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Special Smarandache Curves in the Euclidean Space

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Abstract: In this work, we introduce some special Smarandache curves in the Euclidean space. We study Frenet-Serret invariants of a special case. Besides, we illustrate examples of our main results.

Key Words: Smarandache Curves, Frenet-Serret Trihedra, Euclidean Space.

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§1. Introduction

It is safe to report that the many important results in the theory of the curves in E^3 were initiated by G. Monge; and G. Darboux pionnered the moving frame idea. Thereafter, F. Frenet defined his moving frame and his special equations which play important role in mechanics and kinematics as well as in differential geometry (for more details see [1]).

At the beginning of the 20th century, A. Einstein's theory opened a door to new geometries such as Lorentzian Geometry, which is simultaneously the geometry of special relativity, was established. Thereafter, researchers discovered a bridge between modern differential geometry and the mathematical physics of general relativity by giving an invariant treatment of Lorentzian geometry. They adapted the geometrical models to relativistic motion of charged particles. Consequently, the theory of the curves has been one of the most fascinating topic for such modeling process. As it stands, the Frenet-Serret formalism of a relativistic motion describes the dynamics of the charged particles. The mentioned works are treated in Minkowski space-time.

In the light of the existing literature, in [4] authors introduced special curves by Frenet-Serret frame vector fields in Minkowski space-time. A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a *Smarandache Curve* [4]. In this work, we study special Smarandache Curve in the Euclidean space. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

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§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E^3 are briefly presented (A more complete elementary treatment can be found in [2].)

The Euclidean 3-space E^3 provided with the standard flat metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. Recall that, the norm of an arbitrary vector $a \in \mathrm{E}^3$ is given by $\|a\| = \sqrt{\langle a, a\rangle}$. φ is called an unit speed curve if velocity vector v of φ satisfies $\|v\| = 1$. For vectors $v, w \in E^3$ it is said to be orthogonal if and only if $\langle v, w\rangle = 0$. Let $\vartheta = \vartheta(s)$ be a regular curve in E^3. If the tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called a general helix or an inclined curve. The sphere of radius $r > 0$ and with center in the origin in the space E^3 is defined by

$$S^2 = \left\{p = (p_1, p_2, p_3) \in \mathrm{E}^3 : \langle p, p\rangle = r^2\right\}.$$

Denote by $\{T, N, B\}$ the moving Frenet-Serret frame along the curve φ in the space E^3. For an arbitrary curve $\varphi \in \mathrm{E}^3$, with first and second curvature, κ and τ respectively, the Frenet-Serret formulae is given by [2]

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad (1)$$

where

$$\langle T, T\rangle = \langle N, N\rangle = \langle B, B\rangle = 1,$$
$$\langle T, N\rangle = \langle T, B\rangle = \langle T, N\rangle = \langle N, B\rangle = 0.$$

The first and the second curvatures are defined by $\kappa = \kappa(s) = \|T'(s)\|$ and $\tau(s) = -\langle N, B'\rangle$, respectively.

§3. Special Smarandache Curves in E^3

In [4] authors introduced:

Definition 3.1 *A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve.*

In the light of the above definition, we adapt it to regular curves in the Euclidean space as follows:

Definition 3.2 *Let $\gamma = \gamma(s)$ be a unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving Frenet-Serret frame. Smarandache TN curves are defined by*

$$\zeta = \zeta(s_\zeta) = \frac{1}{\sqrt{2}}(T + N). \quad (2)$$

Let us investigate Frenet-Serret invariants of Smarandache TN curves according to $\gamma = \gamma(s)$. Differentiating (2), we have

$$\zeta' = \frac{d\zeta}{ds_\zeta}\frac{ds_\zeta}{ds} = \frac{1}{\sqrt{2}}\left(-\kappa T + \kappa N + \tau B\right), \tag{3}$$

and hence

$$T_\zeta = \frac{-\kappa T + \kappa N + \tau B}{\sqrt{2\kappa^2 + \tau^2}} \tag{4}$$

where

$$\frac{ds_\zeta}{ds} = \sqrt{\frac{2\kappa^2 + \tau^2}{2}}. \tag{5}$$

In order to determine the first curvature and the principal normal of the curve ζ, we formalize

$$T'_\zeta = \dot{T}_\zeta \frac{ds_\zeta}{ds} = \frac{\delta T + \mu N + \eta B}{(2\kappa^2 + \tau^2)^{\frac{3}{2}}}, \tag{6}$$

where

$$\begin{cases} \delta = -\Big[\kappa^2(2\kappa^2 + \tau^2) + \tau(\tau\kappa' - \kappa\tau')\Big], \\ \mu = -\Big[\kappa^2(2\kappa^2 + 3\tau^2) + \tau(\tau^3 - \tau\kappa' + \kappa\tau')\Big], \\ \eta = \kappa\Big[\tau(2\kappa^2 + \tau^2) - 2(\tau\kappa' - \kappa\tau')\Big]. \end{cases} \tag{7}$$

Then, we have

$$\dot{T}_\zeta = \frac{\sqrt{2}}{(2\kappa^2 + \tau^2)^2}\Big(\delta T + \mu N + \eta B\Big). \tag{8}$$

So, the first curvature and the principal normal vector field are respectively given by

$$\left\|\dot{T}_\zeta\right\| = \frac{\sqrt{2}\sqrt{\delta^2 + \mu^2 + \eta^2}}{\left(2\kappa^2 + \tau^2\right)^2} \tag{9}$$

and

$$N_\zeta = \frac{\delta T + \mu N + \eta B}{\sqrt{\delta^2 + \mu^2 + \eta^2}}. \tag{10}$$

On other hand, we express

$$T_\zeta \times N_\zeta = \frac{1}{vl}\begin{vmatrix} T & N & B \\ -\kappa & \kappa & \tau \\ \delta & \mu & \eta \end{vmatrix}, \tag{11}$$

where $v = \sqrt{2\kappa^2 + \tau^2}$ and $l = \sqrt{\delta^2 + \mu^2 + \eta^2}$. So, the binormal vector is

$$B_\zeta = \frac{\left[\kappa\eta - \tau\mu\right] T + \left[\kappa\eta + \delta\tau\right] N - \kappa\left[\mu + \delta\right] B}{vl}. \tag{12}$$

In order to calculate the torsion of the curve ζ, we differentiate

$$\zeta'' = \frac{1}{\sqrt{2}}\left\{\begin{array}{c} -(\kappa^2 + \kappa')T + \\ (\kappa' - \kappa^2 - \tau^2)N \\ +(\kappa\tau + \tau')B \end{array}\right\} \tag{13}$$

and thus

$$\zeta''' = \frac{\omega T + \phi N + \sigma B}{\sqrt{2}}, \tag{14}$$

where

$$\begin{cases} \omega = \kappa^3 + \kappa(\tau^2 - 3\kappa') - \kappa'', \\ \phi = -\kappa^3 - \kappa(\tau^2 + 3\kappa') - 3\tau\tau' + \kappa'', \\ \sigma = -\kappa^2\tau - \tau^3 + 2\tau\kappa' + \kappa\tau' + \tau''. \end{cases} \tag{15}$$

The torsion is then given by:

$$\tau_\zeta = \frac{\sqrt{2}\Big[(\kappa^2 + \tau^2 - \kappa')(\kappa\sigma + \tau\omega) + \kappa(\kappa\tau + \tau')(\phi - \omega) + (\kappa^2 + \kappa')(\kappa\sigma - \tau\phi)\Big]}{\left[\tau(2\kappa^2 + \tau^2) + \kappa\tau' - \kappa\tau'\right]^2 + (\kappa'\tau - \kappa\tau')^2 + (2\kappa^3 + \kappa\tau^2)^2}. \tag{16}$$

Definition 3.3 *Let $\gamma = \gamma(s)$ be an unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving Frenet-Serret frame. Smarandache NB curves are defined by*

$$\xi = \xi(s_\xi) = \frac{1}{\sqrt{2}}\left(N + B\right). \tag{17}$$

Remark 3.4 The Frenet-Serret invariants of Smarandache NB curves can be easily obtained by the apparatus of the regular curve $\gamma = \gamma(s)$.

Definition 3.5 *Let $\gamma = \gamma(s)$ be an unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving Frenet-Serret frame. Smarandache TNB curves are defined by*

$$\psi = \psi(s_\psi) = \frac{1}{\sqrt{3}}\left(T + N + B\right). \tag{18}$$

Remark 3.6 The Frenet-Serret invariants of Smarandache TNB curves can be easily obtained by the apparatus of the regular curve $\gamma = \gamma(s)$.

§4. Examples

Let us consider the following unit speed curve:

$$\begin{cases} \gamma_1 = \frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s \\ \gamma_2 = -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s \\ \gamma_3 = \frac{6}{65}\sin 10s \end{cases}. \tag{19}$$

It is rendered in Figure 1.

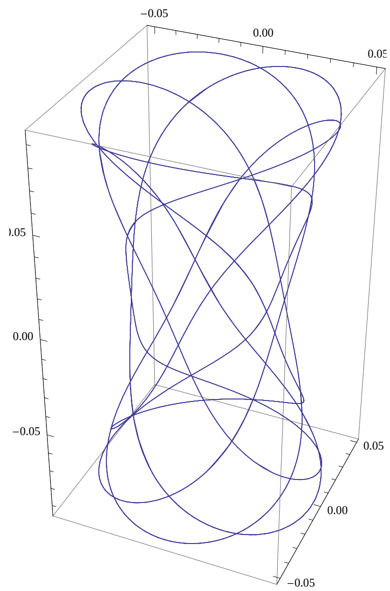

Figure 1: The Curve $\gamma = \gamma(s)$

And, this curve's natural equations are expressed as in [2]:

$$\begin{cases} \kappa(s) = -24 \sin 10s \\ \tau(s) = 24 \cos 10s \end{cases} \tag{20}$$

In terms of definitions, we obtain special Smarandache curves, see Figures $2-4$.

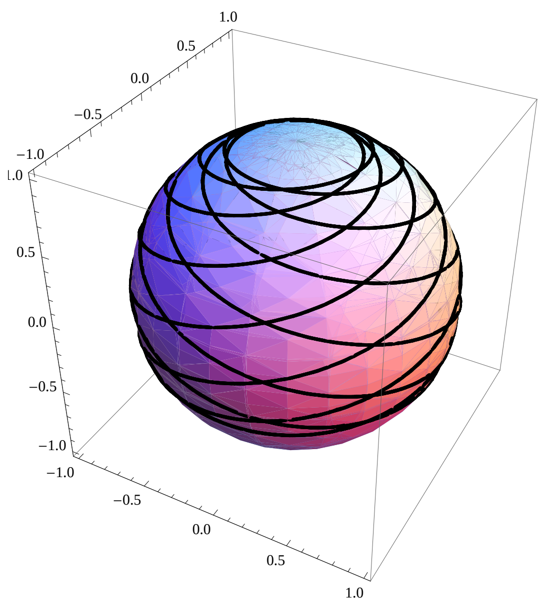

Figure 2: Smarandache TN Curves

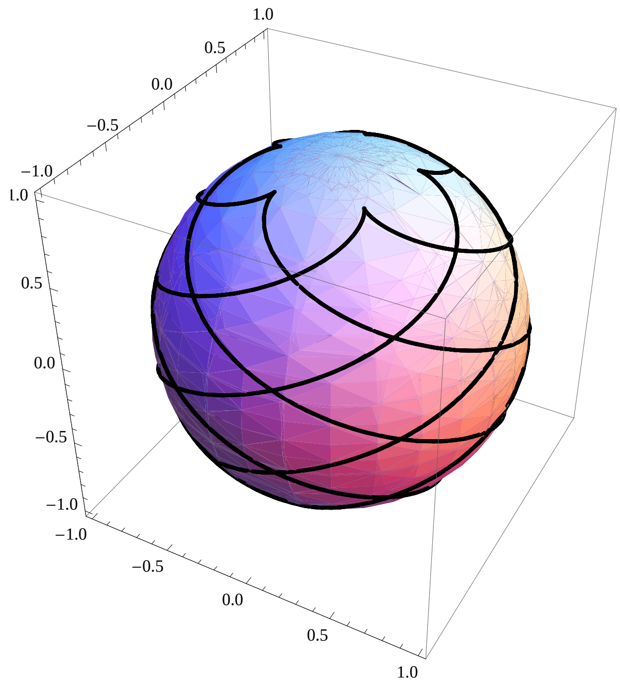

Figure 3: Smarandache NB Curves

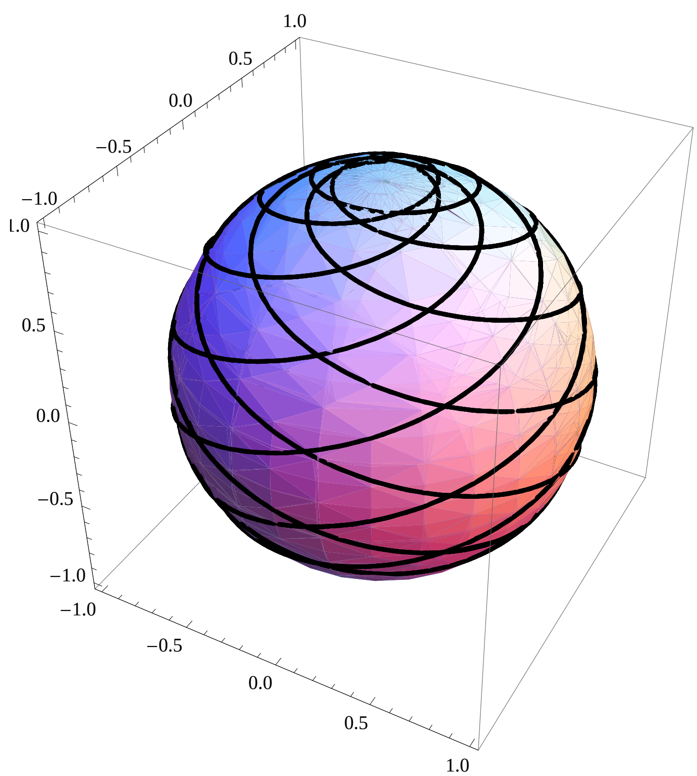

Figure 4: Smarandache TNB Curve

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