} \tag{II.24}$$

We have shown previously that $0 < L_y(p_n) < 1$, so we can insure that if y satisfy (II.24), then $y(p_n) > 1$ for all p_n

We can take the logarithm of booth parts of (II.24):

$$\ln(y) + (y-1).\ln(\ln(p_n)) = \ln\left(\frac{p_n}{g_n}\right) \tag{II.25}$$

We may have a first solution of (II.25), supposing that ln(y) is very small :

$$y_o - 1 \approx \frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln(\ln(p_n))} \tag{II.26}$$

We can see in table II.5 below that the values of y_o correspond to the value of $L_{y_o}(p_n)$ which is normal because we have neglected the term ln(y) into (II.25). For all p_n, $L_{y_o}(p_n) > 1$

We are going to refine this first solution by searching ε such as $y_1 = y_o - \varepsilon_o$

$$y_1 - 1 = \frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln(\ln(p_n))} - \varepsilon_o \qquad \text{and} \quad y_1 = \frac{\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n))} - \varepsilon_o \tag{II.27}$$

We replace (II.27) into (II.24) and we obtain :

$$\left(\frac{\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n))} - \varepsilon_o\right).\ln^{\frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln(\ln(p_n))}-\varepsilon_o}(p_n) = \frac{p_n}{g_n}$$

$$\left(\frac{\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n))} - \varepsilon\right).e^{\ln\left(\frac{p_n}{g_n}\right)-\varepsilon_o.\ln(\ln(p_n))} = \frac{p_n}{g_n}$$

$$\left(\frac{\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n))} - \varepsilon\right).e^{-\varepsilon_o.\ln(\ln(p_n))} = 1$$

Considering $\varepsilon.\ln(\ln(p_n))$ enough small, we can replace $e^{-\varepsilon_o.\ln(\ln(p_n))}$ by its taylor serie expansion

$$\left(\frac{\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n))} - \varepsilon_o\right).\left(1-\varepsilon_o.\ln(\ln(p_n))\right) \approx 1$$

$$\left(\frac{\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n))} - 1\right) \approx \varepsilon_o.\left(\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)+1\right)$$

Which gives an estimation of ε : $\varepsilon_o \approx \dfrac{\ln\left(\frac{p_n}{g_n}\right)}{\ln(\ln(p_n)).\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)}$ then $y_1 - 1 \approx \dfrac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n)).\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)}$ (II.28)

In table II.5, we can see that the values of $L_{y_o}(p_n)$ are closer to 1, with $L_{y_o}(p_n) > L_{y_1}(p_n) > 1$. We are going to iterate a second time this process to have a better estimation of y.

We define $y_2 = y_1 - \varepsilon_1$

$$y_2 - 1 = \frac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n)).\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)} - \varepsilon_1 \tag{II.29}$$

If we replace y_2 into (II.24), we have :

$$\left(\frac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n)).\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)} + 1 - \varepsilon_1\right).e^{\frac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)} - \varepsilon_1.\ln(\ln(p_n))} = \frac{p_n}{g_n}$$

$$\left(\frac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n)).\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)} + 1 - \varepsilon_1\right).e^{\ln\left(\frac{p_n}{g_n}\right) - 1 + \frac{\ln(e.\ln(p_n))}{\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)} - \varepsilon_1.\ln(\ln(p_n))} = \frac{p_n}{g_n}$$

$$\left(\frac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n)).\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)} + 1 - \varepsilon_1\right).e^{-1 + \frac{\ln(e.\ln(p_n))}{\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)} - \varepsilon_1.\ln(\ln(p_n))} = 1$$

$$\left(\frac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n)).\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)} + 1 - \varepsilon_1\right).e^{-\varepsilon_1.\ln(\ln(p_n))} = e^{1 - \frac{\ln(e.\ln(p_n))}{\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)}} \tag{II.30}$$

As before, we can replace $e^{-\varepsilon_o.\ln(\ln(p_n))}$ and the second term of (II.28) by its taylor serie expansion:

$$\left(\frac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n)).\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)} + 1 - \varepsilon_1\right).\left(1 - \varepsilon_1.\ln(\ln(p_n))\right) \approx e^{\frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)}}$$

$$\left(\frac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n)).\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)} + 1 - e^{\frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)}}\right) \approx \varepsilon_1.\left(1 + \frac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)}\right)$$

$$\left(\frac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n)).\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)} + 1 - e^{\frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)}}\right) \approx \varepsilon_1.\left(1 - \frac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)}\right)$$

$$\varepsilon_1 \approx \frac{\left(\frac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n))} + \ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right).\left(1 - e^{\frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)}}\right)\right)}{\left(\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right) + \ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)\right)}$$

$$y_2 \approx 1 + \frac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n)).\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)} - \frac{\left(\frac{\ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)}{\ln(\ln(p_n))} + \ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right).\left(1 - e^{\frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right)}}\right)\right)}{\left(\ln\left(e.\frac{p_n}{g_n}.\ln(p_n)\right) + \ln\left(\frac{p_n}{g_n}\right).\ln\left(\frac{p_n}{g_n}.\ln(p_n)\right)\right)}$$

In table II.5, y_2 seems to give a good estimation of y (represented by y_{ref}), with an accuracy better than 1,5% .

Primes	Gaps	ln(Pn)	Y_o	M_{Yo}	ε_0	Y_1	M_{Y1}	ε_1	Y_2	M_{Y2}	Yref	M_{Yref}	Yref-Y_2
2	1	0,69	-0,89	-0,47	-1,43	0,53	0,23	-0,98	1,52	0,58	3,01	1,00	1,49
3	2	1,10	5,31	10,87	2,88	2,44	1,93	0,99	1,44	0,84	1,62	1,00	0,18
7	4	1,95	1,84	1,60	0,38	1,46	0,95	0,14	1,33	0,77	1,50	1,00	0,17
23	6	3,14	2,18	2,02	0,34	1,84	1,15	0,19	1,65	0,82	1,76	1,00	0,11
89	8	4,49	2,60	2,53	0,33	2,28	1,35	0,22	2,06	0,87	2,13	1,01	0,07
113	14	4,73	2,34	2,25	0,29	2,05	1,25	0,18	1,87	0,85	1,95	1,01	0,08
523	18	6,26	2,84	2,80	0,30	2,54	1,46	0,21	2,33	0,90	2,37	0,99	0,04
887	20	6,79	2,98	2,96	0,30	2,68	1,51	0,22	2,47	0,91	2,50	0,99	0,03
1 129	22	7,03	3,02	3,00	0,29	2,73	1,53	0,22	2,51	0,92	2,55	1,01	0,04
1 327	34	7,19	2,86	2,83	0,28	2,58	1,47	0,20	2,37	0,90	2,41	0,99	0,04
9 551	36	9,16	3,52	3,51	0,29	3,23	1,71	0,23	3,01	0,96	3,02	1,00	0,01
15 683	44	9,66	3,59	3,59	0,28	3,31	1,74	0,23	3,08	0,97	3,09	0,99	0,01
19 609	52	9,88	3,59	3,59	0,28	3,31	1,74	0,22	3,09	0,97	3,10	1,01	0,01
31 397	72	10,35	3,60	3,60	0,28	3,32	1,74	0,22	3,10	0,97	3,11	0,99	0,01
155 921	86	11,96	4,02	4,02	0,28	3,75	1,89	0,23	3,52	1,01	3,52	1,01	0,00
360 653	96	12,80	4,23	4,23	0,27	3,95	1,97	0,23	3,73	1,04	3,71	0,99	-0,02
370 261	112	12,82	4,18	4,18	0,27	3,90	1,95	0,23	3,68	1,03	3,67	1,00	-0,01
492 113	114	13,11	4,25	4,25	0,27	3,98	1,97	0,23	3,75	1,04	3,74	1,00	-0,01
1 349 533	118	14,12	4,53	4,53	0,27	4,26	2,07	0,23	4,03	1,07	4,01	1,01	-0,02
1 357 201	132	14,12	4,49	4,49	0,27	4,22	2,06	0,23	3,99	1,07	3,97	1,00	-0,02
2 010 733	148	14,51	4,56	4,56	0,27	4,29	2,08	0,23	4,06	1,07	4,04	1,01	-0,02
4 652 353	154	15,35	4,78	4,78	0,27	4,51	2,16	0,23	4,28	1,10	4,25	1,01	-0,03
17 051 707	180	16,65	5,07	5,07	0,27	4,81	2,27	0,23	4,58	1,13	4,54	1,01	-0,04
20 831 323	210	16,85	5,07	5,07	0,27	4,81	2,27	0,23	4,58	1,13	4,54	1,01	-0,04
47 326 693	220	17,67	5,28	5,28	0,26	5,01	2,34	0,23	4,78	1,16	4,73	0,99	-0,05
122 164 747	222	18,62	5,52	5,52	0,26	5,26	2,43	0,23	5,03	1,19	4,97	0,99	-0,06
189 695 659	234	19,06	5,62	5,62	0,26	5,35	2,47	0,23	5,12	1,20	5,07	1,01	-0,05
191 912 783	248	19,07	5,60	5,60	0,26	5,34	2,46	0,23	5,11	1,20	5,05	1,00	-0,06
387 096 133	250	19,77	5,78	5,78	0,26	5,51	2,52	0,23	5,28	1,22	5,22	0,99	-0,06
436 273 009	282	19,89	5,77	5,77	0,26	5,50	2,52	0,23	5,28	1,22	5,21	0,99	-0,07
1 294 268 491	288	20,98	6,03	6,03	0,26	5,77	2,62	0,23	5,54	1,25	5,47	0,99	-0,07
1 453 168 141	292	21,10	6,06	6,06	0,26	5,80	2,63	0,23	5,57	1,25	5,50	1,01	-0,07
2 300 942 549	320	21,56	6,14	6,14	0,26	5,88	2,66	0,23	5,65	1,26	5,58	0,99	-0,07
3 842 610 773	336	22,07	6,25	6,25	0,26	5,99	2,70	0,23	5,77	1,28	5,69	1,00	-0,08
4 302 407 359	354	22,18	6,26	6,26	0,26	6,01	2,70	0,23	5,78	1,28	5,70	0,99	-0,08
10 726 904 659	382	23,10	6,46	6,46	0,26	6,21	2,77	0,23	5,98	1,30	5,90	1,01	-0,08
20 678 048 297	384	23,75	6,62	6,62	0,26	6,36	2,83	0,23	6,14	1,32	6,05	1,00	-0,09
22 367 084 959	394	23,83	6,63	6,63	0,26	6,37	2,83	0,23	6,15	1,32	6,06	0,99	-0,09
25 056 082 087	456	23,94	6,61	6,61	0,26	6,36	2,83	0,23	6,13	1,32	6,05	1,00	-0,08
42 652 618 343	464	24,48	6,73	6,73	0,25	6,48	2,87	0,23	6,25	1,34	6,17	1,00	-0,09
127 976 334 671	468	25,58	6,99	6,99	0,25	6,74	2,97	0,23	6,51	1,37	6,42	1,00	-0,09
182 226 896 239	474	25,93	7,07	7,07	0,25	6,82	2,99	0,23	6,59	1,38	6,50	0,99	-0,10
241 160 624 143	486	26,21	7,13	7,13	0,25	6,88	3,02	0,23	6,65	1,39	6,56	1,00	-0,10
297 501 075 799	490	26,42	7,18	7,18	0,25	6,92	3,03	0,23	6,70	1,39	6,60	1,00	-0,10
303 371 455 241	500	26,44	7,18	7,18	0,25	6,92	3,03	0,23	6,70	1,39	6,60	1,00	-0,10
304 599 508 537	514	26,44	7,17	7,17	0,25	6,92	3,03	0,23	6,69	1,39	6,59	0,99	-0,10
416 608 695 821	516	26,76	7,24	7,24	0,25	6,99	3,06	0,23	6,76	1,40	6,66	0,99	-0,10
461 690 510 011	532	26,86	7,25	7,25	0,25	7,00	3,06	0,23	6,78	1,40	6,68	1,01	-0,10
614 487 453 523	534	27,14	7,32	7,32	0,25	7,07	3,09	0,23	6,84	1,41	6,74	0,99	-0,10
738 832 927 927	540	27,33	7,36	7,36	0,25	7,11	3,10	0,23	6,88	1,42	6,78	1,00	-0,10
1 346 294 310 749	582	27,93	7,48	7,48	0,25	7,23	3,14	0,23	7,00	1,43	6,90	1,01	-0,10
1 408 695 493 609	588	27,97	7,48	7,48	0,25	7,23	3,14	0,23	7,01	1,43	6,90	0,99	-0,11
1 968 188 556 461	602	28,31	7,55	7,55	0,25	7,30	3,17	0,23	7,08	1,44	6,97	0,99	-0,11
2 614 941 710 599	652	28,59	7,59	7,59	0,25	7,35	3,19	0,23	7,12	1,45	7,02	1,00	-0,10
7 177 162 611 713	674	29,60	7,82	7,82	0,25	7,57	3,27	0,23	7,34	1,47	7,23	1,00	-0,11
13 829 048 559 701	716	30,26	7,95	7,95	0,25	7,70	3,31	0,23	7,47	1,49	7,36	1,00	-0,11
19 581 334 192 423	766	30,61	8,00	8,01	0,25	7,76	3,34	0,23	7,53	1,50	7,42	1,00	-0,11
42 842 283 925 351	778	31,39	8,18	8,18	0,25	7,93	3,40	0,22	7,71	1,52	7,59	1,01	-0,12
90 874 329 411 493	804	32,14	8,33	8,34	0,25	8,09	3,46	0,22	7,86	1,54	7,75	1,00	-0,12
171 231 342 420 521	806	32,77	8,47	8,46	0,24	8,23	3,50	0,22	8,01	1,56	7,89	1,02	-0,12
218 209 405 436 543	906	33,02	8,49	8,50	0,24	8,25	3,51	0,22	8,03	1,56	7,90	0,99	-0,13

Part 3 : A third variant of Andrica's conjecture

We propose in this part to study the behavior of a third variant of the Generalized Andrica's conjecture represented by the set of function $H_{k,x}(p_n) = (p_{n+1})^x.\ln^k(p_{n+1}) - (p_n)^x.\ln^k(p_n)$ and the values of $x \in \mathbb{R}$ and $k \in \mathbb{N}$ where $H_{k,x}(p_n)$ is bounded.

III.1 – Upper bound functions of $H_{k,x}(p_n)$:

$$H_{k,x}(p_n) = (p_{n+1})^x.\ln^k(p_{n+1}) - (p_n)^x.\ln^k(p_{n+1}) + (p_n)^x.\ln^k(p_{n+1}) - (p_n)^x.\ln^k(p_n)$$

$$H_{k,x}(p_n) = ((p_{n+1})^x - (p_n)^x).\ln^k(p_{n+1}) + (p_n)^x.\left(\ln^k(p_{n+1}) - \ln^k(p_n)\right) \quad \text{(III.1)}$$

$$H_{k,x}(p_n) <.\left(x.(p_n)^x.\frac{g_n}{p_n}\right)\ln^k(p_{n+1}) + (p_n)^x.\left(k.\frac{\ln^{k-1}(p_{n+1})}{p_{n+1}}.g_n.\left(1+\frac{g_n}{p_n}\right)\right)$$

$$H_{k,x}(p_n) < x.(p_n)^x.\frac{\ln^k(p_{n+1})}{p_{n+1}}.g_n.\left(1+\frac{g_n}{p_n}\right) + k.(p_n)^x.\frac{\ln^{k-1}(p_{n+1})}{p_{n+1}}.g_n.\left(1+\frac{g_n}{p_n}\right)$$

$$H_{k,x}(p_n) < (p_n)^x.\frac{\ln^{k-1}(p_{n+1})}{p_{n+1}}.g_n(x.\ln(p_{n+1}) + k).\left(1+\frac{g_n}{p_n}\right)$$

As we have state in (II.2), for all $p_n > e^{k-1}$, $\frac{\ln^{k-1}(p_{n+1})}{p_{n+1}} < \frac{\ln^{k-1}(p_n)}{p_n}$

$$H_{k,x}(p_n) < (p_n)^x.\frac{\ln^{k-1}(p_n)}{p_n}.g_n\left(x.\left(\ln(p_n) + \ln\left(1+\frac{g_n}{p_n}\right)\right) + k\right).\left(1+\frac{g_n}{p_n}\right)$$

$$H_{k,x}(p_n) < (p_n)^x.\frac{\ln^{k-1}(p_n)}{p_n}.g_n\left(x.\left(\ln(p_n) + \frac{g_n}{p_n} - \frac{1}{2}\left(\frac{g_n}{p_n}\right)^2\right) + k\right).\left(1+\frac{g_n}{p_n}\right)$$

$$H_{k,x}(p_n) < (p_n)^x.\frac{\ln^{k-1}(p_n)}{p_n}.g_n\left(x.\ln(p_n) + k + (x.\ln(p_n) + k + x).\frac{g_n}{p_n} + \frac{x}{2}\left(\frac{g_n}{p_n}\right)^2\right)$$

$$H_{k,x}(p_n) < (p_n)^x.\frac{\ln^{k-1}(p_n)}{p_n}.g_n.(x.\ln(p_n) + k).\left(1+\frac{(x.\ln(p_n)+k+2x)}{x.\ln(p_n)+k}.\frac{g_n}{p_n}\right)$$

$$H_{k,x}(p_n) < (p_n)^x.\frac{\ln^{k-1}(p_n)}{p_n}.g_n.(x.\ln(p_n) + k).\left(1+\frac{\left(x+\frac{k+2x}{\ln(p_n)}\right)}{x+\frac{k}{\ln(p_n)}}.\frac{g_n}{p_n}\right)$$

Thanks to BHP result about prime gaps, $g_n < p_n^a$ with $a = 0{,}525$.

$$H_{k,x}(p_n) < (p_n)^x.\frac{\ln^k(p_n)}{p_n}.g_n.\left(x+\frac{k}{\ln(p_n)}\right).\left(1+\frac{1}{p_n^{1-a}}.\frac{\left(x+\frac{k+2x}{\ln(p_n)}\right)}{x+\frac{k}{\ln(p_n)}}\right) \quad \text{(III.2)}$$

We can reformulate $\frac{\left(x+\frac{k+2x}{\ln(p_n)}\right)}{x+\frac{k}{\ln(p_n)}} = 1 + \frac{2x}{x.\ln(p_n)+k} < 2$ for all integer k and all x<1and all prime number

We define $M_{k,x}(p_n) = \frac{\ln^k(p_n)}{p_n^{1-x}}.g_n.\left(x+\frac{k}{\ln(p_n)}\right)$ and $\varepsilon = \frac{2}{p_n^{1-a}}$.

$$H_{k,x}(p_n) < M_{k,x}(p_n).(1+\varepsilon) \quad \text{(III.3)}$$

III.2 – Lower bound functions of $H_{k,x}(p_n)$

We start from (III.1) : $H_{k,x}(p_n) = ((p_{n+1})^x - (p_n)^x).\ln^k(p_{n+1}) + (p_n)^x.\left(\ln^k(p_{n+1}) - \ln^k(p_n)\right)$ (III.4)

As $\ln^k(p_{n+1}) > \ln^k(p_n)$ and $\ln^k(p_{n+1}) - \ln^k(p_n) > k.\ln^{k-1}(p_n).(\ln(p_{n+1}) - \ln(p_{n+1})$

$$H_{k,x}(p_n) > (p_n)^x.\left(\left(1+\frac{g_n}{p_n}\right)^x - 1\right).\ln^k(p_{n+1}) + k.(p_n)^x.\ln^{k-1}(p_n).(\ln(p_{n+1}) - \ln(p_{n+1})$$

$$H_{k,x}(p_n) > (p_n)^x.\left(\left(1+\frac{g_n}{p_n}\right)^x - 1\right).\ln^k(p_{n+1}) + k.(p_n)^x.\ln^{k-1}(p_n).\ln\left(1+\frac{g_n}{p_n}\right)$$

Thanks to $\left(1+\frac{g_n}{p_n}\right)^x - 1 > x.\frac{g_n}{p_n}.\left(1-\frac{1-x}{2}.\frac{g_n}{p_n}\right)$ and $\ln\left(1+\frac{g_n}{p_n}\right) > \frac{g_n}{p_n}.\left(1-\frac{1}{2}.\frac{g_n}{p_n}\right)$

$$H_{k,x}(p_n) > (p_n)^x.x.\frac{g_n}{p_n}.\left(1-\frac{1-x}{2}.\frac{g_n}{p_n}\right).\ln^k(p_{n+1}) + k.(p_n)^x.\ln^{k-1}(p_n).\frac{g_n}{p_n}.\left(1-\frac{1}{2}.\frac{g_n}{p_n}\right)$$

$$H_{k,x}(p_n) > (p_n)^x.\ln^k(p_{n+1})\ \frac{g_n}{p_n}.\left(x.\left(1-\frac{1-x}{2}.\frac{g_n}{p_n}\right) + \frac{k}{\ln(p_n)}.\left(1-\frac{1}{2}.\frac{g_n}{p_n}\right)\right)$$

$$H_{k,x}(p_n) > (p_n)^x.\ln^k(p_{n+1})\ \frac{g_n}{p_n}.\left(x.\left(1-\frac{1-x}{2}.\frac{g_n}{p_n}\right) + \frac{k}{\ln(p_n)}.\left(1-\frac{1}{2}.\frac{g_n}{p_n}\right)\right)$$

$$H_{k,x}(p_n) > (p_n)^x.\ln^k(p_{n+1})\ \frac{g_n}{p_n}.\left(x+\frac{k}{\ln(p_n)} - \frac{g_n}{2p_n}.\left(x(1-x)+\frac{k}{\ln(p_n)}\right)\right)$$

$$H_{k,x}(p_n) > (p_n)^x.\ln^k(p_{n+1})\ \frac{g_n}{p_n}.\left(x+\frac{k}{\ln(p_n)}\right)\left(1-\frac{g_n}{2p_n}.\frac{x(1-x)+\frac{k}{\ln(p_n)}}{x+\frac{k}{\ln(p_n)}}\right)$$

Thanks to BHP result about prime gaps, $g_n < p_n^a$ with $a = 0{,}525$.

$$H_{k,x}(p_n) > (p_n)^x.\ln^k(p_{n+1})\ \frac{g_n}{p_n}.\left(x+\frac{k}{\ln(p_n)}\right)\left(1-\frac{1}{2{p_n}^{1-a}}.\frac{x(1-x)+\frac{k}{\ln(p_n)}}{x+\frac{k}{\ln(p_n)}}\right) > M_{k,x}(p_n).\left(1-\frac{1}{2{p_n}^{1-a}}\right) \quad \text{(III.5)}$$

As we can easily verify that $\frac{x(1-x)+\frac{k}{\ln(p_n)}}{x+\frac{k}{\ln(p_n)}} < 1$ thus $1-\frac{1}{2{p_n}^{1-a}}.\frac{x(1-x)+\frac{k}{\ln(p_n)}}{x+\frac{k}{\ln(p_n)}} > 1-\frac{1}{2{p_n}^{1-a}}$

We define $\varepsilon' = \frac{1}{2{p_n}^{1-a}}$, we can state:

$$M_{k,x}(p_n).(1-\ \varepsilon') < H_{k,x}(p_n) \quad \text{(III.6)}$$

We can finaly give the lower and upper bound functions of $H_{k,x}(p_n)$:

$$M_{k,x}(p_n).(1-\varepsilon') < H_{k,x}(p_n) < M_{k,x}(p_n).(1+\varepsilon) \quad \text{(III.7)}$$

As ε and ε′ tends toward to 0 when n → ∞, we can confirm that $M_{k,x}(p_n)$ represents a good estimator $H_{k,x}(p_n)$ and we can limit our study in the next paragraphs to $M_{k,x}(p_n)$ behavior.

III.3 – Behavior of $M_{k,x}(p_n)$

At first, we can easily state that $M_{k,x}(p_n) = \frac{\ln^k(p_n)}{{p_n}^{1-x}}.g_n.\left(x + \frac{k}{\ln(p_n)}\right)$ is a bounded function, for all integer k and x<1-a, which tends toward 0 when n→∞.

$$0 < M_{k,x}(p_n) < \frac{\ln^k(p_n)}{{p_n}^{1-a-x}}.\left(x + \frac{k}{\ln(p_n)}\right) \text{ and } \quad \lim_{p_n \in \mathbb{P}} M_{k,x}(p_n) < \lim_{p_n \in \mathbb{P}}\left(\frac{\ln^k(p_n)}{{p_n}^{1-a-x}}.\left(x + \frac{k}{\ln(p_n)}\right)\right) = 0$$

We are going compute the derivative function of $M_{k,x}(p_n)$

To derivate $M_{k,x}(p_n)$ we shall consider that g is a derivable function on R which satisfy $g_n = g(p_n)$

$$\frac{dM_{k,x}(p_n)}{dp_n} = \left(\frac{k.\ln^{k-1}(p_n)}{{p_n}^{2-x}} - (1-x)\frac{\ln^k(p_n)}{{p_n}^{2-x}}\right).g_n.\left(x + \frac{k}{\ln(p_n)}\right) - \frac{\ln^k(p_n)}{{p_n}^{1-x}}.g_n.\frac{k}{p_n.\ln^2(p_n)} + \frac{\ln^k(p_n)}{{p_n}^{1-x}}.g'_n.\left(x + \frac{k}{\ln(p_n)}\right)$$

$$\frac{dM_{k,x}(p_n)}{dp_n} = \frac{\ln^{k-2}(p_n)}{{p_n}^{2-x}}.\left(\left(k.\ln(p_n) - (1-x).\ln^2(p_n)\right).\left(x + \frac{k}{\ln(p_n)}\right).g_n - k.g_n + \ln^2(p_n).p_n.g'_n.\left(x + \frac{k}{\ln(p_n)}\right)\right)$$

$$\frac{dM_{k,x}(p_n)}{dp_n} = \frac{\ln^{k-2}(p_n)}{{p_n}^{2-x}}.\left(\left(k.\ln(p_n) - (1-x).\ln^2(p_n)\right).\left(x + \frac{k}{\ln(p_n)}\right).g_n - k.g_n + \ln^2(p_n).p_n.g'_n.\left(x + \frac{k}{\ln(p_n)}\right)\right)$$

$$\frac{dM_{k,x}(p_n)}{dp_n} = \frac{\ln^{k-2}(p_n)}{{p_n}^{2-x}}.g_n.\left(x.\left(p_n.\frac{g'_n}{g_n} - 1 + x\right).\ln^2(p_n) + k.\left(p_n.\frac{g'_n}{g_n} - 1 + 2x\right)\ln(p_n) + k(k-1)\right) \tag{III.8}$$

If we consider that $p_{n,(k,x)}$ is the prime number such as $M_{k,x}(p_n)$ is maximal then $p_{n,(k,x)}$ is a solution of the equality

$$x.\left(p_n.\frac{g'_n}{g_n} - 1 + x\right).\ln^2(p_n) + k.\left(p_n.\frac{g'_n}{g_n} - 1 + 2x\right)\ln(p_n) + k(k-1) = 0 \tag{III.9}$$

The positive solution of this equation is given by ;

$$\ln(p_n) = \frac{-k.\left(p_n.\frac{g'_n}{g_n} - 1 + 2x\right) + \sqrt[2]{k^2\left(p_n.\frac{g'_n}{g_n} - 1 + 2x\right)^2 - 4k(k-1)x.\left(p_n.\frac{g'_n}{g_n} - 1 + x\right)}}{2x.\left(p_n.\frac{g'_n}{g_n} - 1 + x\right)}$$

Or

$$\ln(p_n) = \frac{k.\left(1 - 2x - p_n.\frac{g'_n}{g_n}\right) - \sqrt[2]{k^2\left(1 - 2x - p_n.\frac{g'_n}{g_n}\right)^2 + 4k(k-1)x.\left(1 - x - p_n.\frac{g'_n}{g_n}\right)}}{2x.\left(1 - x - p_n.\frac{g'_n}{g_n}\right)}$$

This previous formulation has the inconvenient to be undetermined when $1 - x - p_n.\frac{g'_n}{g_n} = 0$

For this reason we propose the following formulation which avoid this question:

$$\ln(p_n) = \frac{2(k-1)}{\sqrt[2]{\left(1 - 2x - p_n.\frac{g'_n}{g_n}\right)^2 + \frac{4(k-1)x}{k}.\left(1 - x - p_n.\frac{g'_n}{g_n}\right)} + \left(1 - 2x - p_n.\frac{g'_n}{g_n}\right)} \tag{III.10}$$

(III.10) gives an implicit formulation of $\ln(p_n)$ but we cannot obtain an explicit formulation. But it remains very interesting to estimate the upper bounds of lower or upper bound functions for which we can replace the term $p_n.\frac{g'_n}{g_n}$ by a derivative function. We are going to develop this point below.

III.4 – Lower and upper bound functions independent from prime gaps :

As developed in Part 2, paragraph II.3, we'll limit our study to the set of prime numbers P_n corresponding to the first occurrence of each maximal gap G_n.

Let's define new lower and upper bound functions $H^-_{k,x}(P_n)$ and $H^+_{k,x}(P_n)$ totally independent from g_n.

Thanks to BHP result $G_n < P_n{}^a$ with $a = 0{,}525$: $$H^+_{k,x}(P_n) = \frac{\ln^k(P_n)}{P_n{}^{1-x-a}}.\left(x + \frac{k}{\ln(P_n)}\right) > M_{k,x}(P_n)$$

Thanks to Erik Westzynthius result $G_n > \ln(P_n)$: $$H^-_{k,x}(P_n) = \frac{\ln^{k+1}(P_n)}{P_n{}^{1-x}}.\left(x + \frac{k}{\ln(P_n)}\right) < M_{k,x}(P_n)$$

Thus we can compete (III.6):

$$H^-_{k,x}(P_n).\left(1 - \frac{1}{2P_n{}^{1-a}}\right) > M_{k,x}(P_n).\left(1 - \frac{1}{2P_n{}^{1-a}}\right) < H_{k,x}(P_n) < M_{k,x}(P_n).\left(1 + \frac{2}{P_n{}^{1-a}}\right) < H^+_{k,x}(P_n).\left(1 + \frac{2}{P_n{}^{1-a}}\right)$$

Thanks (III.9) we are not obliged to compute again the first derivative of $H^-_{k,x}(P_n)$ and $H^+_{k,x}(P_n)$

To get the prime P_n corresponding to the maximal value of $H^+_{k,x}(P_n)$ we have only to replace G_n *by* $P_n{}^a$ into (III.9):

$$\ln\left(P^+_{n,(k,x)}\right) = \frac{2(k-1)}{\sqrt[2]{(1-2x-a)^2 + \frac{4(k-1)x}{k}.(1-x-a)} + (1-2x-a)}$$

The table III.1 below gives the values $P^+_{n,(k,x)}$ for k and x. We can compare the values below with the prime numbers corresponding to the upper bound of $H^+_{k,x}(P_n)$ in tables III.3 up to III.6.

k\x	0,10	0,20	0,30	0,40
2	20	124	8 074	39 478 306 814
3	318	5 419	2 793 112	25 876 456 234 780 500
4	4 750	218 245	895 312 015	16 423 661 488 402 600 000 000
5	69 774	8 558 992	279 942 845 553	10 302 943 121 513 000 000 000 000 000
6	1 017 588	331 665 944	86 553 563 248 745	6 427 494 466 447 410 000 000 000 000 000 000
7	14 781 580	12 770 066 641	26 599 709 464 644 400	3 997 491 307 593 910 000 000 000 000 000 000 000 000
8	214 198 646	489 799 265 264	8 145 073 594 203 300 000	2 481 516 795 284 010 000 000 000 000 000 000 000 000 000 000

To get the prime P_n corresponding to the maximal value of $H^-_{k,x}(P_n)$ we have only to compute the derivative :

$$\frac{dH^-_{k,x}(P_n)}{dp_n} = \frac{\ln^{k-1}(p_n)}{p_n{}^{2-x}}\left(-x(1-x).\ln^2(p_n) - k.\left(1 - 2x - \frac{x}{k}\right)\ln(p_n) + k^2\right) \qquad \text{(III.10)}$$

Who has the following solution :

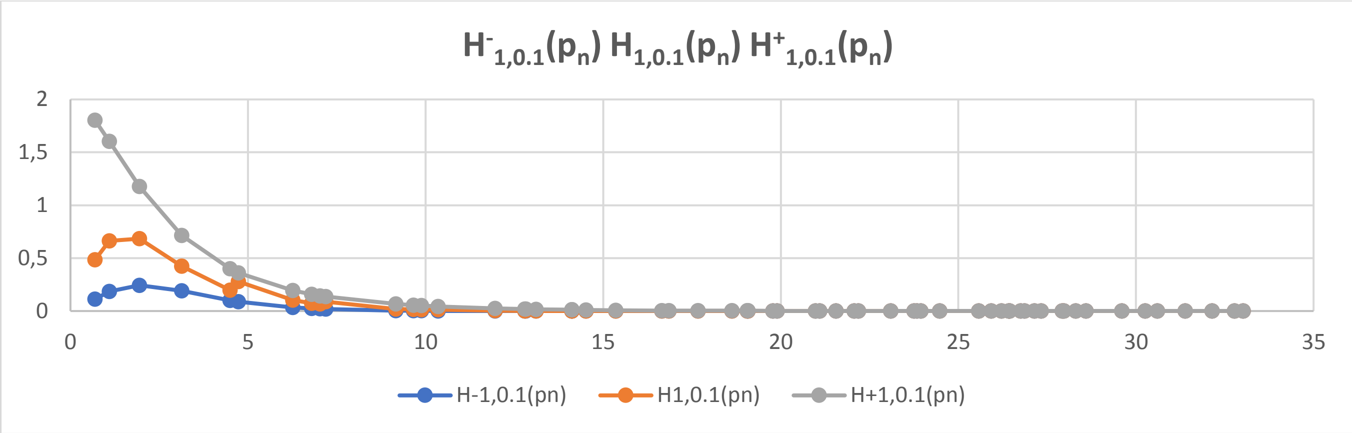
$$\ln\left(P^+_{n,(k,x)}\right) = \frac{2k}{\sqrt[2]{\left(1-2x-\frac{x}{k}\right)^2 + 4x.(1-x)} + \left(1-2x-\frac{x}{k}\right)} \qquad \text{(III.11)}$$

The table III.2 below gives the numerical values of $P^+_{n,(k,x)}$ estimated from (III.11) and must be compared with the true values of $P^+_{n,(k,x)}$ as it is appearing in tables III.3 up to III.6. The values below correspond to the integer which gives the maximal value of $H^-_{k,x}(P_n)$ while tables III.3 up to III.6 gives only the nearest prime corresponding to a first occurrence of the maximal gaps.

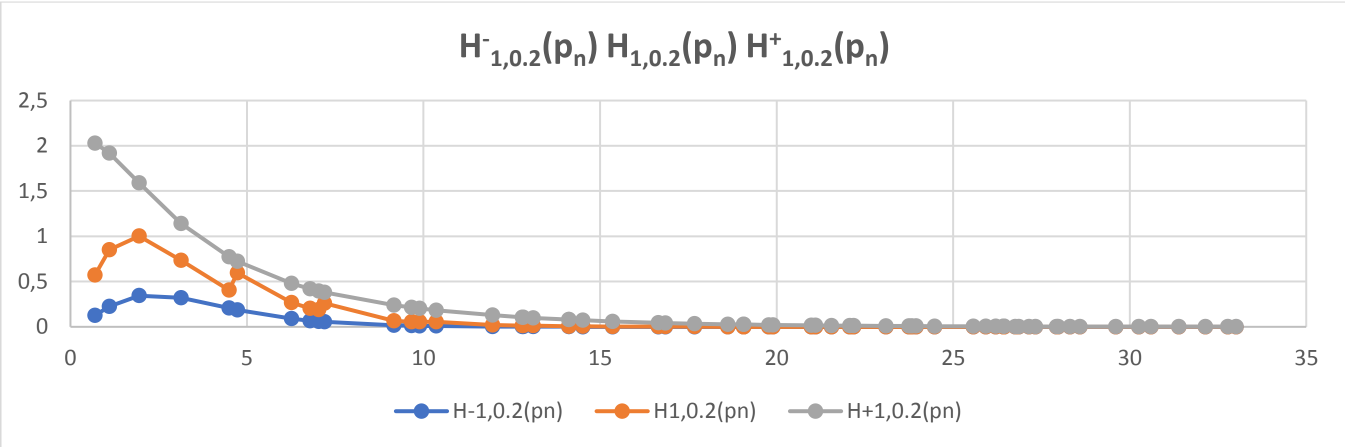
k\x	0,10	0,20	0,30	0,40
1	3	5	7	12
2	10	16	28	59
3	31	55	115	305
4	95	193	476	1594
5	290	671	1978	8371
6	880	2337	8228	44076
7	2671	8150	34257	232451
8	8113	28426	142709	1227131

The table III.3 below give the numeral values of $H^-_{k,x}(p_n)$, $H_{k,x}(p_n)$ and $H^+_{k,x}(p_n)$ for k=1 and x ϵ {0.1, 0.2, 0.3, 0.4} for all prime numbers corresponding to the first occurrence of the maximal gaps G_n.

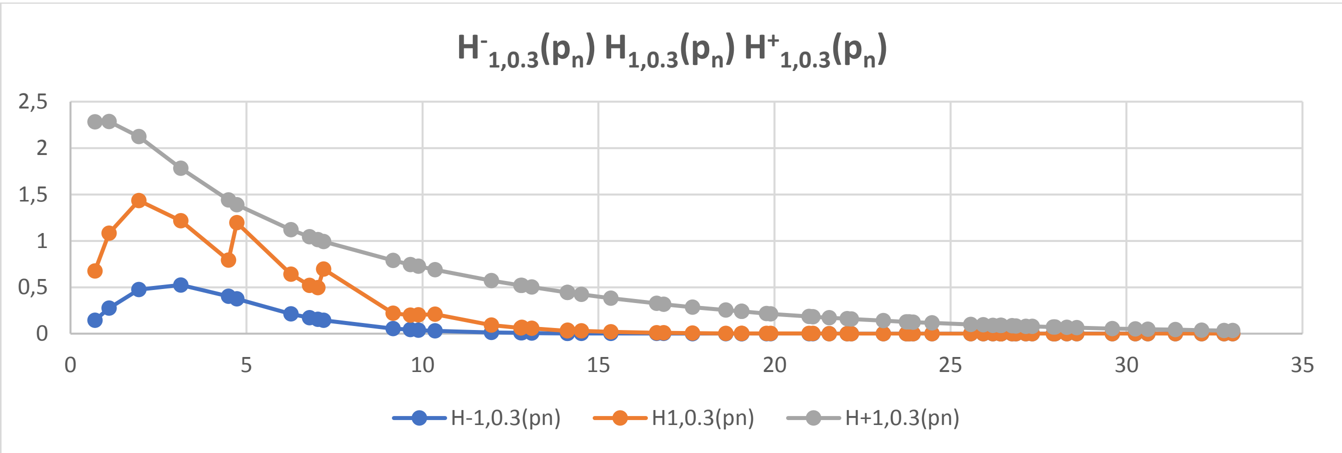
Primes	Gaps	ln(Pn)	$H^-_{1,0.1}(p_n)$	$H_{1,0.1}(p_n)$	$H^+_{1,0.1}(p_n)$	$H^-_{1,0.2}(p_n)$	$H_{1,0.2}(p_n)$	$H^+_{1,0.2}(p_n)$	$H^-_{1,0.3}(p_n)$	$H_{1,0.3}(p_n)$	$H^+_{1,0.3}(p_n)$	$H^-_{1,0.4}(p_n)$	$H_{1,0.4}(p_n)$	$H^+_{1,0.4}(p_n)$
k			1	1	1	1	1	1	1	1	1	1	1	1
x			0,1	0,1	0,1	0,2	0,2	0,2	0,3	0,3	0,3	0,4	0,4	0,4
2	1	0,69	0,1114	0,4833	1,8023	0,1272	0,5724	2,0302	0,1446	0,6741	2,2815	0,1639	0,7903	2,5584
3	2	1,10	0,1844	0,6643	1,6031	0,2262	0,8520	1,9187	0,2752	1,0808	2,2860	0,3326	1,3589	2,7126
7	4	1,95	0,2433	0,6838	1,1766	0,3438	1,0018	1,5885	0,4761	1,4346	2,1231	0,6494	2,0193	2,8140
23	6	3,14	0,1898	0,4252	0,7149	0,3216	0,7331	1,1404	0,5249	1,2150	1,7824	0,8342	1,9591	2,7426
89	8	4,49	0,1009	0,1968	0,4000	0,2070	0,4062	0,7727	0,4010	0,7909	1,4393	0,7484	1,4834	2,6133
113	14	4,73	0,0884	0,2787	0,3617	0,1873	0,5954	0,7227	0,3736	1,1958	1,3881	0,7165	2,3090	2,5938
523	18	6,26	0,0345	0,1032	0,1940	0,0894	0,2678	0,4805	0,2137	0,6412	1,1186	0,4866	1,4626	2,5033
887	20	6,79	0,0243	0,0739	0,1581	0,0673	0,2050	0,4208	0,1710	0,5211	1,0449	0,4124	1,2585	2,4841
1129	22	7,03	0,0207	0,0665	0,1442	0,0589	0,1899	0,3966	0,1538	0,4961	1,0137	0,3808	1,2298	2,4769
1327	34	7,19	0,0185	0,0894	0,1357	0,0538	0,2608	0,3813	0,1431	0,6941	0,9936	0,3605	1,7516	2,4725
9551	36	9,16	0,0045	0,0180	0,0662	0,0168	0,0667	0,2403	0,0555	0,2207	0,7875	0,1727	0,6867	2,4360
15683	44	9,66	0,0031	0,0145	0,0557	0,0123	0,0567	0,2148	0,0431	0,1982	0,7443	0,1414	0,6500	2,4285
19609	52	9,88	0,0027	0,0142	0,0515	0,0107	0,0569	0,2042	0,0384	0,2038	0,7257	0,1289	0,6841	2,4250
31397	72	10,35	0,0019	0,0131	0,0438	0,0080	0,0558	0,1838	0,0300	0,2102	0,6879	0,1059	0,7414	2,4172
155921	86	11,96	0,0006	0,0040	0,0253	0,0028	0,0204	0,1285	0,0127	0,0914	0,5729	0,0528	0,3809	2,3834
360653	96	12,80	0,0003	0,0022	0,0191	0,0016	0,0122	0,1066	0,0080	0,0598	0,5198	0,0362	0,2720	2,3600
370261	112	12,82	0,0003	0,0025	0,0189	0,0016	0,0140	0,1060	0,0078	0,0687	0,5182	0,0357	0,3129	2,3592
492113	114	13,11	0,0002	0,0020	0,0172	0,0013	0,0115	0,0994	0,0067	0,0583	0,5012	0,0314	0,2735	2,3503
1349533	118	14,12	0,0001	0,0009	0,0122	0,0007	0,0056	0,0793	0,0038	0,0316	0,4447	0,0197	0,1646	2,3145
1357201	132	14,12	0,0001	0,0010	0,0122	0,0007	0,0063	0,0792	0,0038	0,0352	0,4444	0,0196	0,1835	2,3143
2010733	148	14,51	0,0001	0,0008	0,0107	0,0005	0,0052	0,0724	0,0030	0,0307	0,4239	0,0163	0,1664	2,2988
4652353	154	15,35	0,0000	0,0004	0,0080	0,0003	0,0029	0,0599	0,0018	0,0186	0,3827	0,0109	0,1098	2,2627
17051707	180	16,65	0,0000	0,0001	0,0052	0,0001	0,0013	0,0445	0,0009	0,0094	0,3257	0,0058	0,0632	2,1998
20831323	210	16,85	0,0000	0,0001	0,0048	0,0001	0,0013	0,0425	0,0008	0,0096	0,3176	0,0053	0,0660	2,1895
47326693	220	17,67	0,0000	0,0001	0,0037	0,0001	0,0007	0,0352	0,0005	0,0059	0,2862	0,0035	0,0441	2,1454
122164747	222	18,62	0,0000	0,0000	0,0027	0,0000	0,0004	0,0282	0,0003	0,0032	0,2533	0,0022	0,0264	2,0915
189695659	234	19,06	0,0000	0,0000	0,0023	0,0000	0,0003	0,0255	0,0002	0,0025	0,2392	0,0018	0,0218	2,0656
191912783	248	19,07	0,0000	0,0000	0,0023	0,0000	0,0003	0,0254	0,0002	0,0027	0,2389	0,0018	0,0229	2,0649
387096133	250	19,77	0,0000	0,0000	0,0018	0,0000	0,0002	0,0216	0,0001	0,0017	0,2178	0,0012	0,0157	2,0225
436273009	282	19,89	0,0000	0,0000	0,0017	0,0000	0,0002	0,0210	0,0001	0,0018	0,2144	0,0012	0,0165	2,0152
1294268491	288	20,98	0,0000	0,0000	0,0012	0,0000	0,0001	0,0162	0,0001	0,0009	0,1856	0,0007	0,0092	1,9474
1453168141	292	21,10	0,0000	0,0000	0,0011	0,0000	0,0001	0,0158	0,0001	0,0008	0,1827	0,0006	0,0088	1,9400
2300942549	320	21,56	0,0000	0,0000	0,0010	0,0000	0,0001	0,0142	0,0000	0,0007	0,1717	0,0005	0,0074	1,9107
3842610773	336	22,07	0,0000	0,0000	0,0008	0,0000	0,0000	0,0125	0,0000	0,0005	0,1602	0,0004	0,0059	1,8778
4302407359	354	22,18	0,0000	0,0000	0,0008	0,0000	0,0000	0,0122	0,0000	0,0005	0,1578	0,0004	0,0058	1,8705
10726904659	382	23,10	0,0000	0,0000	0,0006	0,0000	0,0000	0,0098	0,0000	0,0003	0,1393	0,0002	0,0037	1,8112
20678048297	384	23,75	0,0000	0,0000	0,0005	0,0000	0,0000	0,0084	0,0000	0,0002	0,1273	0,0002	0,0026	1,7684
22367084959	394	23,83	0,0000	0,0000	0,0004	0,0000	0,0000	0,0082	0,0000	0,0002	0,1259	0,0002	0,0026	1,7633
25056082087	456	23,94	0,0000	0,0000	0,0004	0,0000	0,0000	0,0080	0,0000	0,0002	0,1239	0,0001	0,0028	1,7559
42652618343	464	24,48	0,0000	0,0000	0,0004	0,0000	0,0000	0,0070	0,0000	0,0001	0,1151	0,0001	0,0021	1,7211
127976334671	468	25,58	0,0000	0,0000	0,0002	0,0000	0,0000	0,0054	0,0000	0,0001	0,0987	0,0001	0,0011	1,6495
182226896239	474	25,93	0,0000	0,0000	0,0002	0,0000	0,0000	0,0050	0,0000	0,0001	0,0939	0,0001	0,0009	1,6266
241160624143	486	26,21	0,0000	0,0000	0,0002	0,0000	0,0000	0,0046	0,0000	0,0000	0,0903	0,0000	0,0008	1,6084
297501075799	490	26,42	0,0000	0,0000	0,0002	0,0000	0,0000	0,0044	0,0000	0,0000	0,0877	0,0000	0,0007	1,5949
303371455241	500	26,44	0,0000	0,0000	0,0002	0,0000	0,0000	0,0044	0,0000	0,0000	0,0874	0,0000	0,0007	1,5936
304599508537	514	26,44	0,0000	0,0000	0,0002	0,0000	0,0000	0,0044	0,0000	0,0000	0,0874	0,0000	0,0008	1,5934
416608695821	516	26,76	0,0000	0,0000	0,0002	0,0000	0,0000	0,0040	0,0000	0,0000	0,0836	0,0000	0,0006	1,5732
461690510011	532	26,86	0,0000	0,0000	0,0002	0,0000	0,0000	0,0039	0,0000	0,0000	0,0824	0,0000	0,0006	1,5666
614487453523	534	27,14	0,0000	0,0000	0,0001	0,0000	0,0000	0,0037	0,0000	0,0000	0,0791	0,0000	0,0005	1,5483
738832927927	540	27,33	0,0000	0,0000	0,0001	0,0000	0,0000	0,0035	0,0000	0,0000	0,0770	0,0000	0,0005	1,5366
1346294310749	582	27,93	0,0000	0,0000	0,0001	0,0000	0,0000	0,0030	0,0000	0,0000	0,0707	0,0000	0,0004	1,4985
1408695493609	588	27,97	0,0000	0,0000	0,0001	0,0000	0,0000	0,0030	0,0000	0,0000	0,0703	0,0000	0,0004	1,4956
1968188556461	602	28,31	0,0000	0,0000	0,0001	0,0000	0,0000	0,0028	0,0000	0,0000	0,0670	0,0000	0,0003	1,4746
2614941710599	652	28,59	0,0000	0,0000	0,0001	0,0000	0,0000	0,0026	0,0000	0,0000	0,0643	0,0000	0,0003	1,4568
7177162611713	674	29,60	0,0000	0,0000	0,0001	0,0000	0,0000	0,0020	0,0000	0,0000	0,0556	0,0000	0,0002	1,3944
13829048559701	716	30,26	0,0000	0,0000	0,0000	0,0000	0,0000	0,0017	0,0000	0,0000	0,0505	0,0000	0,0001	1,3546
19581334192423	766	30,61	0,0000	0,0000	0,0000	0,0000	0,0000	0,0016	0,0000	0,0000	0,0481	0,0000	0,0001	1,3337
42842283925351	778	31,39	0,0000	0,0000	0,0000	0,0000	0,0000	0,0013	0,0000	0,0000	0,0429	0,0000	0,0001	1,2874
90874329411493	804	32,14	0,0000	0,0000	0,0000	0,0000	0,0000	0,0011	0,0000	0,0000	0,0384	0,0000	0,0000	1,2438
171231342420521	806	32,77	0,0000	0,0000	0,0000	0,0000	0,0000	0,0009	0,0000	0,0000	0,0350	0,0000	0,0000	1,2078
218209405436543	906	33,02	0,0000	0,0000	0,0000	0,0000	0,0000	0,0009	0,0000	0,0000	0,0338	0,0000	0,0000	1,1942



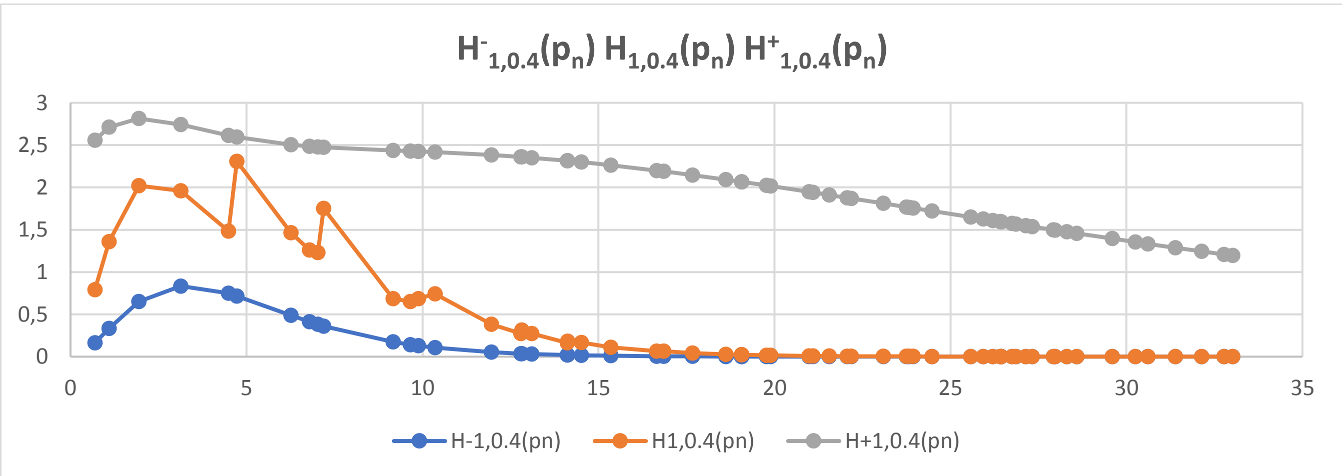
$H^-_{1,0.1}(p_n)$ $H_{1,0.1}(p_n)$ $H^+_{1,0.1}(p_n)$
2
1,5
1
0,5
0
0 5 10 15 20 25 30 35
H-1,0.1(pn) H1,0.1(pn) H+1,0.1(pn)



$H^-_{1,0.2}(p_n)$ $H_{1,0.2}(p_n)$ $H^+_{1,0.2}(p_n)$
2,5
2
1,5
1
0,5
0
0 5 10 15 20 25 30 35
H-1,0.2(pn) H1,0.2(pn) H+1,0.2(pn)



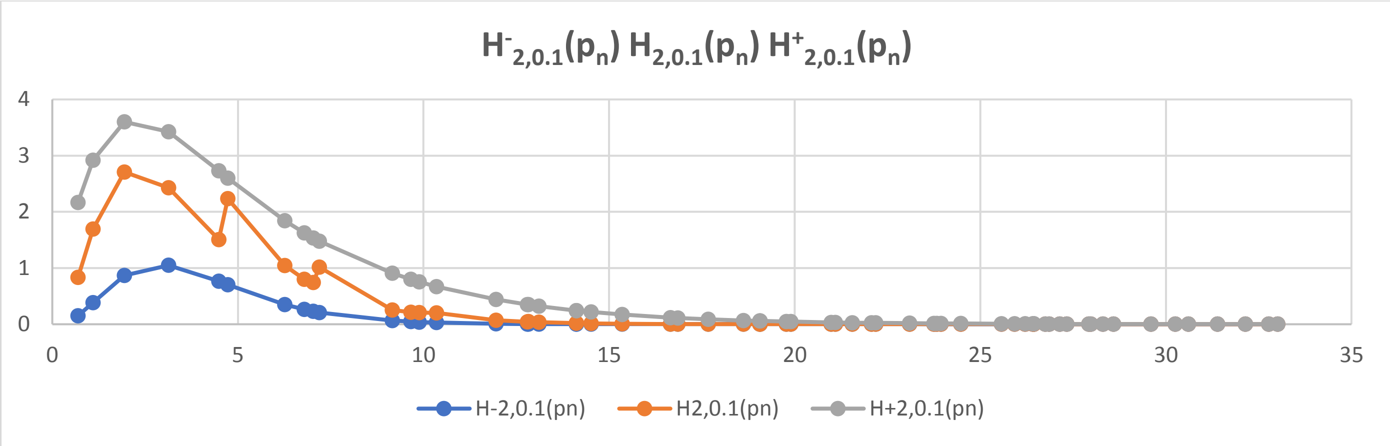
$H^-_{1,0.3}(p_n)$ $H_{1,0.3}(p_n)$ $H^+_{1,0.3}(p_n)$
2,5
2
1,5
1
0,5
0
0 5 10 15 20 25 30 35
H-1,0.3(pn) H1,0.3(pn) H+1,0.3(pn)

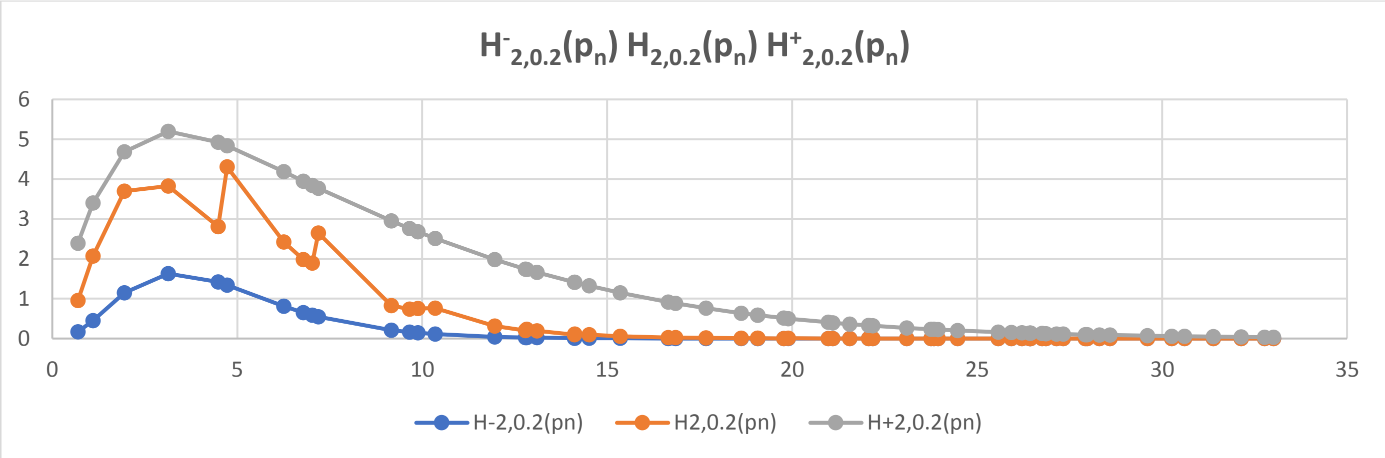


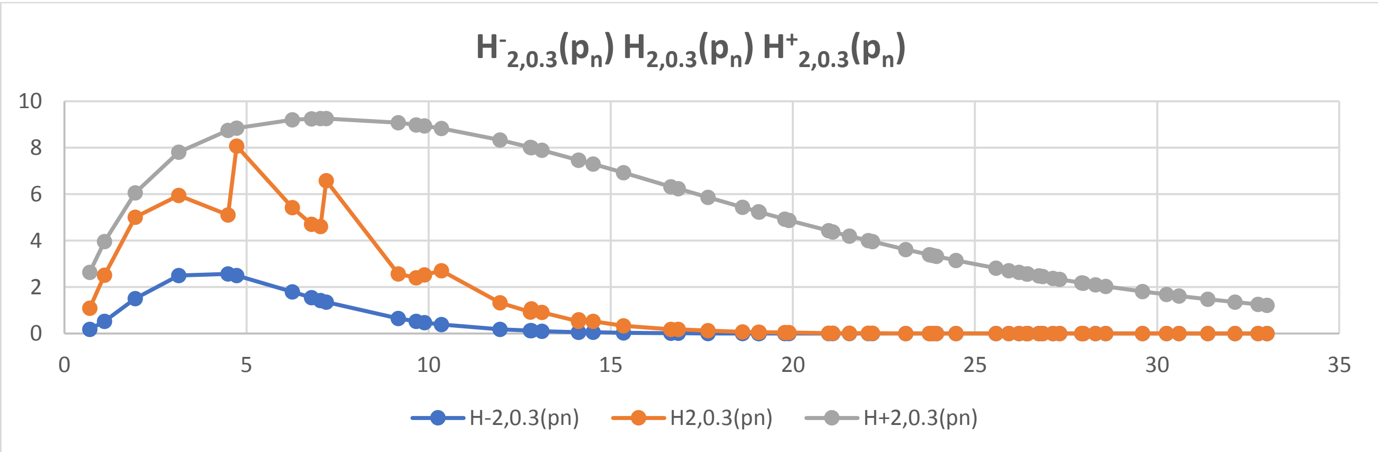
$H^-_{1,0.4}(p_n)$ $H_{1,0.4}(p_n)$ $H^+_{1,0.4}(p_n)$
3
2,5
2
1,5
1
0,5
0
0 5 10 15 20 25 30 35
H-1,0.4(pn) H1,0.4(pn) H+1,0.4(pn)

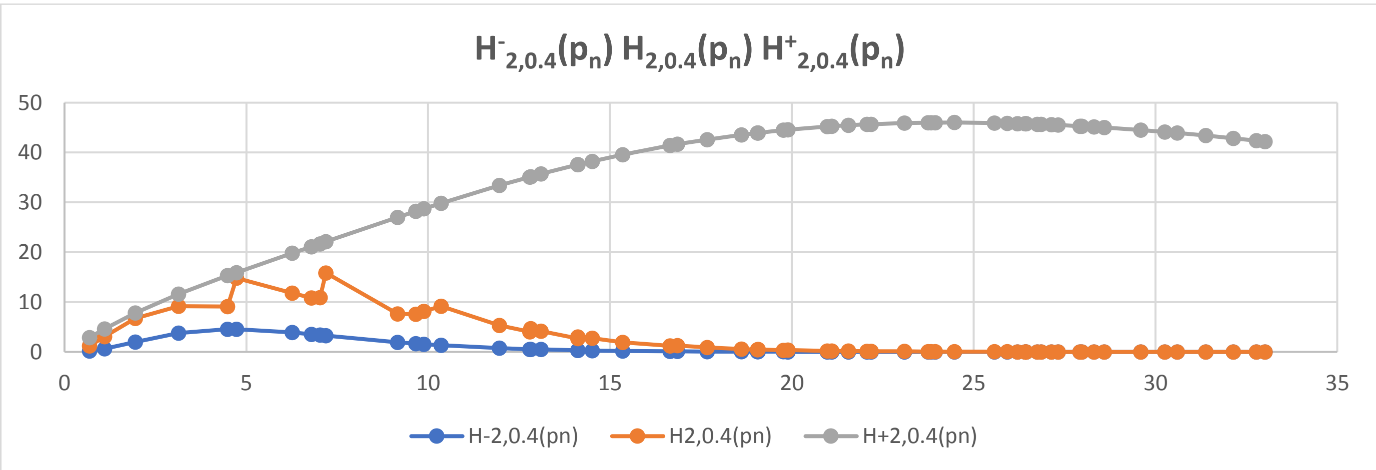
The table III.4 below give the numeral values of $H^-_{k,x}(p_n)$, $H_{k,x}(p_n)$ and $H^+_{k,x}(p_n)$ for k=2 and x ϵ {0.1, 0.2, 0.3, 0.4} for all prime numbers corresponding to the first occurrence of the maximal gaps G_n.

Primes	Gaps	ln(Pn)	$H^-_{2,0.1}(p_n)$	$H_{2,0.1}(p_n)$	$H^+_{2,0.1}(p_n)$	$H^-_{2,0.2}(p_n)$	$H_{2,0.2}(p_n)$	$H^+_{2,0.2}(p_n)$	$H^-_{2,0.3}(p_n)$	$H_{2,0.3}(p_n)$	$H^+_{2,0.3}(p_n)$	$H^-_{2,0.4}(p_n)$	$H_{2,0.4}(p_n)$	$H^+_{2,0.4}(p_n)$
k			2	2	2	2	2	2	2	2	2	2	2	2
x			0,1	0,1	0,1	0,2	0,2	0,2	0,3	0,3	0,3	0,4	0,4	0,4
2	1	0,69	0,1495	0,8322	2,1683	0,1656	0,9516	2,3922	0,1832	1,0866	2,6371	0,2025	1,2390	2,9048
3	2	1,10	0,3852	1,6955	2,9207	0,4523	2,0704	3,4020	0,5298	2,5198	3,9558	0,6192	3,0580	4,5922
7	4	1,95	0,8699	2,7080	3,5997	1,1505	3,7002	4,6827	1,5114	5,0168	6,0649	1,9744	6,7576	7,8248
23	6	3,14	1,0480	2,4263	3,4273	1,6282	3,8293	5,1982	2,4938	5,9530	7,8086	3,7760	9,1445	11,6367
89	8	4,49	0,7655	1,5055	2,7280	1,4189	2,8062	4,9294	2,5671	5,1036	8,7495	4,5608	9,1129	15,3159
113	14	4,73	0,7015	2,2359	2,5977	1,3407	4,3053	4,8413	2,4962	8,0727	8,8479	4,5588	14,8435	15,9291
523	18	6,26	0,3491	1,0454	1,8438	0,8084	2,4260	4,1844	1,8028	5,4205	9,2022	3,9154	11,7951	19,7839
887	20	6,79	0,2634	0,8019	1,6268	0,6509	1,9841	3,9480	1,5427	4,7085	9,2441	3,5528	10,8571	21,1039
1129	22	7,03	0,2304	0,7425	1,5353	0,5864	1,8918	3,8410	1,4287	4,6143	9,2526	3,3790	10,9250	21,7066
1327	34	7,19	0,2103	1,0188	1,4765	0,5458	2,6482	3,7698	1,3546	6,5818	9,2549	3,2613	15,8681	22,1102
9551	36	9,16	0,0633	0,2515	0,9057	0,2080	0,8267	2,9489	0,6444	2,5620	9,0844	1,9223	7,6435	26,9926
15683	44	9,66	0,0459	0,2110	0,7985	0,1599	0,7351	2,7598	0,5233	2,4063	8,9898	1,6462	7,5706	28,1886
19609	52	9,88	0,0396	0,2102	0,7542	0,1417	0,7518	2,6770	0,4754	2,5225	8,9410	1,5316	8,1278	28,7208
31397	72	10,35	0,0290	0,2028	0,6680	0,1095	0,7662	2,5077	0,3867	2,7069	8,8265	1,3099	9,1707	29,8264
155921	86	11,96	0,0097	0,0697	0,4384	0,0439	0,3165	1,9846	0,1845	1,3311	8,3308	0,7403	5,3424	33,3924
360653	96	12,80	0,0053	0,0402	0,3498	0,0267	0,2007	1,7439	0,1229	0,9240	8,0178	0,5386	4,0497	35,1101
370261	112	12,82	0,0052	0,0459	0,3473	0,0263	0,2300	1,7367	0,1213	1,0619	8,0075	0,5331	4,6672	35,1622
492113	114	13,11	0,0043	0,0373	0,3215	0,0221	0,1930	1,6604	0,1054	0,9186	7,8944	0,4773	4,1593	35,7175
1349533	118	14,12	0,0021	0,0173	0,2435	0,0120	0,1002	1,4102	0,0635	0,5312	7,4722	0,3193	2,6723	37,5731
1357201	132	14,12	0,0021	0,0192	0,2431	0,0119	0,1116	1,4089	0,0633	0,5922	7,4697	0,3186	2,9813	37,5830
2010733	148	14,51	0,0015	0,0157	0,2179	0,0094	0,0955	1,3199	0,0517	0,5281	7,2980	0,2713	2,7697	38,2560
4652353	154	15,35	0,0008	0,0083	0,1721	0,0055	0,0555	1,1451	0,0335	0,3359	6,9224	0,1915	1,9222	39,5961
17051707	180	16,65	0,0003	0,0034	0,1187	0,0024	0,0262	0,9123	0,0168	0,1817	6,3279	0,1100	1,1891	41,4092
20831323	210	16,85	0,0003	0,0034	0,1120	0,0021	0,0265	0,8803	0,0151	0,1881	6,2359	0,1008	1,2566	41,6602
47326693	220	17,67	0,0001	0,0018	0,0883	0,0013	0,0156	0,7588	0,0097	0,1204	5,8603	0,0703	0,8754	42,6106
122164747	222	18,62	0,0001	0,0008	0,0668	0,0007	0,0080	0,6369	0,0057	0,0685	5,4329	0,0460	0,5490	43,5533
189695659	234	19,06	0,0001	0,0006	0,0586	0,0005	0,0062	0,5865	0,0045	0,0552	5,2382	0,0377	0,4633	43,9353
191912783	248	19,07	0,0000	0,0006	0,0584	0,0005	0,0065	0,5852	0,0045	0,0581	5,2331	0,0375	0,4882	43,9449
387096133	250	19,77	0,0000	0,0004	0,0474	0,0003	0,0040	0,5122	0,0030	0,0382	4,9289	0,0273	0,3447	44,4815
436273009	282	19,89	0,0000	0,0004	0,0457	0,0003	0,0041	0,5007	0,0028	0,0400	4,8779	0,0258	0,3658	44,5645
1294268491	288	20,98	0,0000	0,0002	0,0329	0,0001	0,0019	0,4057	0,0015	0,0210	4,4267	0,0156	0,2142	45,2074
1453168141	292	21,10	0,0000	0,0001	0,0318	0,0001	0,0018	0,3967	0,0014	0,0198	4,3801	0,0148	0,2046	45,2644
2300942549	320	21,56	0,0000	0,0001	0,0276	0,0001	0,0014	0,3624	0,0011	0,0163	4,1979	0,0119	0,1769	45,4691
3842610773	336	22,07	0,0000	0,0001	0,0236	0,0001	0,0010	0,3275	0,0008	0,0125	4,0001	0,0094	0,1425	45,6583
4302407359	354	22,18	0,0000	0,0001	0,0228	0,0001	0,0010	0,3202	0,0008	0,0123	3,9574	0,0089	0,1416	45,6946
10726904659	382	23,10	0,0000	0,0000	0,0172	0,0000	0,0006	0,2667	0,0005	0,0075	3,6226	0,0057	0,0951	45,9176
20678048297	384	23,75	0,0000	0,0000	0,0141	0,0000	0,0003	0,2335	0,0003	0,0050	3,3945	0,0042	0,0678	46,0037
22367084959	394	23,83	0,0000	0,0000	0,0137	0,0000	0,0003	0,2298	0,0003	0,0049	3,3679	0,0040	0,0668	46,0100
25056082087	456	23,94	0,0000	0,0000	0,0133	0,0000	0,0004	0,2246	0,0003	0,0053	3,3297	0,0038	0,0729	46,0177
42652618343	464	24,48	0,0000	0,0000	0,0112	0,0000	0,0002	0,2014	0,0002	0,0038	3,1550	0,0030	0,0561	46,0306
127976334671	468	25,58	0,0000	0,0000	0,0080	0,0000	0,0001	0,1605	0,0001	0,0019	2,8159	0,0017	0,0317	45,9429
182226896239	474	25,93	0,0000	0,0000	0,0071	0,0000	0,0001	0,1491	0,0001	0,0016	2,7130	0,0015	0,0266	45,8835
241160624143	486	26,21	0,0000	0,0000	0,0065	0,0000	0,0001	0,1407	0,0001	0,0014	2,6335	0,0013	0,0236	45,8263
297501075799	490	26,42	0,0000	0,0000	0,0061	0,0000	0,0001	0,1346	0,0001	0,0012	2,5752	0,0011	0,0212	45,7775
303371455241	500	26,44	0,0000	0,0000	0,0061	0,0000	0,0001	0,1341	0,0001	0,0012	2,5698	0,0011	0,0215	45,7727
304599508537	514	26,44	0,0000	0,0000	0,0061	0,0000	0,0001	0,1339	0,0001	0,0012	2,5687	0,0011	0,0220	45,7717
416608695821	516	26,76	0,0000	0,0000	0,0055	0,0000	0,0001	0,1254	0,0001	0,0010	2,4838	0,0010	0,0187	45,6892
461690510011	532	26,86	0,0000	0,0000	0,0053	0,0000	0,0000	0,1227	0,0000	0,0010	2,4564	0,0009	0,0183	45,6599
614487453523	534	27,14	0,0000	0,0000	0,0049	0,0000	0,0000	0,1155	0,0000	0,0008	2,3816	0,0008	0,0157	45,5722
738832927927	540	27,33	0,0000	0,0000	0,0046	0,0000	0,0000	0,1111	0,0000	0,0007	2,3343	0,0007	0,0144	45,5112
1346294310749	582	27,93	0,0000	0,0000	0,0038	0,0000	0,0000	0,0978	0,0000	0,0005	2,1857	0,0005	0,0113	45,2890
1408695493609	588	27,97	0,0000	0,0000	0,0037	0,0000	0,0000	0,0969	0,0000	0,0005	2,1748	0,0005	0,0111	45,2708
1968188556461	602	28,31	0,0000	0,0000	0,0034	0,0000	0,0000	0,0902	0,0000	0,0004	2,0957	0,0004	0,0095	45,1303
2614941710599	652	28,59	0,0000	0,0000	0,0031	0,0000	0,0000	0,0849	0,0000	0,0004	2,0304	0,0004	0,0089	45,0028
7177162611713	674	29,60	0,0000	0,0000	0,0022	0,0000	0,0000	0,0683	0,0000	0,0002	1,8121	0,0002	0,0053	44,4923
13829048559701	716	30,26	0,0000	0,0000	0,0018	0,0000	0,0000	0,0593	0,0000	0,0002	1,6813	0,0002	0,0040	44,1158
19581334192423	766	30,61	0,0000	0,0000	0,0016	0,0000	0,0000	0,0550	0,0000	0,0001	1,6152	0,0001	0,0035	43,9027
42842283925351	778	31,39	0,0000	0,0000	0,0012	0,0000	0,0000	0,0463	0,0000	0,0001	1,4748	0,0001	0,0024	43,3916
90874329411493	804	32,14	0,0000	0,0000	0,0010	0,0000	0,0000	0,0393	0,0000	0,0001	1,3501	0,0001	0,0016	42,8625
171231342420521	806	32,77	0,0000	0,0000	0,0008	0,0000	0,0000	0,0342	0,0000	0,0000	1,2523	0,0000	0,0011	42,3901
218209405436543	906	33,02	0,0000	0,0000	0,0007	0,0000	0,0000	0,0324	0,0000	0,0000	1,2166	0,0000	0,0011	42,2034


$H^-_{2,0.1}(p_n)$ $H_{2,0.1}(p_n)$ $H^+_{2,0.1}(p_n)$
0 1 2 3 4
0 5 10 15 20 25 30 35
H-2,0.1(pn)
H2,0.1(pn)
H+2,0.1(pn)

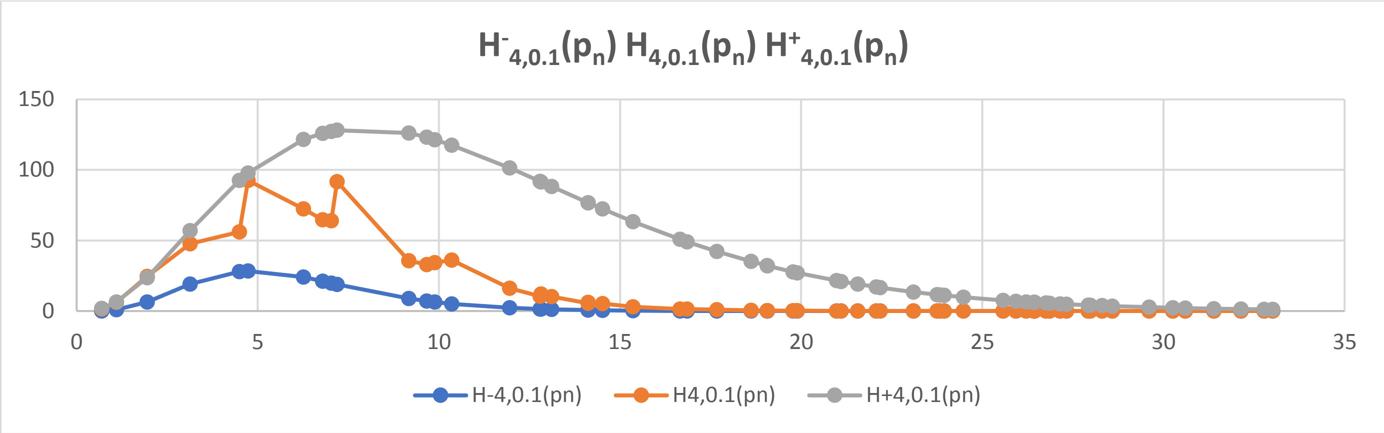

$H^-_{2,0.2}(p_n)$ $H_{2,0.2}(p_n)$ $H^+_{2,0.2}(p_n)$
0 1 2 3 4 5 6
0 5 10 15 20 25 30 35
H-2,0.2(pn)
H2,0.2(pn)
H+2,0.2(pn)

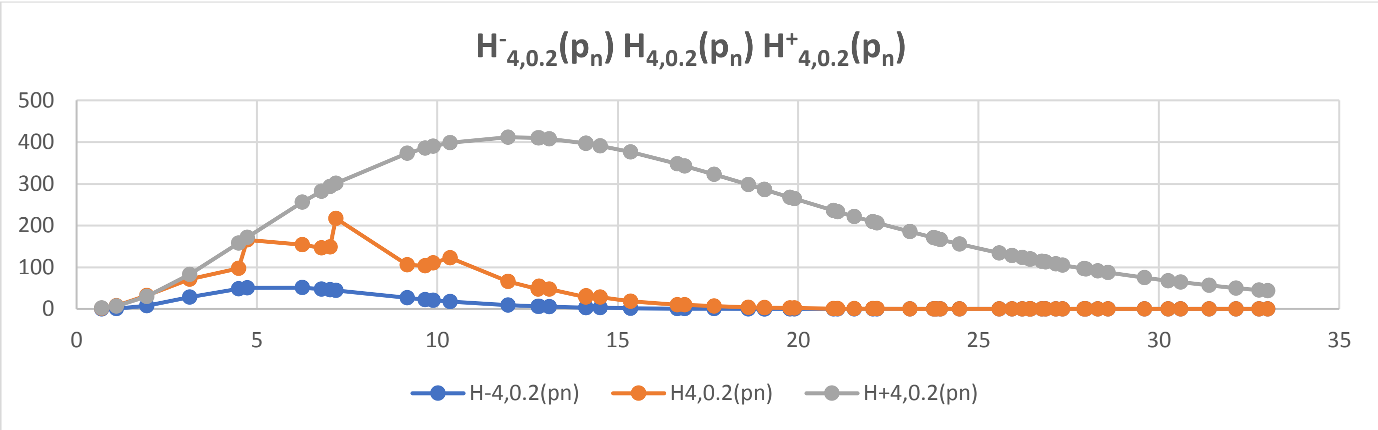

$H^-_{2,0.3}(p_n)$ $H_{2,0.3}(p_n)$ $H^+_{2,0.3}(p_n)$
0 2 4 6 8 10
0 5 10 15 20 25 30 35
H-2,0.3(pn)
H2,0.3(pn)
H+2,0.3(pn)

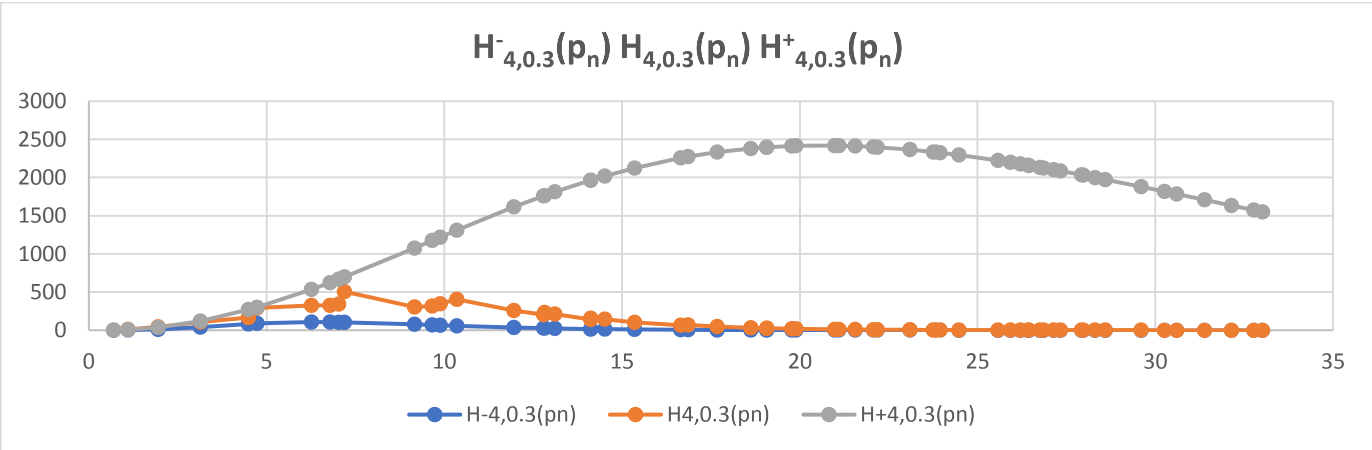

$H^-_{2,0.4}(p_n)$ $H_{2,0.4}(p_n)$ $H^+_{2,0.4}(p_n)$
0 10 20 30 40 50
0 5 10 15 20 25 30 35
H-2,0.4(pn)
H2,0.4(pn)
H+2,0.4(pn)

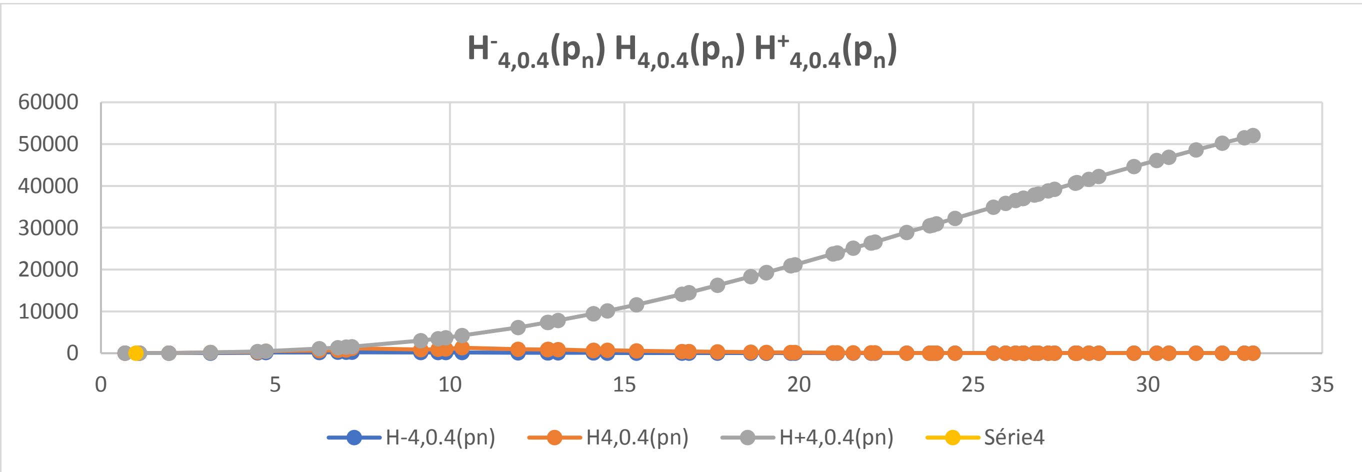
The table III.5 below give the numeral values of $H^-_{k,x}(p_n)$, $H_{k,x}(p_n)$ and $H^+_{k,x}(p_n)$ for k=4 and x ϵ {0.1, 0.2, 0.3, 0.4} for all prime numbers corresponding to the first occurrence of the maximal gaps G_n.

Primes	Gaps	ln(Pn)	$H^-_{4,0.1}(p_n)$	$H_{4,0.1}(p_n)$	$H^+_{4,0.1}(p_n)$	$H^-_{4,0.2}(p_n)$	$H_{4,0.2}(p_n)$	$H^+_{4,0.2}(p_n)$	$H^-_{4,0.3}(p_n)$	$H_{4,0.3}(p_n)$	$H^+_{4,0.3}(p_n)$	$H^-_{4,0.4}(p_n)$	$H_{4,0.4}(p_n)$	$H^+_{4,0.4}(p_n)$
k			4	4	4	4	4	4	4	4	4	4	4	4
x			0,1	0,1	0,1	0,2	0,2	0,2	0,3	0,3	0,3	0,4	0,4	0,4
2	1	0,69	0,1412	1,3785	1,9249	0,1539	1,5495	2,0958	0,1677	1,7412	2,2814	0,1827	1,9560	2,4829
3	2	1,10	0,9056	6,2553	6,3239	1,0378	7,4428	7,2299	1,1884	8,8485	8,2610	1,3601	10,5122	9,4340
7	4	1,95	6,2958	24,6023	23,5531	8,0030	32,2474	29,7854	10,1532	42,1743	37,6085	12,8579	55,0466	47,4181
23	6	3,14	19,2095	47,7880	57,0093	28,1944	71,1639	83,0055	41,1918	105,4614	120,4170	59,9385	155,6287	174,1264
89	8	4,49	28,0179	56,1295	92,5496	48,3192	97,3002	158,1961	82,6306	167,2317	268,5198	140,3105	285,3712	453,0728
113	14	4,73	28,3587	92,5531	97,7462	50,3071	165,3386	171,8742	88,4271	292,6146	299,9011	154,2490	513,8584	519,9005
523	18	6,26	24,0955	72,5281	121,5469	51,1567	154,2877	256,1542	107,0669	323,5323	532,9815	221,5411	670,7064	1097,6091
887	20	6,79	21,1981	64,7299	125,9694	47,8547	146,3169	282,4806	106,2976	325,4163	624,2001	233,1290	714,5713	1363,2695
1129	22	7,03	19,8084	63,9972	127,3897	45,9847	148,7344	293,8588	104,9476	339,8138	667,3608	236,3429	766,0756	1497,0200
1327	34	7,19	18,8730	91,7305	128,1407	44,6398	217,2857	301,2345	103,7393	505,6714	696,7438	237,7933	1160,7222	1591,1085
9551	36	9,16	8,9619	35,6263	126,2296	26,5850	105,7062	373,0930	76,9162	305,8950	1076,5731	218,4329	868,8813	3051,1414
15683	44	9,66	7,1722	32,9766	123,1723	22,5109	103,5176	385,3828	68,7796	316,3351	1174,8350	206,0269	947,7135	3513,1693
19609	52	9,88	6,4632	34,2914	121,5437	20,8064	110,4084	390,1355	65,1489	345,7598	1219,0293	199,8858	1060,9894	3734,2505
31397	72	10,35	5,1542	36,0795	117,6754	17,5011	122,5238	398,5644	57,6964	403,9773	1311,6145	186,1706	1303,6852	4226,5925
155921	86	11,96	2,2441	16,1932	101,4034	9,1262	65,8549	411,7784	35,8148	258,4472	1614,3622	137,0600	989,0825	6173,5272
360653	96	12,80	1,4076	10,5838	91,8914	6,2868	47,2721	409,9819	27,0108	203,1034	1760,1761	112,9572	849,3771	7357,0977
370261	112	12,82	1,3867	12,1394	91,5894	6,2119	54,3794	409,8388	26,7645	234,3035	1764,5678	112,2385	982,5817	7395,9983
492113	114	13,11	1,1786	10,2711	88,3179	5,4497	47,4925	407,9752	24,2112	210,9957	1811,3290	104,6253	911,7998	7823,7981
1349533	118	14,12	0,6522	5,4587	76,8217	3,3733	28,2339	397,0777	16,7008	139,7829	1965,0270	80,2539	671,7148	9439,8360
1357201	132	14,12	0,6500	6,0833	76,7582	3,3640	31,4839	396,9992	16,6649	155,9678	1965,8335	80,1291	749,9367	9449,3387
2010733	148	14,51	0,5131	5,2372	72,3972	2,7736	28,3111	391,1371	14,3307	146,2762	2020,1502	71,8077	732,9596	10119,8836
4652353	154	15,35	0,3067	3,0784	63,4522	1,8188	18,2559	376,1413	10,2775	103,1599	2124,9257	56,2274	564,3802	11623,1583
17051707	180	16,65	0,1350	1,4597	50,8521	0,9234	9,9853	347,7688	5,9906	64,7800	2255,8086	37,5324	405,8610	14131,6277
20831323	210	16,85	0,1187	1,4793	49,0629	0,8298	10,3440	342,9823	5,4989	68,5465	2272,5209	35,1774	438,5068	14536,3918
47326693	220	17,67	0,0696	0,8664	42,1797	0,5322	6,6266	322,5744	3,8467	47,8976	2331,3485	26,8008	333,7095	16241,7158
122164747	222	18,62	0,0371	0,4427	35,1325	0,3149	3,7553	297,9588	2,5161	30,0013	2380,2286	19,3426	230,6370	18297,3031
189695659	234	19,06	0,0276	0,3394	32,1866	0,2460	3,0198	286,3610	2,0581	25,2695	2396,1393	16,5600	203,3215	19278,8717
191912783	248	19,07	0,0274	0,3567	32,1116	0,2443	3,1776	286,0542	2,0472	26,6226	2396,5026	16,4917	214,4662	19305,0109
387096133	250	19,77	0,0171	0,2156	27,8313	0,1640	2,0731	267,5487	1,4789	18,6987	2413,1453	12,8105	161,9736	20902,8446
436273009	282	19,89	0,0157	0,2228	27,1501	0,1531	2,1703	264,4070	1,3984	19,8237	2414,9626	12,2637	173,8548	21178,8243
1294268491	288	20,98	0,0074	0,1022	21,5660	0,0815	1,1191	236,2503	0,8346	11,4569	2418,5155	8,1893	112,4159	23730,2969
1453168141	292	21,10	0,0069	0,0951	21,0330	0,0762	1,0545	233,3109	0,7894	10,9267	2417,5693	7,8389	108,5013	24006,0029
2300942549	320	21,56	0,0050	0,0740	19,0270	0,0581	0,8630	221,7897	0,6321	9,3838	2411,4321	6,5813	97,7004	25106,6598
3842610773	336	22,07	0,0035	0,0530	16,9852	0,0429	0,6531	209,2382	0,4921	7,4926	2400,2624	5,4017	82,2408	26345,7042
4302407359	354	22,18	0,0032	0,0513	16,5618	0,0401	0,6401	206,5194	0,4656	7,4298	2397,2125	5,1698	82,5056	26620,1179
10726904659	382	23,10	0,0017	0,0279	13,4654	0,0232	0,3835	185,2361	0,2961	4,8968	2365,2358	3,6115	59,7346	28852,4159
20678048297	384	23,75	0,0011	0,0171	11,5698	0,0156	0,2518	170,7676	0,2130	3,4429	2334,7990	2,7789	44,9271	30467,3622
22367084959	394	23,83	0,0010	0,0165	11,3598	0,0148	0,2455	169,0853	0,2047	3,3840	2330,7718	2,6925	44,5166	30660,9614
25056082087	456	23,94	0,0009	0,0175	11,0623	0,0139	0,2639	166,6717	0,1933	3,6805	2324,8094	2,5721	48,9842	30940,9848
42652618343	464	24,48	0,0006	0,0119	9,7595	0,0100	0,1897	155,6607	0,1475	2,7959	2294,7549	2,0730	39,2981	32254,2067
127976334671	468	25,58	0,0003	0,0052	7,4994	0,0051	0,0928	134,5112	0,0838	1,5342	2222,6822	1,3189	24,1339	34964,7607
182226896239	474	25,93	0,0002	0,0040	6,8816	0,0041	0,0744	128,1695	0,0698	1,2758	2196,9703	1,1383	20,8086	35833,7223
241160624143	486	26,21	0,0002	0,0033	6,4253	0,0034	0,0634	123,3003	0,0603	1,1183	2175,8066	1,0122	18,7700	36521,0251
297501075799	490	26,42	0,0002	0,0028	6,1020	0,0030	0,0556	119,7438	0,0540	1,0022	2159,5259	0,9267	17,1878	37034,8386
303371455241	500	26,44	0,0002	0,0028	6,0727	0,0030	0,0560	119,4168	0,0535	1,0115	2157,9929	0,9191	17,3820	37082,6027
304599508537	514	26,44	0,0002	0,0029	6,0667	0,0030	0,0574	119,3492	0,0534	1,0375	2157,6756	0,9175	17,8355	37092,4765
416608695821	516	26,76	0,0001	0,0023	5,6147	0,0024	0,0468	114,2028	0,0453	0,8734	2132,7036	0,8039	15,5034	37856,4922
461690510011	532	26,86	0,0001	0,0022	5,4734	0,0023	0,0450	112,5518	0,0429	0,8499	2124,3537	0,7696	15,2449	38106,5523
614487453523	534	27,14	0,0001	0,0018	5,0974	0,0019	0,0373	108,0548	0,0369	0,7260	2100,7368	0,6816	13,4093	38800,5656
738832927927	540	27,33	0,0001	0,0015	4,8679	0,0017	0,0334	105,2312	0,0335	0,6615	2085,2291	0,6301	12,4510	39246,4293
1346294310749	582	27,93	0,0001	0,0010	4,1857	0,0012	0,0241	96,4382	0,0243	0,5074	2033,3331	0,4872	10,1531	40689,0903
1408695493609	588	27,97	0,0000	0,0010	4,1380	0,0011	0,0236	95,7989	0,0238	0,4996	2029,3341	0,4778	10,0434	40797,4102
1968188556461	602	28,31	0,0000	0,0008	3,8013	0,0009	0,0193	91,1839	0,0199	0,4228	1999,5001	0,4136	8,7950	41594,0821
2614941710599	652	28,59	0,0000	0,0007	3,5357	0,0008	0,0172	87,4063	0,0171	0,3894	1973,7495	0,3657	8,3383	42266,6412
7177162611713	674	29,60	0,0000	0,0003	2,7265	0,0004	0,0090	75,0072	0,0099	0,2256	1879,7695	0,2352	5,3559	44621,1418
13829048559701	716	30,26	0,0000	0,0002	2,2983	0,0003	0,0061	67,7686	0,0069	0,1642	1817,1603	0,1761	4,1674	46117,2032
19581334192423	766	30,61	0,0000	0,0002	2,0980	0,0002	0,0052	64,1764	0,0057	0,1437	1783,6172	0,1509	3,7773	46898,7149
42842283925351	778	31,39	0,0000	0,0001	1,7060	0,0001	0,0031	56,6821	0,0037	0,0926	1707,5668	0,1064	2,6375	48625,4649
90874329411493	804	32,14	0,0000	0,0001	1,3961	0,0001	0,0019	50,2102	0,0025	0,0617	1634,2322	0,0758	1,8969	50238,2726
171231342420521	806	32,77	0,0000	0,0000	1,1775	0,0000	0,0012	45,2694	0,0017	0,0426	1572,5505	0,0569	1,3984	51559,8337
218209405436543	906	33,02	0,0000	0,0000	1,1029	0,0000	0,0012	43,4954	0,0015	0,0416	1549,0261	0,0510	1,3972	52056,1053


$H^-_{4,0.1}(p_n)$ $H_{4,0.1}(p_n)$ $H^+_{4,0.1}(p_n)$
150
100
50
0
0
5
10
15
20
25
30
35
H-4,0.1(pn)
H4,0.1(pn)
H+4,0.1(pn)

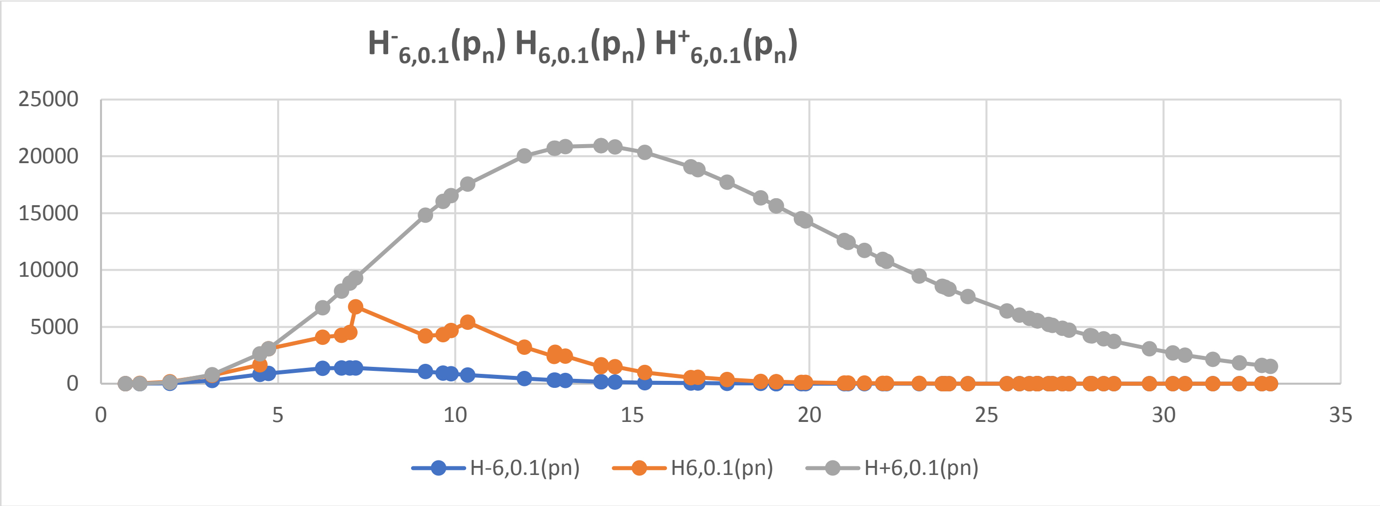

$H^-_{4,0.2}(p_n)$ $H_{4,0.2}(p_n)$ $H^+_{4,0.2}(p_n)$
500
400
300
200
100
0
0
5
10
15
20
25
30
35
H-4,0.2(pn)
H4,0.2(pn)
H+4,0.2(pn)


$H^-_{4,0.3}(p_n)$ $H_{4,0.3}(p_n)$ $H^+_{4,0.3}(p_n)$
3000
2500
2000
1500
1000
500
0
0
5
10
15
20
25
30
35
H-4,0.3(pn)
H4,0.3(pn)
H+4,0.3(pn)

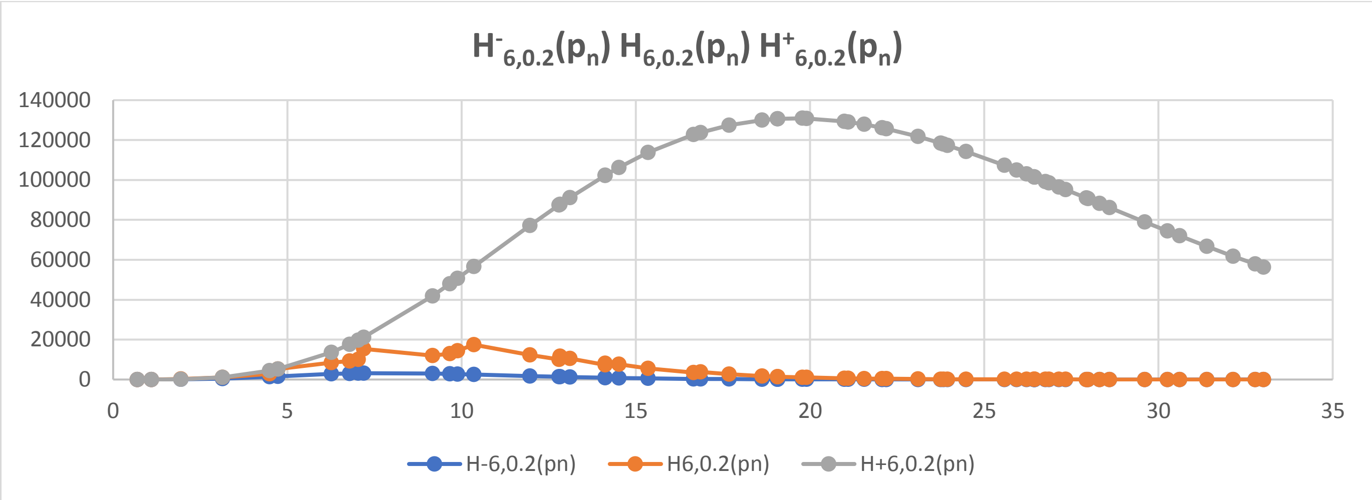

$H^-_{4,0.4}(p_n)$ $H_{4,0.4}(p_n)$ $H^+_{4,0.4}(p_n)$
60000
50000
40000
30000
20000
10000
0
0
5
10
15
20
25
30
35
H-4,0.4(pn)
H4,0.4(pn)
H+4,0.4(pn)
Série4

The table III.6 below give the numeral values of $H^{-}_{k,x}(p_n)$, $H_{k,x}(p_n)$ and $H^{+}_{k,x}(p_n)$ for k=6 and x є {0.1, 0.2, 0.3, 0.4} for all prime numbers corresponding to the first occurrence of the maximal gaps G_n.

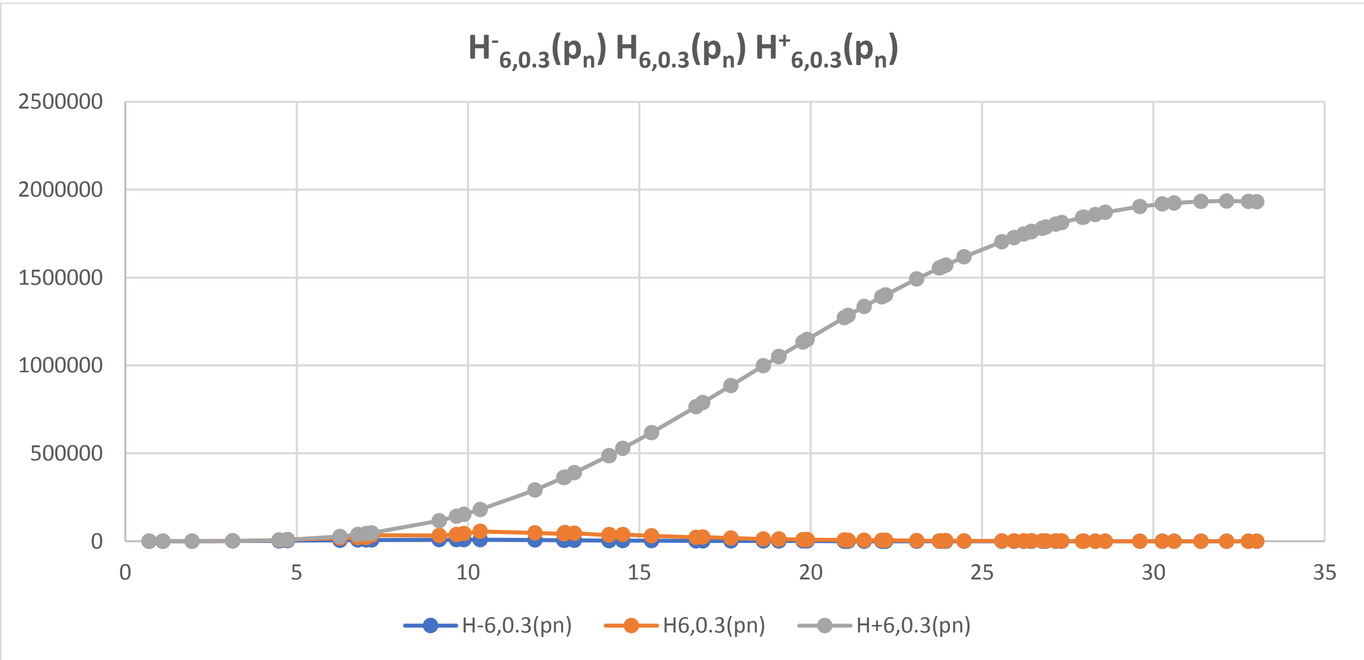
Primes	Gaps	Ln(Pn)	$H^{-}_{6,0.1}(p_n)$	$H_{6,0.1}(p_n)$	$H^{+}_{6,0.1}(p_n)$	$H^{-}_{6,0.2}(p_n)$	$H_{6,0.2}(p_n)$	$H^{+}_{6,0.2}(p_n)$	$H^{-}_{6,0.3}(p_n)$	$H_{6,0.3}(p_n)$	$H^{+}_{6,0.3}(p_n)$	$H^{-}_{6,0.4}(p_n)$	$H_{6,0.4}(p_n)$	$H^{+}_{6,0.4}(p_n)$
k			6	6	6	6	6	6	6	6	6	6	6	6
x			0,1	0,1	0,1	0,2	0,2	0,2	0,3	0,3	0,3	0,4	0,4	0,4
2	1	0,69	0,1012	1,8435	1,3491	0,1097	2,0628	1,4617	0,1189	2,3080	1,5835	0,1289	2,5821	1,7153
3	2	1,10	1,6249	18,4523	11,0107	1,8462	21,7892	12,4964	2,0970	25,7222	14,1787	2,3811	30,3567	16,0832
7	4	1,95	35,2061	175,6583	126,7576	44,1124	226,9630	158,4278	55,2204	292,9661	197,8552	69,0653	377,8216	246,9110
23	6	3,14	276,4181	741,2009	789,6860	396,9987	1079,6680	1129,6739	568,9033	1569,0082	1612,9772	813,5806	2275,2833	2299,0437
89	8	4,49	818,2723	1670,7662	2621,9458	1371,0701	2813,4369	4373,6000	2287,5916	4717,2357	7268,4543	3802,5337	7879,3599	12039,6189
113	14	4,73	917,1572	3067,6745	3071,4789	1578,9401	5317,0823	5264,2110	2705,6455	9172,3927	8985,4963	4617,5203	15757,7493	15282,0393
523	18	6,26	1352,3021	4092,1280	6694,3586	2767,7572	8391,4008	13649,3834	5622,5643	17078,8152	27639,2746	11349,8532	34539,7706	55640,5140
887	20	6,79	1394,2012	4271,0115	8154,5310	3027,9920	9287,6348	17649,2333	6520,3630	20024,3067	37895,7331	13940,5329	42864,0226	80823,4948
1129	22	7,03	**1394,9036**	4518,8007	8840,3565	3112,5999	10094,2330	19661,5077	6882,9137	22345,2257	43358,9233	15105,8644	49092,3532	94940,4540
1327	34	7,19	1389,4212	**6776,4415**	9303,8144	**3157,0108**	15419,1836	21072,6264	7106,2480	34756,1612	47308,5026	15871,4540	77733,0544	105427,7152
9551	36	9,16	1058,8608	4210,9849	14814,4846	2998,3521	11926,6430	41868,5913	**8374,1425**	33316,9186	116756,0355	23131,6676	92048,9763	322110,5030
15683	44	9,66	938,8883	4318,0568	16035,1271	2809,0383	12921,1014	47894,7019	8279,6409	38090,7397	140984,0625	24116,5990	110965,7605	410217,9358
19609	52	9,88	884,5170	4694,1667	16549,8877	2712,6983	14398,5088	50676,7788	8191,7631	43486,6309	152845,6337	24436,6903	129742,4693	455503,0975
31397	72	10,35	772,0976	5405,8406	17553,9798	2494,5427	**17467,7510**	56637,6293	7926,8751	55513,9182	179787,8425	**24863,4530**	174146,3181	563451,6609
155921	86	11,96	444,3551	3206,5064	20035,2244	1713,0989	12362,2513	77178,8616	6470,3246	46693,2464	291326,8593	24058,0869	173620,8018	1082709,5063
360653	96	12,80	317,7653	2389,3676	20713,3281	1343,1972	10100,0274	87503,0153	5550,8140	41739,3461	361449,2809	22550,9585	169574,3151	1467936,9792
370261	112	12,82	314,3025	2751,4607	20727,6991	1332,3946	11664,2144	87816,9561	5521,7011	**48339,5458**	363770,7134	22494,9818	**196934,6993**	1481472,7957
492113	114	13,11	278,7088	2428,8714	20856,9997	1218,9158	10622,6562	91167,9118	5207,6508	45384,4051	389348,5984	21861,5267	190524,3903	1633975,9713
1349533	118	14,12	177,9693	1489,5677	**20944,8509**	869,1100	7274,3119	102245,7588	4135,6542	34614,9270	486404,6818	19305,1558	161582,5692	2270064,2816
1357201	132	14,12	177,5051	1661,2734	20943,7946	867,3795	8117,8615	102303,9439	4129,9144	38652,3391	486975,9219	19289,8388	180536,7186	2274084,2295
2010733	148	14,51	147,7417	1508,0269	20831,1346	753,5763	7691,9253	106217,9072	3741,5591	38191,0994	527257,5309	18212,0817	185896,3288	2565984,7872
4652353	154	15,35	98,4099	987,7817	20350,2262	549,9724	5520,3173	113702,4449	2985,5155	29966,9999	617129,0301	15867,2057	159266,7563	3279465,5773
17051707	180	16,65	50,6445	547,6499	19072,8521	325,9007	3524,1630	122717,5188	2030,3929	21955,8915	764466,3158	12359,4356	133650,2788	4653129,8511
20831323	210	16,85	45,5584	567,9119	18830,0763	299,6009	3734,7004	123813,9216	1906,5093	23765,7645	787817,3766	11850,1794	147719,5667	4896461,5809
47326693	220	17,67	29,2667	364,4137	17738,8254	210,3372	2619,0051	127475,0024	1459,7371	18175,8635	884616,7918	9882,8468	123055,9405	5988828,1845
122164747	222	18,62	17,2673	205,8925	16335,9764	137,4785	1639,2682	130054,7526	1054,4356	12572,8939	997452,3903	7878,4414	93941,0758	7452440,0769
189695659	234	19,06	13,4439	165,0618	15652,5173	112,2364	1378,0222	130668,2320	901,6492	11070,3164	1049682,0700	7051,7356	86580,1753	8209271,6526
191912783	248	19,07	13,3547	173,6712	15634,2863	111,6318	1451,7139	130679,5618	897,8888	11676,5851	1051057,5311	7030,7975	91431,9308	8229946,3463
387096133	250	19,77	8,8998	112,5274	14522,7544	80,2295	1014,4060	**130913,6210**	694,7116	8783,7933	1133556,5656	5850,3362	73970,4759	9545769,8981
436273009	282	19,89	8,2985	117,6429	14332,0829	75,7773	1074,2495	130867,4403	664,4543	9419,5695	1147482,7138	5665,2946	80313,4850	9783527,1072
1294268491	288	20,98	4,3504	59,7183	12606,6929	44,6461	612,8665	129374,4799	438,7852	6023,2967	1271480,7611	4186,8183	57473,3385	12132125,1839
1453168141	292	21,10	4,0572	56,1575	12425,4083	42,1579	583,5262	129107,8162	419,3888	5804,9432	1284349,7473	4049,9355	56056,9294	12402516,3501
2300942549	320	21,56	3,0705	45,5820	11713,8774	33,5161	497,5529	127860,9073	349,8576	5193,6993	1334657,1475	3542,7977	52593,4758	13515149,1216
3842610773	336	22,07	2,2426	34,1438	10938,1964	25,8610	393,7361	126134,1802	284,8278	4336,5358	1389203,7926	3041,1162	46301,3427	14832468,4837
4302407359	354	22,18	2,0916	33,3802	10770,2674	24,4135	389,6144	125709,0009	272,0817	4342,1495	1400979,5879	2939,1070	46905,1844	15133691,3546
10726904659	382	23,10	1,1840	19,5840	9459,4605	15,2380	252,0354	121736,5635	186,8279	3090,1211	1492560,8112	2217,5349	36677,8711	17715712,9182
20678048297	384	23,75	0,7821	12,6445	8574,9918	10,7957	174,5351	118361,8258	141,7420	2291,5476	1554016,3737	1800,0516	29101,4854	19735149,0004
22367084959	394	23,83	0,7440	12,3013	8472,6422	10,3563	171,2254	117932,5998	137,0905	2266,5696	1561106,8293	1755,1084	29017,8838	19986090,3490
25056082087	456	23,94	0,6921	13,1811	8326,0044	9,7513	185,7072	117303,0690	130,6178	2487,5307	1571256,4653	1691,8964	32221,0723	20352473,2553
42652618343	464	24,48	0,4923	9,3333	7660,4505	7,3410	139,1659	114221,8600	103,9358	1970,3347	1617166,0096	1422,0214	26957,5928	22125580,0685
127976334671	468	25,58	0,2415	4,4184	6401,3368	4,0468	74,0525	107286,4625	64,2334	1175,4166	1702925,1440	983,8860	18004,2588	26084262,6671
182226896239	474	25,93	0,1915	3,5015	6029,8147	3,3330	60,9312	104927,6248	54,8832	1003,3253	1727788,6686	871,7316	15936,2129	27443128,1677
241160624143	486	26,21	0,1593	2,9535	5746,7596	2,8554	52,9486	103023,1865	48,4066	897,6295	1746531,3479	791,2968	14673,4238	28550284,8996
297501075799	490	26,42	0,1387	2,5717	5541,2988	2,5417	47,1426	101579,1608	44,0390	816,8153	1760005,0773	735,5779	13643,1504	29397101,8352
303371455241	500	26,44	0,1369	2,5886	5522,4639	2,5143	47,5506	101444,0932	43,6523	825,5555	1761234,0743	730,5811	13816,8320	29476665,7242
304599508537	514	26,44	0,1365	2,6535	5518,5760	2,5087	48,7649	101416,1549	43,5727	846,9945	1761487,6326	729,5519	14181,4883	29493131,2235
416608695821	516	26,76	0,1109	2,1392	5223,5065	2,1073	40,6408	99237,2497	37,8105	729,2070	1780585,4932	653,7391	12607,9082	30786114,8650
461690510011	532	26,86	0,1036	2,0520	5129,3871	1,9897	39,4126	98517,1476	36,0842	714,7490	1786611,1877	630,5012	12488,8496	31217520,1450
614487453523	534	27,14	0,0856	1,6846	4874,4257	1,6953	33,3507	96502,2039	31,6691	623,0229	1802748,6286	569,8174	11209,9336	32436520,4074
738832927927	540	27,33	0,0757	1,4960	4715,4219	1,5284	30,2011	95195,9753	29,1030	575,0688	1812658,1746	533,6319	10544,4336	33236778,7560
1346294310749	582	27,93	0,0506	1,0545	4226,0359	1,0886	22,6862	90915,7047	22,0592	459,6941	1842235,6365	430,1242	8963,3906	35921083,3120
1408695493609	588	27,97	0,0491	1,0317	4190,8089	1,0610	22,3015	90591,4319	21,5998	454,0237	1844301,5177	423,1260	8894,0313	36128643,6096
1968188556461	602	28,31	0,0392	0,8327	3938,0807	0,8769	18,6490	88196,0679	18,4825	393,0483	1858822,7315	374,6764	7967,7969	37681865,1845
2614941710599	652	28,59	0,0323	0,7365	3733,2967	0,7454	16,9978	86161,3997	16,1791	368,9390	1870151,5276	337,6625	7699,8906	39030475,9758
7177162611713	674	29,60	0,0162	0,3692	3075,6456	0,4163	9,4797	78977,6558	10,0316	228,4092	1902956,4695	232,1461	5285,7656	44037010,5108
13829048559701	716	30,26	0,0103	0,2443	2703,1794	0,2841	6,7221	74389,6306	7,3248	173,3252	1918107,5621	181,2597	4289,2188	47465799,3260
19581334192423	766	30,61	0,0081	0,2031	2521,8511	0,2317	5,7985	71993,1527	6,1921	154,9834	1924209,6679	158,7734	3973,8750	49339515,3268
42842283925351	778	31,39	0,0047	0,1167	2151,7719	0,1460	3,6185	66712,7640	4,2296	104,8652	1933153,6028	117,4828	2912,4063	53695357,9960
90874329411493	804	32,14	0,0028	0,0695	1842,0381	0,0933	2,3341	61815,4184	2,9222	73,0879	1935675,6475	87,6405	2192,0000	58054301,1021
171231342420521	806	32,77	0,0018	0,0437	1612,4370	0,0638	1,5690	57839,5095	2,1341	52,4375	1933367,4552	68,2781	1677,7500	61856175,3942
218209405436543	906	33,02	0,0015	0,0411	1531,5679	0,0552	1,5127	56356,9856	1,8911	51,8633	1931448,2166	62,0170	1700,2500	63341530,8038



$H^-_{6,0.1}(p_n)$ $H_{6,0.1}(p_n)$ $H^+_{6,0.1}(p_n)$
25000
20000
15000
10000
5000
0
0 5 10 15 20 25 30 35
H-6,0.1(pn)
H6,0.1(pn)
H+6,0.1(pn)



$H^-_{6,0.2}(p_n)$ $H_{6,0.2}(p_n)$ $H^+_{6,0.2}(p_n)$
140000
120000
100000
80000
60000
40000
20000
0
0 5 10 15 20 25 30 35
H-6,0.2(pn)
H6,0.2(pn)
H+6,0.2(pn)



$H^-_{6,0.3}(p_n)$ $H_{6,0.3}(p_n)$ $H^+_{6,0.3}(p_n)$
2500000
2000000
1500000
1000000
500000
0
0 5 10 15 20 25 30 35
H-6,0.3(pn)
H6,0.3(pn)
H+6,0.3(pn)

III.5 – $H_{y,x}(p_n) = (p_{n+1})^x . \ln^y(p_{n+1}) - (p_n)^x . \ln^y(p_n) = 1$ has a set of solutions (x,y) for all p_n

This question represent a generalization of the cases studied in paragraphs I.1 and II.7.

We are going to study the field of $(x,y) \in \mathbb{R}^2$ which satisfy to $H_{y,x}(p_n) = (p_{n+1})^x . \ln^y(p_{n+1}) - (p_n)^x . \ln^y(p_n) = 1$

We have state in III.3 that that $M_{y,x}(p_n) = \frac{\ln^y(p_n)}{{p_n}^{1-x}} . g_n . \left(x + \frac{y}{\ln(p_n)}\right)$ was representing a good estimation of $H_{k,x}(p_n)$. So we are going to study the relation :

$$\frac{\ln^y(p_n)}{{p_n}^{1-x}} . g_n . \left(x + \frac{y}{\ln(p_n)}\right) = 1 \quad \Leftrightarrow \quad {p_n}^x . \ln^y(p_n) . \left(x + \frac{y}{\ln(p_n)}\right) = \frac{p_n}{g_n} \qquad \text{(III.12)}$$

As before we start by taking the logarithm of the two parts of (III.12) :

$$x . \ln(p_n) + y . \ln(\ln(p_n)) + \ln\left(x + \frac{y}{\ln(p_n)}\right) = \ln\left(\frac{p_n}{g_n}\right) \qquad \text{(III.13)}$$

(III.12) gives an implicit relation between y, x and p_n and we have to build an as good as possible estimation of $y = y(x, p_n)$. We can obtain a first approximation y_0 of y, neglecting the term $\ln\left(x + \frac{y}{\ln(p_n)}\right)$.

$$y_0 = \frac{\left(\ln\left(\frac{p_n}{g_n}\right) - x.\ln(p_n)\right)}{\ln(\ln(p_n))}$$

We define

$$y_1 = y_0 - \varepsilon = \frac{\left(\ln\left(\frac{p_n}{g_n}\right) - x.\ln(p_n)\right)}{\ln(\ln(p_n))} - \varepsilon \qquad \text{(III.14)}$$

We now replace (III.14) into (III.12):

$${p_n}^x . e^{\left(\ln\left(\frac{p_n}{g_n}\right) - x.\ln(p_n)\right) - \varepsilon.\ln(\ln(p_n))} . \left(x + \frac{\left(\ln\left(\frac{p_n}{g_n}\right) - x.\ln(p_n)\right) - \varepsilon.\ln(\ln(p_n))}{\ln(p_n).\ln(\ln(p_n))} \right) = \frac{p_n}{g_n}$$

$$e^{-\varepsilon.\ln(\ln(p_n))} . \left(x + \frac{\left(\ln\left(\frac{p_n}{g_n}\right) - x.\ln(p_n)\right) - \varepsilon.\ln(\ln(p_n))}{\ln(p_n).\ln(\ln(p_n))} \right) = 1$$

Considering ε small, we apply a Taylor serie expansion of $e^{-\varepsilon.\ln(\ln(p_n))}$

$$\left(1 - \varepsilon.\ln(\ln(p_n))\right) . \left(x.\ln(p_n).\ln(\ln(p_n)) + \ln\left(\frac{p_n}{g_n}\right) - x.\ln(p_n) - \varepsilon.\ln(\ln(p_n)) \right) = \ln(p_n).\ln(\ln(p_n))$$

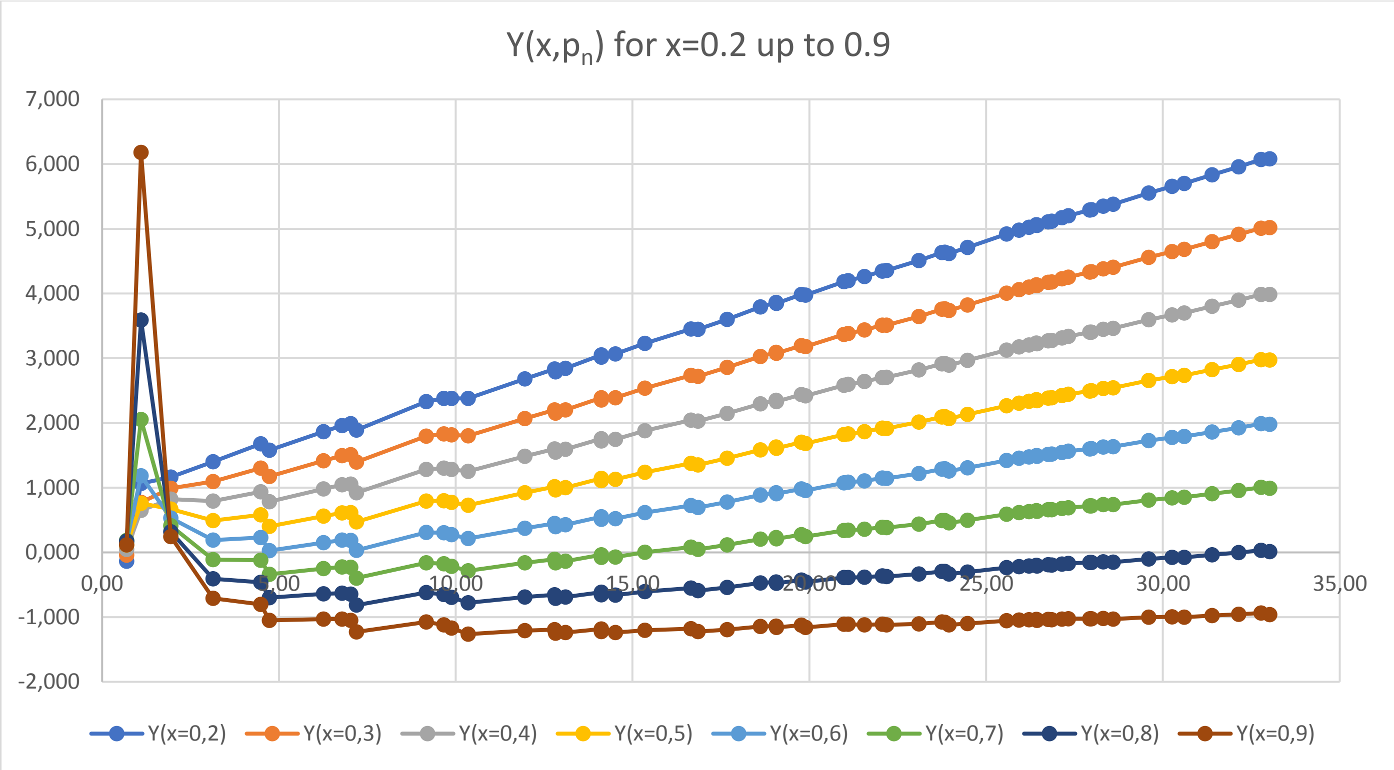
$$\varepsilon = \frac{(x-1).\ln(p_n).\ln(\ln(p_n)) + \ln\left(\frac{p_n}{g_n}\right) - x.\ln(p_n)}{\ln(\ln(p_n)).\left(x.\ln(p_n).\ln(\ln(p_n)) + \ln\left(\frac{p_n}{g_n}\right) - x.\ln(p_n) + 1\right)}$$

And

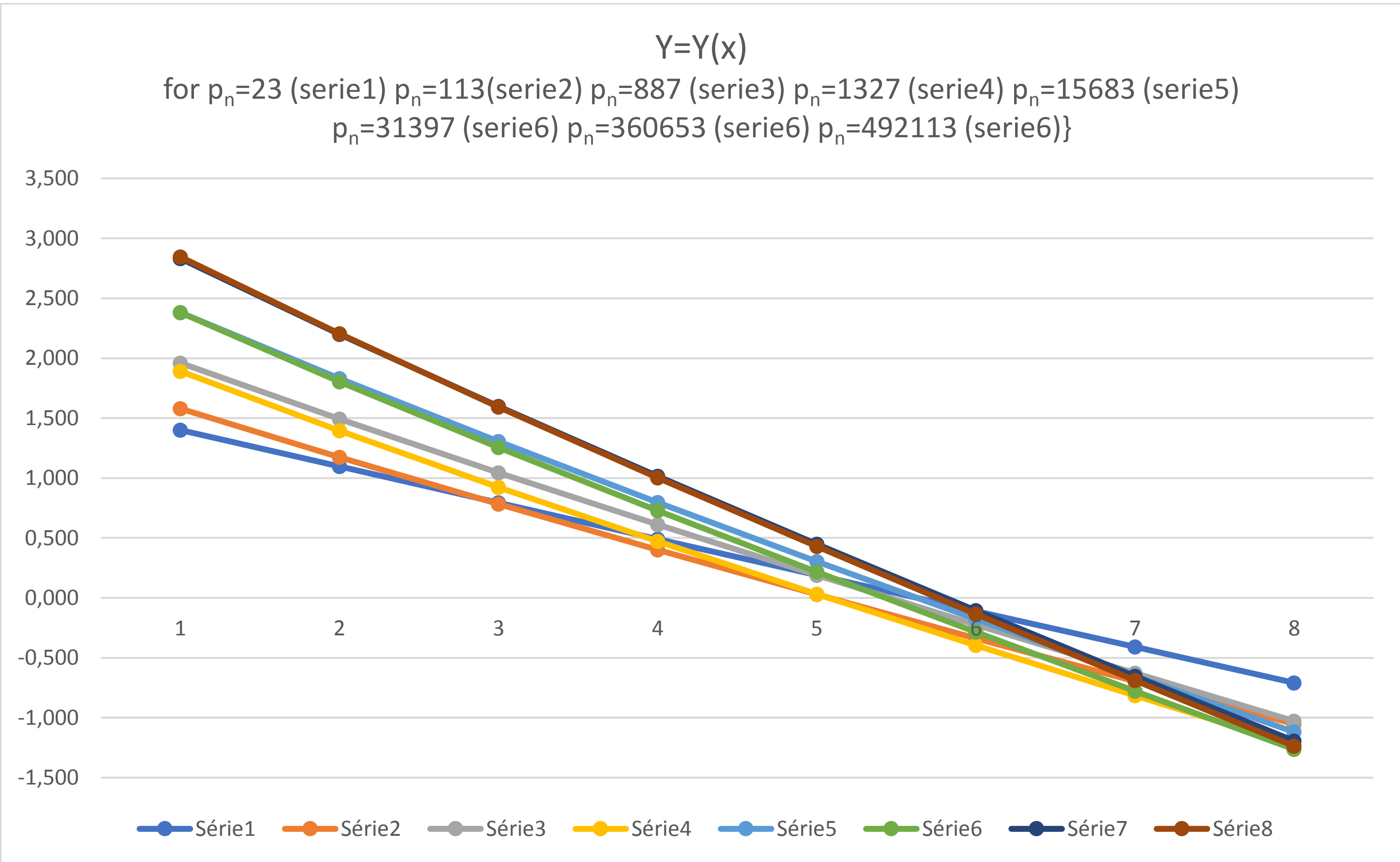
$$y_1 = \frac{\left(\ln\left(\frac{p_n}{g_n}\right) - x.\ln(p_n)\right)}{\ln(\ln(p_n))} - \frac{(x-1).\ln(p_n).\ln(\ln(p_n)) + \ln\left(\frac{p_n}{g_n}\right) - x.\ln(p_n)}{\ln(\ln(p_n)).\left(x.\ln(p_n).\ln(\ln(p_n)) + \ln\left(\frac{p_n}{g_n}\right) - x.\ln(p_n) + 1\right)}$$

In table III.7 below, we have the estimation of y given by y_1 for different values of x and in red the corresponding value of $H_{y,x}(p_n)$. We see that y_1 gives a better estimation of y for the highest values of x.

Primes	Gaps	ln(Pn)	Y(x=0,2)	$H_{0.2,Y}(P_n)$	Y(x=0,3)	$H_{0.3,Y}(P_n)$	Y(x=0,4)	$H_{0.4,Y}(P_n)$	Y(x=0,5)	$H_{0.5,Y}(P_n)$	Y(x=0,6)	$H_{0.6,Y}(P_n)$	Y(x=0,7)	$H_{0.7,Y}(P_n)$	Y(x=0,8)	$H_{0.8,Y}(P_n)$	Y(x=0,9)	$H_{0.9,Y}(P_n)$
2	1	0,69	-0,138	0,021	-0,040	0,136	0,045	0,261	0,114	0,394	0,163	0,535	0,185	0,678	0,174	0,815	0,118	0,931
3	2	1,10	1,066	0,915	0,773	0,846	0,652	0,946	0,760	1,351	1,183	2,452	2,053	5,580	3,589	16,624	6,181	75,844
7	4	1,95	1,166	1,271	0,989	1,412	0,823	1,593	0,670	1,827	0,533	2,137	0,415	2,554	0,318	3,131	0,247	3,950
23	6	3,14	1,400	1,460	1,096	1,427	0,793	1,397	0,491	1,369	0,190	1,344	-0,110	1,321	-0,410	1,300	-0,708	1,281
89	8	4,49	1,676	1,529	1,304	1,413	0,939	1,323	0,580	1,253	0,228	1,198	-0,120	1,154	-0,464	1,119	-0,804	1,091
113	14	4,73	1,579	1,911	1,174	1,684	0,782	1,519	0,401	1,398	0,028	1,306	-0,338	1,235	-0,697	1,180	-1,052	1,137
523	18	6,26	1,866	1,819	1,414	1,570	0,980	1,403	0,560	1,287	0,151	1,204	-0,249	1,144	-0,642	1,100	-1,029	1,067
887	20	6,79	1,960	1,818	1,493	1,558	1,045	1,388	0,611	1,271	0,189	1,190	-0,223	1,131	-0,629	1,089	-1,028	1,058
1 129	22	7,03	1,988	1,841	1,512	1,568	1,056	1,392	0,616	1,272	0,187	1,189	-0,232	1,130	-0,643	1,087	-1,049	1,056
1 327	34	7,19	1,891	2,068	1,395	1,704	0,924	1,478	0,471	1,330	0,032	1,228	-0,396	1,156	-0,815	1,106	-1,228	1,069
9 551	36	9,16	2,332	1,875	1,796	1,560	1,284	1,368	0,791	1,244	0,311	1,161	-0,159	1,104	-0,621	1,065	-1,076	1,038
15 683	44	9,66	2,381	1,914	1,830	1,578	1,304	1,376	0,797	1,248	0,304	1,162	-0,178	1,104	-0,652	1,064	-1,119	1,037
19 609	52	9,88	2,378	1,959	1,818	1,601	1,285	1,389	0,771	1,255	0,272	1,167	-0,216	1,107	-0,696	1,066	-1,169	1,038
31 397	72	10,35	2,381	2,043	1,803	1,644	1,255	1,412	0,728	1,268	0,215	1,174	-0,286	1,111	-0,778	1,068	-1,263	1,039
155 921	86	11,96	2,683	1,992	2,070	1,602	1,486	1,379	0,923	1,243	0,375	1,154	-0,161	1,095	-0,689	1,056	-1,211	1,030
360 653	96	12,80	2,832	1,980	2,200	1,590	1,597	1,369	1,016	1,234	0,449	1,147	-0,106	1,090	-0,653	1,052	-1,194	1,027
370 261	112	12,82	2,789	2,021	2,153	1,612	1,549	1,382	0,965	1,242	0,397	1,152	-0,160	1,093	-0,708	1,054	-1,250	1,029
492 113	114	13,11	2,846	2,012	2,204	1,605	1,593	1,377	1,003	1,238	0,429	1,150	-0,134	1,091	-0,689	1,053	-1,237	1,028
1 349 533	118	14,12	3,052	1,978	2,389	1,582	1,757	1,360	1,146	1,226	0,549	1,140	-0,036	1,084	-0,613	1,048	-1,183	1,024
1 357 201	132	14,12	3,018	2,004	2,354	1,596	1,720	1,368	1,108	1,231	0,511	1,143	-0,075	1,086	-0,653	1,049	-1,224	1,025
2 010 733	148	14,51	3,067	2,016	2,392	1,600	1,749	1,370	1,128	1,231	0,521	1,143	-0,074	1,086	-0,660	1,048	-1,240	1,024
4 652 353	154	15,35	3,232	1,996	2,539	1,585	1,879	1,359	1,239	1,223	0,615	1,137	0,001	1,081	-0,604	1,045	-1,202	1,022
17 051 707	180	16,65	3,454	1,990	2,733	1,577	2,044	1,351	1,376	1,217	0,723	1,132	0,081	1,077	-0,553	1,042	-1,181	1,020
20 831 323	210	16,85	3,448	2,015	2,721	1,589	2,026	1,358	1,353	1,220	0,695	1,134	0,048	1,079	-0,590	1,043	-1,222	1,020
47 326 693	220	17,67	3,602	2,003	2,858	1,580	2,146	1,351	1,455	1,215	0,779	1,130	0,114	1,076	-0,542	1,041	-1,192	1,019
122 164 747	222	18,62	3,791	1,985	3,027	1,568	2,295	1,343	1,584	1,209	0,888	1,126	0,203	1,072	-0,474	1,038	-1,145	1,017
189 695 659	234	19,06	3,864	1,986	3,090	1,567	2,349	1,341	1,628	1,208	0,923	1,124	0,228	1,071	-0,459	1,038	-1,139	1,017
191 912 783	248	19,07	3,849	1,995	3,074	1,572	2,332	1,344	1,611	1,209	0,905	1,125	0,210	1,072	-0,477	1,038	-1,158	1,017
387 096 133	250	19,77	3,986	1,985	3,197	1,565	2,440	1,339	1,704	1,205	0,983	1,123	0,273	1,070	-0,429	1,037	-1,125	1,016
436 273 009	282	19,89	3,974	2,001	3,181	1,573	2,421	1,343	1,682	1,208	0,958	1,124	0,245	1,071	-0,460	1,037	-1,158	1,017
1 294 268 491	288	20,98	4,182	1,989	3,367	1,564	2,584	1,336	1,822	1,203	1,075	1,120	0,339	1,068	-0,389	1,035	-1,112	1,015
1 453 168 141	292	21,10	4,200	1,989	3,383	1,564	2,597	1,336	1,833	1,203	1,084	1,120	0,345	1,068	-0,386	1,035	-1,111	1,015
2 300 942 549	320	21,56	4,263	1,996	3,435	1,566	2,640	1,337	1,865	1,203	1,106	1,120	0,357	1,068	-0,383	1,035	-1,118	1,015
3 842 610 773	336	22,07	4,348	1,996	3,509	1,565	2,703	1,336	1,918	1,202	1,148	1,119	0,388	1,067	-0,364	1,034	-1,109	1,015
4 302 407 359	354	22,18	4,355	2,002	3,513	1,568	2,704	1,337	1,917	1,202	1,144	1,120	0,382	1,067	-0,373	1,034	-1,121	1,015
10 726 904 659	382	23,10	4,508	2,001	3,647	1,566	2,819	1,335	2,012	1,200	1,220	1,118	0,439	1,066	-0,335	1,034	-1,103	1,014
20 678 048 297	384	23,75	4,632	1,996	3,757	1,561	2,916	1,332	2,095	1,198	1,289	1,116	0,494	1,065	-0,293	1,033	-1,075	1,014
22 367 084 959	394	23,83	4,639	1,998	3,763	1,563	2,920	1,332	2,097	1,198	1,290	1,116	0,493	1,065	-0,296	1,033	-1,079	1,014
25 056 082 087	456	23,94	4,619	2,015	3,739	1,570	2,893	1,337	2,068	1,201	1,258	1,118	0,458	1,066	-0,334	1,033	-1,120	1,014
42 652 618 343	464	24,48	4,714	2,012	3,824	1,568	2,967	1,335	2,131	1,200	1,309	1,117	0,498	1,065	-0,304	1,033	-1,101	1,013
127 976 334 671	468	25,58	4,918	2,004	4,005	1,562	3,126	1,330	2,267	1,196	1,423	1,114	0,589	1,063	-0,237	1,031	-1,057	1,013
182 226 896 239	474	25,93	4,980	2,003	4,060	1,561	3,173	1,329	2,307	1,195	1,456	1,114	0,615	1,063	-0,218	1,031	-1,046	1,012
241 160 624 143	486	26,21	5,025	2,004	4,100	1,561	3,207	1,329	2,335	1,195	1,478	1,113	0,631	1,062	-0,208	1,031	-1,042	1,012
297 501 075 799	490	26,42	5,062	2,003	4,132	1,560	3,235	1,329	2,358	1,195	1,497	1,113	0,646	1,062	-0,198	1,031	-1,036	1,012
303 371 455 241	500	26,44	5,060	2,005	4,129	1,561	3,231	1,329	2,355	1,195	1,493	1,113	0,641	1,062	-0,203	1,031	-1,041	1,012
304 599 508 537	514	26,44	5,053	2,008	4,122	1,563	3,224	1,330	2,347	1,196	1,485	1,114	0,633	1,063	-0,211	1,031	-1,049	1,012
416 608 695 821	516	26,76	5,110	2,007	4,172	1,562	3,268	1,329	2,385	1,195	1,516	1,113	0,658	1,062	-0,192	1,031	-1,037	1,012
461 690 510 011	532	26,86	5,120	2,009	4,181	1,563	3,274	1,329	2,389	1,195	1,518	1,113	0,657	1,062	-0,195	1,031	-1,042	1,012
614 487 453 523	534	27,14	5,172	2,008	4,226	1,561	3,314	1,329	2,423	1,194	1,546	1,113	0,680	1,062	-0,179	1,030	-1,032	1,012
738 832 927 927	540	27,33	5,202	2,008	4,253	1,561	3,337	1,328	2,442	1,194	1,562	1,112	0,692	1,062	-0,171	1,030	-1,027	1,012
1 346 294 310 749	582	27,93	5,291	2,012	4,330	1,562	3,401	1,328	2,494	1,194	1,601	1,112	0,718	1,061	-0,156	1,030	-1,026	1,012
1 408 695 493 609	588	27,97	5,297	2,013	4,334	1,563	3,405	1,329	2,496	1,194	1,603	1,112	0,719	1,061	-0,157	1,030	-1,027	1,012
1 968 188 556 461	602	28,31	5,351	2,013	4,382	1,562	3,446	1,328	2,530	1,194	1,630	1,112	0,739	1,061	-0,144	1,030	-1,020	1,012
2 614 941 710 599	652	28,59	5,381	2,020	4,405	1,565	3,463	1,329	2,542	1,194	1,635	1,112	0,739	1,061	-0,150	1,030	-1,033	1,012
7 177 162 611 713	674	29,60	5,555	2,018	4,559	1,563	3,596	1,327	2,654	1,193	1,727	1,111	0,810	1,060	-0,099	1,029	-1,003	1,011
13 829 048 559 701	716	30,26	5,656	2,021	4,647	1,563	3,670	1,327	2,715	1,192	1,775	1,110	0,844	1,060	-0,078	1,029	-0,995	1,011
19 581 334 192 423	766	30,61	5,700	2,025	4,683	1,566	3,700	1,328	2,738	1,193	1,790	1,111	0,852	1,060	-0,078	1,029	-1,002	1,011
42 842 283 925 351	778	31,39	5,835	2,024	4,803	1,564	3,804	1,327	2,826	1,191	1,863	1,110	0,909	1,059	-0,037	1,029	-0,977	1,011
90 874 329 411 493	804	32,14	5,960	2,025	4,913	1,563	3,898	1,325	2,905	1,190	1,927	1,109	0,958	1,058	-0,003	1,028	-0,959	1,010
171 231 342 420 521	806	32,77	6,071	2,020	5,011	1,561	3,984	1,323	2,979	1,189	1,987	1,107	1,006	1,057	0,032	1,027	-0,936	1,010
218 209 405 436 543	906	33,02	6,082	2,033	5,017	1,563	3,985	1,325	2,974	1,191	1,978	1,108	0,992	1,058	0,013	1,027	-0,961	1,009



The horizontal axis correspond to $\ln(P_n)$for all P_n corresponding to the first occurrence of the maximal gaps.



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