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SPECIAL SMARANDACHE CURVES IN $\mathbb{R}^3_1$

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Abstract. In this study, we determine **TN**-Smarandache curves whose position vector is composed by Frenet frame vectors of another regular curve in Minkowski 3-space $\mathbb{R}^3_1$. Then, we present some characterisations of Smarandache curves and calculate Frenet invariants of these curves. Moreover, we classify $\mathbf{TN}, \mathbf{TB}, \mathbf{NB}$ and **TNB**-Smarandache curves of a regular curve parametrized by arc length α by presenting a brief table with respect to the causal character of α. Also, we will give some examples related to results.

1. Introduction

In differential geometry, there are many important consequences and properties of curves studied by some authors [1, 2, 3]. Researchers always introduce some new curves by using the existing studies. Special Smarandache curves are one of them. Smarandache curve is defined as a regular curve whose position vector is composed by Frenet frame vectors of another regular curve in Minkowski spacetime in [4]. Special Smarandache curves have been studied by some authors [4, 5, 6]. M. Turgut and S. Yılmaz have identified a special case of such curves and called them $\mathbf{TB_2}$-Smarandache curves in the space $\mathbb{R}^4_1$ [4]. They have dealt with a special Smarandache curve which is defined by the tangent and second binormal vector fields. Besides, they have computed formulae of this kind of curves by the method expressed in [4]. A. T. Ali has introduced some special Smarandache curves in the Euclidean space [5]. Special Smarandache curves such as $\mathbf{TN}_1$, $\mathbf{TN}_2$, $\mathbf{N}_1\mathbf{N}_2$ and $\mathbf{TN}_1\mathbf{N}_2$-Smarandache curves according to Bishop frame in Euclidean 3-space have been investigated by Çetin and Tunçer [6]. Furthermore, they have studied differential geometric properties of these special curves and they have calculated the first and second curvature (natural curvatures) of these curves. Also, they have found the centres of the curvature spheres and osculating spheres of Smarandache curves.

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In addition, special Smarandache curves according to Darboux frame in Euclidean 3-space have been introduced in [7]. They have investigated special Smarandache curves such as $\mathbf{Tg}, \mathbf{Tn}, \mathbf{gn}$ and $\mathbf{Tgn}$-Smarandache curves. Furthermore, they have found some properties of these special curves and calculated normal curvature, geodesic curvature and geodesic torsion of these curves.

In this study, we firstly mention the fundamental properties and Frenet invariants of curves in $\mathbb{R}_1^3$. Then, we give the definition of $\mathbf{TN}, \mathbf{TB}, \mathbf{NB}$ and $\mathbf{TNB}$-Smarandache curves of a regular curve parametrized by arc length α in $\mathbb{R}_1^3$. In sections 2-4, we investigate and calculate Frenet invariants of $\mathbf{TN}$-Smarandache curves of a timelike, spacelike and null curves in $\mathbb{R}_1^3$. In section 5, we obtain the characterisations of $\mathbf{TN}, \mathbf{TB}, \mathbf{NB}$ and $\mathbf{TNB}$-Smarandache curves of a regular curve parametrized by arc length α by presenting a brief table. We classify general results of these Smarandache curves with respect to the causal character of α. Also, we present some examples.

2. Preliminaries

The Minkowski 3-space $\mathbb{R}_1^3$ is the real vector space provided with the standard flat metric given by

$$<,>= -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{R}_1^3$. An arbitrary vector $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}_1^3$ can have one of three Lorentzian causal characters; it can be spacelike if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ or $\mathbf{v} = \mathbf{0}$, timelike if $\langle \mathbf{v}, \mathbf{v} \rangle < 0$ and null (lightlike) if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ and $\mathbf{v} \neq \mathbf{0}$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ can be locally spacelike, timelike or null (lightlike) if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null (lightlike), respectively. We say that a timelike vector is future pointing or past pointing if the first component of the vector is positive or negative, respectively. Also, the vector product of any vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}_1^3$ is defined by

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2, x_1 y_3 - x_3 y_1, x_2 y_1 - x_1 y_2),$$

where

$$e_1 \times e_2 = -e_3,\ e_2 \times e_3 = e_1,\ e_3 \times e_1 = -e_2.$$

Let $\alpha = \alpha(s)$ be a regular curve parametrized by arc length in $\mathbb{R}_1^3$ and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}, \boldsymbol{\kappa}, \boldsymbol{\tau}\}$ be its Frenet invariants, where $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}, \kappa$ and τ are moving Frenet frame, curvature and torsion of $\alpha(s)$, respectively. The Frenet formulae of $\alpha = \alpha(s)$ are described with respect to the causal character of $\alpha'(s)$. For an arbitrary timelike curve $\alpha(s)$ in the Minkowski space $\mathbb{R}_1^3$, the following Frenet formulae are given as follows ([8]):

$$\begin{cases} \mathbf{T}' = \kappa\mathbf{N} \\ \mathbf{N}' = \kappa\mathbf{T} + \tau\mathbf{B} \\ \mathbf{B}' = -\tau\mathbf{N}, \end{cases} \tag{2.1}$$

where $\langle\mathbf{T},\mathbf{T}\rangle = -1$, $\langle\mathbf{N},\mathbf{N}\rangle = \langle\mathbf{B},\mathbf{B}\rangle = 1$, $\langle\mathbf{T},\mathbf{N}\rangle = \langle\mathbf{T},\mathbf{B}\rangle = \langle\mathbf{N},\mathbf{B}\rangle = 0$. Then, we write Frenet invariants in this way: $\mathbf{T}(s) = \alpha'(s)$, $\kappa(s) = ||\mathbf{T}'(s)||$, $\mathbf{N}(\mathbf{s}) = \mathbf{T}'(s)/\kappa(s)$, $\mathbf{B}(\mathbf{s}) = \mathbf{T}(\mathbf{s}) \times \mathbf{N}(\mathbf{s})$ and $\tau(s) = \langle\mathbf{N}(\mathbf{s})', \mathbf{B}(\mathbf{s})\rangle$.

For an arbitrary spacelike curve $\alpha(s)$ in the space $\mathbb{R}_1^3$, the Frenet formulae are given as below ([8]):

$$\begin{cases} \mathbf{T}' = \kappa\mathbf{N} \\ \mathbf{N}' = -\varepsilon\kappa\mathbf{T} + \tau\mathbf{B} \\ \mathbf{B}' = \tau\mathbf{N}, \end{cases} \tag{2.2}$$

where $\varepsilon = \pm 1$. If $\varepsilon = 1$, then $\alpha(s)$ is a spacelike curve with spacelike principal normal $\mathbf{N}$ and timelike binormal $\mathbf{B}$. Also, $\langle\mathbf{T},\mathbf{T}\rangle = \langle\mathbf{N},\mathbf{N}\rangle = 1, \langle\mathbf{B},\mathbf{B}\rangle = -1$ and $\langle\mathbf{T},\mathbf{N}\rangle = \langle\mathbf{T},\mathbf{B}\rangle = \langle\mathbf{N},\mathbf{B}\rangle = 0$. If $\varepsilon = -1$, then $\alpha(s)$ is a spacelike curve with timelike principal normal $\mathbf{N}$ and spacelike binormal $\mathbf{B}$. Besides, $\langle\mathbf{T},\mathbf{T}\rangle = \langle\mathbf{B},\mathbf{B}\rangle = 1, \langle\mathbf{N},\mathbf{N}\rangle = -1$ and $\langle\mathbf{T},\mathbf{N}\rangle = \langle\mathbf{T},\mathbf{B}\rangle = \langle\mathbf{N},\mathbf{B}\rangle = 0$. We define Frenet invariants for spacelike curves in this way: $\mathbf{T}(s) = \alpha'(s), \kappa(s) = \sqrt{\varepsilon < \mathbf{T}'(s), \mathbf{T}'(s) >}, \mathbf{N}(s) = \mathbf{T}'(s)/\kappa(s), \mathbf{B}(\mathbf{s}) = \mathbf{T}(\mathbf{s}) \times \mathbf{N}(\mathbf{s})$ and $\tau(s) = -\varepsilon < \mathbf{N}'(s), \mathbf{B}(s) >$.

In addition, the Frenet formulae for a null curve parametrized by distinguished parameter $\alpha(s)$ in $\mathbb{R}_1^3$, are given as follows:

$$\begin{cases} \mathbf{T}' = \kappa\mathbf{B} \\ \mathbf{N}' = -\tau\mathbf{B} \\ \mathbf{B}' = -\tau\mathbf{T} + \kappa\mathbf{N}, \end{cases} \tag{2.3}$$

where $\langle\mathbf{T},\mathbf{T}\rangle = \langle\mathbf{N},\mathbf{N}\rangle = \langle\mathbf{T},\mathbf{B}\rangle = \langle\mathbf{N},\mathbf{B}\rangle = 0, \langle\mathbf{B},\mathbf{B}\rangle = 1$, $\langle\mathbf{T},\mathbf{N}\rangle = 1$ and $\mathbf{T} \times \mathbf{B} = -\mathbf{T}$, $\mathbf{T} \times \mathbf{N} = -\mathbf{B}, \mathbf{B} \times \mathbf{N} = -\mathbf{N}$. In this state, $\mathbf{T}$ and $\mathbf{N}$ are null vectors and $\mathbf{B}$ is a spacelike vector. We describe Frenet invariants $\mathbf{T}(s) = \alpha'(s)$, $\kappa(s) = ||\mathbf{T}'(s)||, \mathbf{N}(s) = ((1/\kappa)(\tau\mathbf{T} + (1/\kappa)'\mathbf{T}' + (1/\kappa)\mathbf{T}''))(s), \mathbf{B}(\mathbf{s}) = ((1/\kappa)\mathbf{T}')(s)$ and $\tau(s) = (1/2)[(1/\kappa^3)||\mathbf{T}''||^2 - ((1/\kappa)')^2\kappa](s)$ ([9]).

Definition 2.1 ([4])**.** *Let $\alpha = \alpha(s)$ be a regular curve parametrized by arc length in $\mathbb{R}_1^3$ with its Frenet frame $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$. Then,* $\mathbf{TN}, \mathbf{TB}, \mathbf{NB}$ *and* $\mathbf{TNB}$*-Smarandache curves of α are defined, respectively, as follows:*

$$\beta_{TN} = \frac{1}{\sqrt{2}}(\mathbf{T}+\mathbf{N}),$$

$$\beta_{TB} = \frac{1}{\sqrt{2}}(\mathbf{T}+\mathbf{B}),$$

$$\beta_{NB} = \frac{1}{\sqrt{2}}(\mathbf{N}+\mathbf{B}),$$

$$\beta_{TNB} = \frac{1}{\sqrt{3}}\left(\mathbf{T} + \mathbf{N} + \mathbf{B}\right).$$

In the following sections, we investigate special **TN**-Smarandache curves for a given regular curve α in $\mathbb{R}_1^3$. We consider Frenet invariants $\{\mathbf{T}, \mathbf{N}, \mathbf{B}, \boldsymbol{\kappa}, \boldsymbol{\tau}\}$ and $\{\mathbf{T}^*, \mathbf{N}^*, \mathbf{B}^*, \boldsymbol{\kappa}^*, \boldsymbol{\tau}^*\}$ for α and its **TN**-Smarandache curve β_{TN}, respectively.

3. **TN**-Smarandache curves of a timelike curve in $\mathbb{R}_1^3$

Definition 3.1. *Let the curve $\alpha = \alpha(s)$ be a timelike curve parametrized by arc length in $\mathbb{R}_1^3$. Then,* **TN***-Smarandache curves of α can be determined by the Frenet vectors of α such as:*

$$\beta_{TN} = \frac{1}{\sqrt{2}}\left(\mathbf{T} + \mathbf{N}\right), \tag{3.1}$$

where $\mathbf{T}$ *is timelike,* $\mathbf{N}$ *and* $\mathbf{B}$ *are spacelike.*

Let us investigate the causal character of **TN**-Smarandache curve β_{TN}. By differentiating Equation (3.1) with respect to s and using Equation (2.1), we get

$$\beta_{TN}^{'} = \frac{d\beta_{TN}}{ds} = \frac{1}{\sqrt{2}}\left(\kappa\mathbf{T} + \kappa\mathbf{N} + \tau\mathbf{B}\right), \tag{3.2}$$

and

$$\langle\beta_{TN}^{'}, \beta_{TN}^{'}\rangle = \frac{\tau^2}{2}.$$

Therefore, there are two possibilities for the causal character of β_{TN} under the conditions $\tau \neq 0$ and $\tau = 0$. Because of the fact that $\frac{\tau^2}{2} \neq 1$ for at least one $\tau \in \mathbb{R}$, we know that s is not arc-length of β_{TN}. Assume that s^* is arc-length of β_{TN}.

Case 3.1. *If $\tau \neq 0$, then* **TN***-Smarandache curve β_{TN} is a spacelike curve.*

If we rearrange Equation (3.2), we obtain the tangent vector of β_{TN} as below:

$$\mathbf{T}^* = \frac{1}{|\tau|}\left(\kappa\mathbf{T} + \kappa\mathbf{N} + \tau\mathbf{B}\right), \tag{3.3}$$

where

$$\frac{ds^*}{ds} = \frac{|\tau|}{\sqrt{2}}. \tag{3.4}$$

i) Let β_{TN} be a spacelike curve with timelike binormal.

If we differentiate Equation (3.3) with respect to s, we obtain

$$\frac{d\mathbf{T}^*}{ds^*}\frac{ds^*}{ds} = \frac{1}{\tau^2}\left(\Gamma_1\mathbf{T}+\Gamma_2\mathbf{N} + \Gamma_3\mathbf{B}\right), \tag{3.5}$$

where

$$\begin{cases} \Gamma_1 = -\kappa^{'}\tau^{'} - \kappa^2\tau^{'} \\ \Gamma_2 = -\kappa^2\tau^{'} - \kappa^{'}\tau^{'} + \tau^2\tau^{'} \\ \Gamma_3 = -\kappa\tau\tau^{'} - \tau'^2. \end{cases}$$

Substituting Equation (3.4) with Equation (3.5), we gain

$$\dot{\mathbf{T}}^* = \frac{\sqrt{2}}{|\tau^3|}\left(\Gamma_1\mathbf{T}+\Gamma_2\mathbf{N}+\Gamma_3\mathbf{B}\right).$$

According to the definition of the Frenet invariants of a spacelike curve with timelike binormal, we can calculate the curvature and principal normal vector fields of β_{TN} such as:

$$\kappa^* = \frac{\sqrt{2}}{|\tau^3|}\sqrt{(-\Gamma_1^2+\Gamma_2^2+\Gamma_3^2)}$$

and

$$\mathbf{N}^* = \frac{1}{\sqrt{-\Gamma_1^2+\Gamma_2^2+\Gamma_3^2}}\left(\Gamma_1\mathbf{T}+\Gamma_2\mathbf{N}+\Gamma_3\mathbf{B}\right),$$

respectively. Furthermore, we have

$$\mathbf{B}^* = \mathbf{T}^*\times\mathbf{N}^* = \frac{1}{|\tau|\sqrt{-\Gamma_1^2+\Gamma_2^2+\Gamma_3^2}}\begin{vmatrix} \mathbf{T} & -\mathbf{N} & -\mathbf{B} \\ \kappa & \kappa & \tau \\ \Gamma_1 & \Gamma_2 & \Gamma_3 \end{vmatrix}.$$

Based upon this calculation, the binormal vector of β_{TN} is

$$\mathbf{B}^* = \frac{1}{|\tau|\sqrt{-\Gamma_1^2+\Gamma_2^2+\Gamma_3^2}}\left(\mu_1\mathbf{T}+\mu_2\mathbf{N}+\mu_3\mathbf{B}\right),$$

where

$$\begin{cases} \mu_1 = \kappa\Gamma_3-\tau\Gamma_2 \\ \mu_2 = -\tau\Gamma_1+\kappa\Gamma_3 \\ \mu_3 = -\kappa\Gamma_2+\kappa\Gamma_1. \end{cases}$$

So, the torsion of β_{TN} is given as below:

$$\tau^* = \frac{\begin{array}{l} -\kappa\Gamma_1^3\mu_2+\Gamma_2^3(\kappa\mu_1+\mu_3\tau)-\Gamma_3^2(\tau\Gamma_3\mu_2+\mu_1\Gamma_1^{'}+\mu_2\Gamma_2^{'}) \\ -\Gamma_2\Gamma_3(\Gamma_3(\kappa\mu_1-\mu_3\tau)+\mu_3\Gamma_2^{'}+\mu_2\Gamma_3^{'}) \\ -\Gamma_2^2(\tau\Gamma_3\mu_2+\mu_1\Gamma_1^{'}+\mu_3\Gamma_3^{'})+\Gamma_1^2(\tau\Gamma_3\mu_2+\Gamma_2(\kappa\mu_1-\mu_3\tau)-\mu_2\Gamma_2^{'}-\mu_3\Gamma_3^{'}) \\ +\Gamma_1(\kappa\Gamma_2^2\mu_2+\Gamma_2(\mu_2\Gamma_1^{'}+\mu_1\Gamma_2^{'}+\Gamma_3(\kappa\Gamma_3\mu_2+\mu_3\Gamma_1^{'}+\mu_1\Gamma_3^{'})) \end{array}}{|\tau|\left(-\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\right)^2}.$$

ii) Let β_{TN} be a spacelike curve with timelike principal normal, then the Frenet invariants of β_{TN} are given as below:

$$\begin{aligned}
\mathbf{T}^* &= \frac{1}{|\tau|}\left(\kappa\mathbf{T}+\kappa\mathbf{N}+\tau\mathbf{B}\right),\\
\mathbf{N}^* &= \frac{1}{\sqrt{\Gamma_1^2-\Gamma_2^2-\Gamma_3^2}}\left(\Gamma_1\mathbf{T}+\Gamma_2\mathbf{N}+\Gamma_3\mathbf{B}\right),\\
\mathbf{B}^* &= \frac{1}{|\tau|\sqrt{\Gamma_1^2-\Gamma_2^2-\Gamma_3^2}}\left(\mu_1\mathbf{T}+\mu_2\mathbf{N}+\mu_3\mathbf{B}\right),\\
\kappa^* &= \frac{\sqrt{2}}{|\tau^3|}\sqrt{(-\Gamma_1^2+\Gamma_2^2+\Gamma_3^2)},
\end{aligned}$$

$$\tau^*=\frac{\begin{array}{l}\kappa\Gamma_1^3\mu_2+\Gamma_2^3(\kappa\mu_1-\mu_3\tau)+\Gamma_3^2(\tau\Gamma_3\mu_2+\mu_1\Gamma_1^{'}+\mu_2\Gamma_2^{'})\\
+\Gamma_2\Gamma_3(\Gamma_3(\kappa\mu_1-\mu_3\tau)\\
+\mu_3\Gamma_2^{'}+\mu_2\Gamma_3^{'})+\Gamma_2^2(\tau\Gamma_3\mu_2+\mu_1\Gamma_1^{'}-\mu_3\Gamma_3^{'})\\
+\Gamma_1^2(-\tau\Gamma_3\mu_2+\Gamma_2(-\kappa\mu_1-\mu_3\tau)+\mu_2\Gamma_2^{'}-\mu_3\Gamma_3^{'})-\\
\Gamma_1(\kappa\Gamma_2^2\mu_2+\Gamma_2(\mu_2\Gamma_1^{'}+\mu_1\Gamma_2^{'}+\Gamma_3(\kappa\Gamma_3\mu_2+\mu_3\Gamma_1^{'}+\mu_1\Gamma_3^{'}))\end{array}}{|\tau|\left(-\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\right)^{2}}.$$

Case 3.2. *If $\tau=0$, then* **TN**-*Smarandache curve β_{TN} is a null curve.*

If we differentiate Equation (3.1) with respect to s, the tangent vector of β_{TN} can be written namely:

$$\mathbf{T}^*=\frac{1}{\sqrt{2}}\frac{ds}{ds^*}\kappa(\mathbf{T}+\mathbf{N}). \tag{3.6}$$

By differentiating Equation (3.6), we obtain

$$\dot{\mathbf{T}}^*=\frac{1}{\sqrt{2}}\Gamma^*(\mathbf{T}+\mathbf{N}),$$

where

$$\Gamma^*=\left(\frac{d^2s}{ds^{*2}}\kappa+\left(\frac{ds}{ds^*}\right)^2\left(\kappa'+\kappa^2\right)\right).$$

From the calculations of Frenet invariants of a null curve, we find that the curvature κ^* of β_{TN} is equal to zero. So, there is no calculation for $\mathbf{N}^*$ and $\mathbf{B}^*$ in this case.

4. TN-Smarandache curves of a spacelike curve in $\mathbb{R}_1^3$

4.1. TN-Smarandache curves of a spacelike curve with timelike binormal.

Definition 4.1. *Let the curve $\alpha=\alpha(s)$ be a spacelike curve parametrized by arc length with timelike binormal in $\mathbb{R}_1^3$. Then,* **TN**-*Smarandache curves of α can be defined by the frame vectors of α such as:*

$$\beta_{TN}=\frac{1}{\sqrt{2}}\left(\mathbf{T}+\mathbf{N}\right), \tag{4.1}$$

where **B** *is timelike,* **N** *and* **T** *are spacelike.*

Let us analyze the causal character of β_{TN}. By differentiating Equation (4.1) with respect to s, we have

$$\beta_{TN}^{'}=\frac{1}{\sqrt{2}}\left(-\kappa\mathbf{T}+\kappa\mathbf{N}+\tau\mathbf{B}\right),$$

and

$$\langle\beta_{TN}^{'},\beta_{TN}^{'}\rangle=\frac{2\kappa^2-\tau^2}{2}.$$

In this situation, there are three possibilities for the causal character of β_{TN} under the circumstances $2\kappa^2-\tau^2<0,\ 2\kappa^2-\tau^2>0$ and $\sqrt{2}|\kappa|=|\tau|$. Because of the fact that $\frac{2\kappa^2-\tau^2}{2}\neq 1$ for at least one κ or τ, we know that s is not arc-length of β_{TN}. Suppose that s^* is arc-length of β_{TN}.

Case 4.1. *If $2\kappa^2-\tau^2<0$, then* **TN**-*Smarandache curve β_{TN} is a timelike curve.*

If we differentiate Equation (4.1) with respect to s, we obtain the tangent vector of β_{TN} as given below:

$$\mathbf{T}^*=\frac{1}{\sqrt{|2\kappa^2-\tau^2|}}\left(-\kappa\mathbf{T}+\kappa\mathbf{N}+\tau\mathbf{B}\right), \tag{4.2}$$

where

$$\frac{ds^*}{ds}=\frac{\sqrt{|2\kappa^2-\tau^2|}}{\sqrt{2}}. \tag{4.3}$$

By differentiating Equation (4.2) with respect to s, we have

$$\frac{d\mathbf{T}^*}{ds^*}\frac{ds^*}{ds}=\frac{1}{\sqrt{|2\kappa^2-\tau^2|^3}}\left(\Lambda_1\mathbf{T}+\Lambda_2\mathbf{N}+\Lambda_3\mathbf{B}\right), \tag{4.4}$$

where

$$\begin{cases}\Lambda_1=-4\kappa^2\kappa'+2\kappa\tau\tau'-\left|2\kappa^2-\tau^2\right|(\kappa^2+\kappa')\\ \Lambda_2=4\kappa^2\kappa'-2\kappa\tau\tau'-\left|2\kappa^2-\tau^2\right|(\kappa^2+\kappa'+\tau^2)\\ \Lambda_3=4\kappa\kappa'\tau-2\tau^2\tau'-\left|2\kappa^2-\tau^2\right|(\kappa\tau+\tau').\end{cases}$$

Substituting Equation (4.3) with Equation (4.4), we gain

$$\dot{\mathbf{T}}^*=\frac{\sqrt{2}}{(2\kappa^2-\tau^2)^2}\left(\Lambda_1\mathbf{T}+\Lambda_2\mathbf{N}+\Lambda_3\mathbf{B}\right).$$

By using the definition of Frenet invariants of a timelike curve, the curvature and principal normal vector fields of β_{TN} are

$$\kappa^*=\left\|\dot{\mathbf{T}}^*\right\|=\frac{\sqrt{2}}{(2\kappa^2-\tau^2)^2}\sqrt{(\Lambda_1^2+\Lambda_2^2-\Lambda_3^2)}$$

and

$$\mathbf{N}^*=\frac{1}{\sqrt{\Lambda_1^2+\Lambda_2^2-\Lambda_3^2}}\left(\Lambda_1\mathbf{T}+\Lambda_2\mathbf{N}+\Lambda_3\mathbf{B}\right),$$

respectively. Additionally, we gain

$$\mathbf{B}^* = \mathbf{T}^* \times \mathbf{N}^* = \frac{\sqrt{2}}{\sqrt{|2\kappa^2 - \tau^2|}\sqrt{\Lambda_1^2+\Lambda_2^2-\Lambda_3^2}} \begin{vmatrix} \mathbf{T} & -\mathbf{N} & -\mathbf{B} \\ -\kappa & \kappa & \tau \\ \Lambda_1 & \Lambda_2 & \Lambda_3 \end{vmatrix}.$$

From this calculation, the binormal vector of β_{TN} is

$$\mathbf{B}^* = \frac{\sqrt{2}}{\sqrt{|2\kappa^2 - \tau^2|}\sqrt{\Lambda_1^2+\Lambda_2^2-\Lambda_3^2}} \left(\eta_1\mathbf{T}+\eta_2\mathbf{N}+\eta_3\mathbf{B}\right),$$

where

$$\begin{cases} \eta_1 = \kappa\Lambda_3 - \tau\Lambda_2 \\ \eta_2 = -\kappa\Lambda_3 - \tau\Lambda_1 \\ \eta_3 = \kappa(\Lambda_1+\Lambda_2). \end{cases}$$

So, the torsion of β_{TN} is given by:

$$\tau^* = \frac{\begin{array}{l} -\kappa\Lambda_1^3\eta_2+\Lambda_2^3(-\kappa\eta_1+\eta_3\tau) - \Lambda_3^2(\tau\Lambda_3\eta_2+\eta_1\Lambda_1^{'}+\eta_2\Lambda_2^{'}-3\eta_3\Lambda_3^{'}) \\ +\Lambda_2\,\Lambda_3(\Lambda_3(\kappa\eta_1+\eta_3\tau)-2\eta_3\Lambda_2^{'}-2\eta_2\Lambda_3^{'}) \\ -\Lambda_2^2(\tau\Lambda_3\eta_2+\eta_1\Lambda_1^{'}-\eta_3\Lambda_3^{'}+2\eta_2\Lambda_2^{'})+\Lambda_1^2(\tau\Lambda_3\eta_2 \\ +\Lambda_2(-\kappa\eta_1+\eta_3\tau)+\eta_2\Lambda_2^{'}+\eta_1\Lambda_1^{'}) \\ +\Lambda_1(\kappa\Lambda_2^2\eta_2-\kappa\Lambda_3^2\eta_2+2\eta_2\Lambda_1^{'}+2\Lambda_2\eta_1\eta_2^{'}-\Lambda_3(\eta_3\Lambda_1^{'}+2\eta_1\Lambda_3^{'})) \end{array}}{|2\kappa^2 - \tau^2|\,(\Lambda_1^2+\Lambda_2^2-\Lambda_3^2)^2}.$$

Case 4.2. *If* $2\kappa^2 - \tau^2 > 0$, *then* $\mathbf{TN}$*-Smarandache curve* β_{TN} *is a spacelike curve.*

i) Let β_{TN} be a spacelike curve with timelike binormal, then the Frenet invariants of β_{TN} are given as below:

$\mathbf{T}^* = \frac{1}{\sqrt{2\kappa^2 - \tau^2}}\left(-\kappa\mathbf{T}+\kappa\mathbf{N}+\tau\mathbf{B}\right),$

$\mathbf{N}^* = \frac{1}{\sqrt{\Lambda_1^2+\Lambda_2^2-\Lambda_3^2}}\left(\Lambda_1\mathbf{T}+\Lambda_2\mathbf{N}+\Lambda_3\mathbf{B}\right),$

$\mathbf{B}^* = \frac{\sqrt{2}}{\sqrt{2\kappa^2 - \tau^2}\sqrt{\Lambda_1^2+\Lambda_2^2-\Lambda_3^2}}\left(\eta_1\mathbf{T}+\eta_2\mathbf{N}+\eta_3\mathbf{B}\right),$

$\kappa^* = \frac{\sqrt{2}}{(2\kappa^2 - \tau^2)^2}\sqrt{(\Lambda_1^2+\Lambda_2^2-\Lambda_3^2)},$

$$\tau^* = -\frac{\begin{array}{l} -\kappa\Lambda_1^3\eta_2+\Lambda_2^3(-\kappa\eta_1+\eta_3\tau) - \Lambda_3^2(\tau\Lambda_3\eta_2+\eta_1\Lambda_1^{'}+\eta_2\Lambda_2^{'}-3\eta_3\Lambda_3^{'}) \\ +\Lambda_2\,\Lambda_3(\Lambda_3(\kappa\eta_1+\eta_3\tau)-2\eta_3\Lambda_2^{'}-2\eta_2\Lambda_3^{'})- \\ \Lambda_2^2(\tau\Lambda_3\eta_2+\eta_1\Lambda_1^{'}-\eta_3\Lambda_3^{'}+2\eta_2\Lambda_2^{'})+\Lambda_1^2(\tau\Lambda_3\eta_2 \\ +\Lambda_2(-\kappa\eta_1+\eta_3\tau)+\eta_2\Lambda_2^{'}+\eta_1\Lambda_1^{'}) \\ +\Lambda_1(\kappa\Lambda_2^2\eta_2-\kappa\Lambda_3^2\eta_2+2\eta_2\Lambda_1^{'}+2\Lambda_2\eta_1\eta_2^{'}-\Lambda_3(\eta_3\Lambda_1^{'}+2\eta_1\Lambda_3^{'})) \end{array}}{|2\kappa^2 - \tau^2|\left(\Lambda_1^2+\Lambda_2^2-\Lambda_3^2\right)^2}.$$

ii) Let β_{TN} be a spacelike curve with timelike principal normal, then the Frenet invariants of β_{TN} are namely:

$$\mathbf{T}^* = \frac{1}{\sqrt{2\kappa^2 \ -\tau^2}} \left(-\kappa \mathbf{T} + \kappa \mathbf{N} + \tau \mathbf{B}\right),$$
$$\mathbf{N}^* = \frac{1}{\sqrt{-\Lambda_1^2 - \Lambda_2^2 + \Lambda_3^2}} \left(\Lambda_1 \mathbf{T} + \Lambda_2 \mathbf{N} + \Lambda_3 \mathbf{B}\right),$$
$$\mathbf{B}^* = \frac{\sqrt{2}}{\sqrt{2\kappa^2 \ -\tau^2}\sqrt{-\Lambda_1^2 - \Lambda_2^2 + \Lambda_3^2}} \left(\eta_1 \mathbf{T} + \eta_2 \mathbf{N} + \eta_3 \mathbf{B}\right),$$
$$\kappa^* = \frac{\sqrt{2}}{(2\kappa^2 \ -\tau^2)^2} \sqrt{(-\Lambda_1^2 - \Lambda_2^2 + \Lambda_3^2)},$$
$$\tau^* = \frac{\begin{array}{l} -\kappa\Lambda_1^3\eta_2 + \Lambda_2^3(-\kappa\eta_1 + \eta_3\tau) - \Lambda_3^2(\tau\Lambda_3\eta_2 + \eta_1\Lambda_1' + \eta_2\Lambda_2' - 3\eta_3\Lambda_3') \\ +\Lambda_2\ \Lambda_3(\Lambda_3(\kappa\eta_1 + \eta_3\tau) - 2\eta_3\Lambda_2' - 2\eta_2\Lambda_3') - \\ \Lambda_2^2(\tau\Lambda_3\eta_2 + \eta_1\Lambda_1' - \eta_3\Lambda_3' + 2\eta_2\Lambda_2') + \Lambda_1^2(\tau\Lambda_3\eta_2 \\ +\Lambda_2(-\kappa\eta_1 + \eta_3\tau) + \eta_2\Lambda_2' + \eta_1\Lambda_1') \\ +\Lambda_1(\kappa\Lambda_2^2\eta_2 - \kappa\Lambda_3^2\eta_2 + 2\eta_2\Lambda_1' + 2\Lambda_2\eta_1\eta_2' - \Lambda_3(\eta_3\Lambda_1' + 2\eta_1\Lambda_3')) \end{array}}{|2\kappa^2 \ -\tau^2|\left(\Lambda_1^2 + \Lambda_2^2 - \Lambda_3^2\right)^2}.$$

Case 4.3. *If $\sqrt{2}|\kappa| = |\tau|$, then* **TN**-*Smarandache curve β_{TN} is a null curve.*

The tangent vector of β_{TN} and its derivative with respect to s^* are obtained, respectively,

$$\begin{cases} \mathbf{T}^* &= \frac{1}{\sqrt{2}} \frac{ds}{ds^*}(-\kappa \mathbf{T} + \kappa \mathbf{N} + \tau \mathbf{B}) \\ \dot{\mathbf{T}}^* &= \frac{1}{\sqrt{2}}(\Lambda_1^* \mathbf{T} + \Lambda_2^* \mathbf{N} + \Lambda_3^* \mathbf{B}), \end{cases}$$

where

$$\begin{cases} \Lambda_1^* = \frac{d^2s}{ds^{*2}}(-\kappa) + \left(\frac{ds}{ds^*}\right)^2 \left(-\kappa' - \kappa^2\right) \\ \Lambda_2^* = \frac{d^2s}{ds^{*2}}(\kappa) + \left(\frac{ds}{ds^*}\right)^2 \left(\kappa' - \kappa^2 + \tau^2\right) \\ \Lambda_3^* = \frac{d^2s}{ds^{*2}}(\tau) + \left(\frac{ds}{ds^*}\right)^2 \left(\tau' + \kappa\tau\right). \end{cases}$$

Based upon these calculations, the Frenet invariants of β_{TN} are namely:

$$\mathbf{N}^* = \frac{\begin{array}{l} ((\Lambda_3^*)^2 + 2\Lambda_1^*\Lambda_2^*)(-\sqrt{2}\kappa\tau\frac{ds}{ds^*} + \eta_1^*) \\ \quad -[\Lambda_1^*\Lambda_3^*(\Lambda_3^*)' + (\Lambda_1^*)^2(\Lambda_2^*)' + (\Lambda_1^*)'\Lambda_2^*\Lambda_1^*])\mathbf{T} \\ +((\Lambda_3^*)^2 + 2\Lambda_1^*\Lambda_2^*)(\sqrt{2}\kappa\tau\frac{ds}{ds^*} + \eta_2^*) \\ \quad -[\Lambda_2^*\Lambda_3^*(\Lambda_3^*)' + (\Lambda_1^*)'(\Lambda_2^*)^2 + (\Lambda_2^*)'\Lambda_2^*\Lambda_1^*])\mathbf{N} \\ +((\Lambda_3^*)^2 + 2\Lambda_1^*\Lambda_2^*)(\sqrt{2}\tau^2\frac{ds}{ds^*} + \eta_3^*) \\ \quad -[(\Lambda_3^*)^2(\Lambda_3^*)' + (\Lambda_1^*)'\Lambda_2^* + \Lambda_3^*\Lambda_2^*\Lambda_1^* + \Lambda_3^*])\mathbf{B} \end{array}}{\left((\Lambda_3^*)^2 + 2\Lambda_1^*\Lambda_2^*\right)^{\frac{3}{2}}},$$

$$\mathbf{B}^* = \frac{1}{\sqrt{(\Lambda_3^*)^2 + 2\Lambda_1^*\Lambda_2^*}} \Lambda_1^* \mathbf{T} + \Lambda_2^* \mathbf{N} + \Lambda_3^* \mathbf{B},$$

$$\kappa^* = \frac{1}{\sqrt{2}} \sqrt{(\Lambda_3^*)^2 + 2\Lambda_1^*\Lambda_2^*},$$

$$\tau^* = \frac{((\Lambda_3^*)^2 + 2\Lambda_1^*\Lambda_2^*)^3(\eta_3^* + 2\eta_1^*\eta_2^*) - 4(\Lambda_3^*(\Lambda_3^*)' + \Lambda_1^*(\Lambda_2^*)' + (\Lambda_1^*)'\Lambda_2^*)^2}{8\sqrt{2}((\Lambda_1^*)^2 + 2\Lambda_1^*\Lambda_2^*)^{\frac{3}{2}}}.$$

4.2. TN-Smarandache curves of a spacelike curve with timelike principal normal.

Definition 4.2. *Let the curve $\alpha = \alpha(s)$ be a spacelike curve parametrized by arc length with timelike principal normal in $\mathbb{R}^3_1$. Then,* **TN**-*Smarandache curves of α can be defined by the Frenet vectors of α such as:*

$$\beta_{TN} = \frac{1}{\sqrt{2}}(\mathbf{T}+\mathbf{N}), \tag{4.5}$$

where $\mathbf{N}$ *is timelike,* $\mathbf{T}$ *and* $\mathbf{B}$ *are spacelike.*

Let us search the causal character of β_{TN}. If we differentiate Equation (4.5) with respect to s, we get

$$\beta'_{TN} = \frac{1}{\sqrt{2}}(\kappa\mathbf{T}+\kappa\mathbf{N}+\tau\mathbf{B}).$$

Then, we can obtain

$$\langle \beta'_{TN}, \beta'_{TN} \rangle = \frac{\tau^2}{2}.$$

Hence, there are two possibilities for the causal character of β_{TN} under the conditions $\tau \neq 0$ and $\tau = 0$. Because of the fact that $\frac{\tau^2}{2} \neq 1$ for at least one $\tau \in \mathbb{R}$, we suppose that s^* is arc-length of β_{TN}.

Case 4.4. *If $\tau \neq 0$, then* **TN**-*Smarandache curve β_{TN} is a spacelike curve.*

i) Let β_{TN} be a spacelike curve with timelike binormal, then the Frenet invariants of β_{TN} are given as follows:

$$\begin{aligned}
\mathbf{T}^* &= \frac{1}{|\tau|}(\kappa\mathbf{T}+\kappa\mathbf{N}+\tau\mathbf{B}),\\
\mathbf{N}^* &= \frac{1}{\sqrt{\lambda_1^2-\lambda_2^2+\lambda_3^2}}(\lambda_1\mathbf{T}+\lambda_2\mathbf{N}+\lambda_3\mathbf{B}),\\
\mathbf{B}^* &= \frac{\sqrt{2}}{|\tau|\sqrt{\lambda_1^2-\lambda_2^2+\lambda_3^2}}(\omega_1\mathbf{T}+\omega_2\mathbf{N}+\omega_3\mathbf{B}),\\
\kappa^* &= \frac{\sqrt{2}}{|\tau^3|}\sqrt{\left(\lambda_1^2-\lambda_2^2+\lambda_3^2\right)},
\end{aligned}$$

$$\tau^* = -\frac{\begin{array}{l}-\kappa\lambda_1^3\mu_2+\lambda_2^3(\kappa\omega_1+\mu_3\tau)+\lambda_3^2(\tau\lambda_3\omega_2-\omega_1\lambda_1^{'}+\omega_2\lambda_2^{'})\\ -\lambda_2\lambda_3(\lambda_3(\kappa\omega_1+\omega_3\tau)+\omega_3\lambda_2^{'}+\omega_2\lambda_3^{'})+\\ \lambda_2^2(-\tau\lambda_3\omega_2+\omega_1\lambda_1^{'}+\omega_3\lambda_3^{'})+\lambda_1^2(-\tau\lambda_3\omega_2\\ +\lambda_2(\kappa\omega_1+\omega_3\tau)-\omega_2\lambda_2^{'}+\omega_3\lambda_3^{'})\\ +\lambda_1(-\kappa\lambda_2^2\mu_2+\lambda_2(\mu_2\lambda_1^{'}-\omega_1\lambda_2^{'}+\lambda_3(\kappa\lambda_3\omega_2-\omega_3\lambda_1^{'}+\omega_1\lambda_3^{'}))\end{array}}{|\tau|\left(\lambda_1^2-\lambda_2^2+\lambda_3^2\right)^{2}},$$

where
$$\begin{cases} \lambda_1 = -\kappa\tau' + \kappa'|\tau| + \kappa^2|\tau| \\ \lambda_2 = -\kappa\tau' + \kappa^{'}|\tau| + \kappa^2|\tau| + \tau^2|\tau| \\ \lambda_3 = -\tau\tau' + \kappa\tau|\tau| + \tau'|\tau| \end{cases}$$
and
$$\begin{cases} \omega_1 = \kappa\lambda_3 - \tau\lambda_2 \\ \omega_2 = -\tau\lambda_1 + \kappa\lambda_3 \\ \omega_3 = \kappa\lambda_1 - \kappa\lambda_2. \end{cases}$$

ii) Let β_{TN} be a spacelike curve with timelike principal normal, then the Frenet invariants of β_{TN} are as follows:

$$\begin{aligned}
\mathbf{T}^* &= \frac{1}{|\tau|}\left(\kappa\mathbf{T} + \kappa\mathbf{N} + \tau\mathbf{B}\right), \\
\mathbf{N}^* &= \frac{\sqrt{1}}{\sqrt{-\lambda_1^2 + \lambda_2^2 - \lambda_3^2}}\left(\lambda_1\mathbf{T} + \lambda_2\mathbf{N} + \lambda_3\mathbf{B}\right), \\
\mathbf{B}^* &= \frac{\sqrt{2}}{|\tau|\sqrt{-\lambda_1^2 + \lambda_2^2 - \lambda_3^2}}\left(\omega_1\mathbf{T} + \omega_2\mathbf{N} + \omega_3\mathbf{B}\right), \\
\kappa^* &= \frac{\sqrt{2}}{|\tau^3|}\left(-\lambda_1^2 + \lambda_2^2 - \lambda_3^2\right), \\
\tau^* &= \frac{\begin{array}{l} \kappa\lambda_1^3\omega_2 - \lambda_2^3(\kappa\omega_1 + \omega_3\tau) - \lambda_3^2(\tau\lambda_3\omega_2 - \omega_1\lambda_1^{'} + \omega_2\lambda_2^{'}) \\ +\lambda_2\lambda_3(\lambda_3(\kappa\omega_1 + \omega_3\tau) + \omega_3\lambda_2^{'} + \omega_2\lambda_3^{'}) + \\ \lambda_2^2(\tau\lambda_3\omega_2 - \omega_1\lambda_1^{'} - \omega_3\lambda_3^{'}) - \lambda_1^2(-\tau\lambda_3\omega_2 \\ +\lambda_2(\kappa\omega_1 + \omega_3\tau) - \omega_2\lambda_2^{'} + \omega_3\lambda_3^{'}) \\ +\lambda_1(\kappa\lambda_2^2\omega_2 + \lambda_2(-\omega_2\lambda_1^{'} + \omega_1\lambda_2^{'} - \lambda_3(\kappa\lambda_3\omega_2 - \omega_3\lambda_1^{'} + \omega_1\lambda_3^{'})) \end{array}}{|\tau|\left(\lambda_1^2 - \lambda_2^2 + \lambda_3^2\right)^2}.
\end{aligned}$$

Case 4.5. *If $\tau = 0$, then* **TN***-Smarandache curve β_{TN} is a null curve.*

If we differentiate Equation (4.5) with respect to s, we obtain the tangent vector of β_{TN} as below:
$$\mathbf{T}^* = \frac{1}{\sqrt{2}}\frac{ds}{ds^*}\kappa(\mathbf{T} + \mathbf{N}). \tag{4.6}$$
Then, we get
$$\dot{\mathbf{T}^*} = \frac{1}{\sqrt{2}}\frac{ds}{ds*}\lambda^*(\mathbf{T} + \mathbf{N}),$$
where
$$\lambda^* = \left(\frac{d^2s}{ds^{*2}}\kappa + \left(\frac{ds}{ds^*}\right)^2\left(\kappa' + \kappa^2\right)\right)$$

From the calculations of Frenet invariants of a null curve, the curvature κ^* of β_{TN} is equal to zero. So, there is no calculation for $\mathbf{N}^*$ and $\mathbf{B}^*$ in this case.

5. TN-Smarandache curves of a null curve in $\mathbb{R}_1^3$

Definition 5.1. *Let the curve $\alpha = \alpha(s)$ be a null curve parametrized by pseudo arc length in $\mathbb{R}_1^3$. Then,* **TN**-*Smarandache curves of α can be defined by the Frenet vectors of α such as:*

$$\beta_{TN} = \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{N}), \tag{5.1}$$

where $\mathbf{T}$ *and* $\mathbf{N}$ *are null and* $\mathbf{B}$ *is spacelike.*

Let us investigate the causal character of β_{TN}. By differentiating Equation (5.1) with respect to s, we have

$$\beta'_{TN} = \frac{d\beta_{TN}}{ds} = \frac{1}{\sqrt{2}}(\kappa - \tau)\mathbf{B},$$

and

$$\langle \beta'_{TN}, \beta'_{TN} \rangle = \frac{(\kappa - \tau)^2}{2}.$$

In this state, there are two possibilities for the causal character of β_{TN} under the condition $\kappa \neq \tau$ and $\kappa = \tau$. Since $\frac{(\kappa-\tau)^2}{2} \neq 1$, for at least one κ or τ, we know that s is not arc-length of β_{TN}. Let s^* be arc-length of β_{TN}.

Case 5.1. *If $\kappa \neq \tau$, then* **TN**-*Smarandache curve β_{TN} is a spacelike curve.*

If we differentiate Equation (5.1) with respect to s, we obtain the tangent vector of β_{TN} as follows:

$$\mathbf{T}^* = \frac{1}{\sqrt{2}} \frac{ds}{ds*}(\kappa - \tau)\mathbf{B}. \tag{5.2}$$

i) Let β_{TN} be a spacelike curve with timelike binormal.

If we differentiate Equation (5.2) with respect to s, we obtain

$$\dot{\mathbf{T}}^* = \frac{1}{\sqrt{2}}(\delta_1\mathbf{T} + \delta_2\mathbf{N} + \delta_3\mathbf{B}),$$

where

$$\begin{cases} \delta_1 = \left(\frac{ds}{ds^*}\right)^2(-\kappa\tau + \tau^2) \\ \delta_2 = \left(\frac{ds}{ds^*}\right)^2(\kappa^2 - \tau\kappa) \\ \delta_3 = \frac{d^2s}{ds^{*2}} + (\kappa - \tau)\left(\frac{ds}{ds^*}\right)^2(\kappa' + \tau'). \end{cases}$$

Then, Frenet invariants of β_{TN} are as follows:

$$\begin{aligned}
\mathbf{N}^* &= \frac{\delta_1\mathbf{T}+\delta_2\mathbf{N}+\delta_3\mathbf{B}}{\sqrt{\delta_3^2+\delta_1\delta_2}}, \\
\mathbf{B}^* &= \frac{1}{\sqrt{2}}\frac{ds}{ds*}\frac{1}{\sqrt{\delta_3^2+\delta_1\delta_2}}(\kappa-\tau)(-\delta_2\mathbf{T}+\delta_1\mathbf{N}), \\
\boldsymbol{\kappa}^* &= \frac{1}{\sqrt{2}}\sqrt{(\delta_3^2+\delta_1\delta_2)},
\end{aligned}$$

$$\begin{aligned}
\tau^* = -\frac{1}{\sqrt{2}}\frac{ds}{ds*}(\kappa-\tau)&\left[\frac{\delta_1^2-\delta_2^2}{2\left(\delta_3^2+\delta_1\delta_2\right)^2}\left(2\delta_3\delta_3'+\delta_1'\delta_2+\delta_1\delta_2'\right)\right. \\
&\left.+\left[\frac{1}{\left(\delta_3^2+\delta_1\delta_2\right)}\left(\,-\delta_2\delta_2'-\delta_2\delta_3\kappa+\delta_1'-\delta_3\tau\right)\right]\right]
\end{aligned}$$

ii) Let β_{TN} be a spacelike curve with timelike principal normal.

Then, Frenet invariants of β_{TN} are given below:

$$\begin{aligned}
\mathbf{T}^* &= \frac{1}{\sqrt{2}}\frac{ds}{ds*}(\kappa-\tau)\mathbf{B}, \\
\mathbf{N}^* &= \frac{\delta_1\mathbf{T}+\delta_2\mathbf{N}+\delta_3\mathbf{B}}{\sqrt{-\delta_3^2-\delta_1\delta_2}}, \\
\mathbf{B}^* &= \frac{1}{\sqrt{2}}\frac{ds}{ds*}\frac{1}{\sqrt{-\delta_3^2-\delta_1\delta_2}}(\kappa-\tau)(-\delta_2\mathbf{T}+\delta_1\mathbf{N}), \\
\boldsymbol{\kappa}^* &= \frac{1}{\sqrt{2}}\sqrt{(-\delta_3^2-\delta_1\delta_2)},
\end{aligned}$$

$$\begin{aligned}
\tau^* \quad = \quad -\frac{1}{\sqrt{2}}\frac{ds}{ds*}(\kappa-\tau)&\left[\frac{\delta_1^2-\delta_2^2}{2\left(\delta_3^2+\delta_1\delta_2\right)^2}\left(2\delta_3\delta_3'+\delta_1'\delta_2+\delta_1\delta_2'\right)\right. \\
&\left.+\frac{1}{\left(\delta_3^2+\delta_1\delta_2\right)}\left(\,\delta_2\delta_2'+\delta_2\delta_3\kappa-\delta_1'+\delta_3\tau\right)\right]
\end{aligned}$$

Case 5.2. *If $\kappa=\tau$, then* **TN**-*Smarandache curve β_{TN} is a null curve. In this case, there is no calculation of Frenet invariants of β_{TN}.*

6. General results and examples

In this section, we will obtain the characterisations of **TN**, **TB**, **NB** and **TNB**-Smarandache curves of a regular curve parametrized by arc length α in a brief table given in the following without calculations. The calculations can be done by using the similar way giving in Section 2-5. We will classify general results of these Smarandache curves with respect to the causal character of α. Also, we will give some examples related to Table 6.1.

TABLE 6.1. The characterisations of Smarandache curves in Minkowski Space

The casual charater of β				
The casual character of α	β_{TN}	β_{TB}	β_{NB}	β_{TNB}
Timelike	$\tau \neq 0 \Rightarrow$ spacelike $\tau = 0 \Rightarrow$ null	$\kappa \neq \tau \Rightarrow$ spacelike $\kappa = \tau \Rightarrow$ null	$2\kappa^2 - \tau^2 < 0$ $\Rightarrow$ timelike $2\kappa^2 - \tau^2 > 0$ $\Rightarrow$ spacelike $2\kappa^2 - \tau^2 = 0$ $\Rightarrow$ null	$\tau^2 - \kappa\tau < 0$ $\Rightarrow$ timelike $\tau^2 - \kappa\tau > 0$ $\Rightarrow$ spacelike $\tau^2 - \kappa\tau = 0$ $\Rightarrow$ null
Spacelike with timelike binormal	$2\kappa^2 - \tau^2 < 0$ $\Rightarrow$ timelike $2\kappa^2 - \tau^2 > 0 \Rightarrow$ spacelike $2\kappa^2 - \tau^2 = 0$ $\Rightarrow$ null	$\kappa \neq -\tau \Rightarrow$ spacelike $\kappa = -\tau \Rightarrow$ null	$\kappa \neq 0 \Rightarrow$ spacelike $\kappa = 0 \Rightarrow$ null	$\kappa^2 + \kappa\tau < 0$ $\Rightarrow$ timelike $\kappa^2 + \kappa\tau > 0$ $\Rightarrow$ spacelike $\kappa^2 + \kappa\tau = 0$ $\Rightarrow$ null
Spacelike with timelike principle normal	$\tau \neq 0 \Rightarrow$ spacelike $\tau = 0 \Rightarrow$ null	$\kappa \neq -\tau \Rightarrow$ timelike $\kappa = -\tau \Rightarrow$ null	$\kappa \neq 0 \Rightarrow$ spacelike $\kappa = 0 \Rightarrow$ null	$\kappa\tau > 0 \Rightarrow$ timelike $\kappa\tau < 0 \Rightarrow$ spacelike $\kappa\tau = 0 \Rightarrow$ null
Null	$\kappa \neq \tau \Rightarrow$ spacelike $\kappa = \tau \Rightarrow$ null	$\kappa^2 - 2\kappa\tau < 0$ $\Rightarrow$ timelike $\kappa^2 - 2\kappa\tau > 0$ $\Rightarrow$ spacelike $\kappa^2 - 2\kappa\tau = 0$ $\Rightarrow$ null	$\tau^2 - 2\kappa\tau < 0$ $\Rightarrow$ timelike $\tau^2 - 2\kappa\tau > 0$ $\Rightarrow$ spacelike $\tau^2 - 2\kappa\tau = 0$ $\Rightarrow$ null	$\kappa^2 - 4\kappa\tau + \tau^2 < 0$ $\Rightarrow$ timelike $\kappa^2 - 4\kappa\tau + \tau^2 > 0$ $\Rightarrow$ spacelike $\kappa^2 - 4\kappa\tau + \tau^2 = 0$ $\Rightarrow$ null

Example 1. *Let*

$$\alpha(s) = \left(\frac{\sqrt{7}}{\sqrt{2}}s, \frac{\sqrt{5}}{\sqrt{2}}\cos s, \frac{\sqrt{5}}{\sqrt{2}}\sin s\right)$$

a timelike curve parametrized by arc length.

From the calculating of the Frenet invariants of a spacelike curve with timelike principal normal, we get $\kappa = \sqrt{5/2} > 0$ and $\tau = -\sqrt{7/2} \neq 0$. Hence, it follows that $\kappa \neq \tau$, $2\kappa^2 - \tau^2 > 0$ and $\tau^2 - \kappa\tau > 0$. From Table 6.1, under this conditions all of $\mathbf{TN}, \mathbf{TB}, \mathbf{NB}$ and $\mathbf{TNB}$-Smarandache curves of α are spacelike curves. Indeed, they can be given by using Definition 2.1 respectively, as follows:

$$\begin{aligned}
\beta_{TN}(s) &= \frac{1}{2}\left(\sqrt{7}, -\left(\sqrt{5}\sin s + \sqrt{2}\cos s\right), \sqrt{5}\cos s - \sqrt{2}\sin s\right) \\
\beta_{TB}(s) &= \left(\sqrt{7} + \sqrt{5}\right)(1, -\sin s, \cos s) \\
\beta_{NB}(s) &= \frac{1}{2}\left(\sqrt{5}, -\left(\sqrt{2}\cos s + \sqrt{7}\sin s\right), -\sqrt{2}\sin s + \sqrt{7}\cos s\right)
\end{aligned}$$

$$\begin{aligned}
&\beta_{TNB}(s) \\
&= \frac{1}{\sqrt{6}}\left(\left(\sqrt{7}+\sqrt{5}\right), -\left(\sqrt{7}+\sqrt{5}\right)\sin s - \sqrt{2}\cos s, \left(\sqrt{7}+\sqrt{5}\right)\cos s - \sqrt{2}\sin s\right)
\end{aligned}$$

It is obvious that all of the above curves are spacelike curves. They are shown in Figure 1.

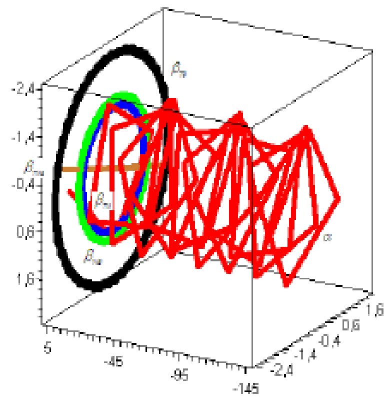

Figure 1. Smarandache curves of a timelike curve

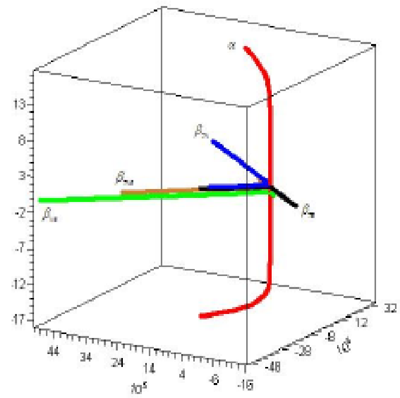

Figure 2. Smarandache curves of a spacelike curve with timelike binormal

Example 2. *Let*

$$\alpha(s) = \left(\frac{1}{\sqrt{5}}\sinh s, \frac{1}{\sqrt{5}}\cosh s, \frac{\sqrt{6}}{\sqrt{5}}s\right)$$

a spacelike curve parametrized by arc length with timelike binormal.

If we calculate Frenet invariants of a timelike curve, we get $\kappa = 1/\sqrt{5} \neq 0$ and $\tau = -\sqrt{6/5}$. Hence, we have $2\kappa^2 - \tau^2 < 0$, $\kappa \neq \tau < 0$ and $\kappa^2 + \kappa\tau < 0$. From Table 6.1, $\mathbf{TN}, \mathbf{TNB}$-Smarandache curves of α are timelike curves, and $\mathbf{TB}, \mathbf{NB}$-Smarandache curves of α are spacelike curves given as below:

$$\begin{aligned}
\beta_{TN}(s) &= \tfrac{1}{\sqrt{10}}\left(\cosh s + \sqrt{5}\sinh s, \sinh s + \sqrt{5}\cosh s, \sqrt{6}\right) \\
\beta_{TB}(s) &= \tfrac{1-\sqrt{6}}{\sqrt{10}}\left(\cosh s, \sinh s, -1\right) \\
\beta_{NB}(s) &= \tfrac{1}{\sqrt{10}}\left(\sqrt{5}\sinh s - \sqrt{6}\cosh s, \sqrt{5}\cosh s - \sqrt{6}\sinh s, -1\right)
\end{aligned}$$

$$\begin{aligned}
&\beta_{TNB}(s) \\
&\quad = \frac{1}{\sqrt{15}}\left(\left(1-\sqrt{6}\right)\cosh s + \sqrt{5}\sinh s, \left(1-\sqrt{6}\right)\sinh s + \sqrt{5}\cosh s, \sqrt{6}-1\right)
\end{aligned}$$

They are shown in Figure 2.

Example 3. *Let*

$$\alpha(s) = \left(\frac{\sqrt{3}}{2} \cosh s, \frac{\sqrt{3}}{2} \sinh s, \frac{s}{2} \right)$$

be a nonplanar spacelike curve parametrized by arc length with timelike principal normal.

From the calculating of Frenet invariants of a timelike curve, we have $\kappa = \sqrt{3}/2 \neq 0$ and $\tau = -1/2 \neq 0$. Hence, it follows that $\kappa \neq -\tau$, $\kappa\tau < 0$. Thus, from Table 6.1, **TN**, **NB** and **TNB**-Smarandache curves of α are spacelike curves and **TB** Smarandache curve of α is a timelike curve given as below:

$$\begin{aligned}
\beta_{TN}(s) &= \tfrac{1}{2\sqrt{2}} \left(\sqrt{3} + 2\sinh s + \cosh s, \sqrt{3}\cosh s + 2\sinh s, 1 \right) \\
\beta_{TB}(s) &= \tfrac{1}{2\sqrt{2}} \left(\left(\sqrt{3} - 1\right)\sinh s, \left(\sqrt{3} - 1\right)\cosh s, \sqrt{3} + 1 \right) \\
\beta_{NB}(s) &= \tfrac{1}{2\sqrt{2}} \left(2\cosh s - \sinh s, 2\sinh s - \cosh s, \sqrt{3} \right) \\
\beta_{TNB}(s) &= \tfrac{1}{2\sqrt{3}} \left(\left(\sqrt{3} - 1\right)\sinh s + 2\cosh s, \left(\sqrt{3} - 1\right)\cosh s + 2\sinh s, \sqrt{3} + 1 \right).
\end{aligned}$$

They are indicated in Figure 3.

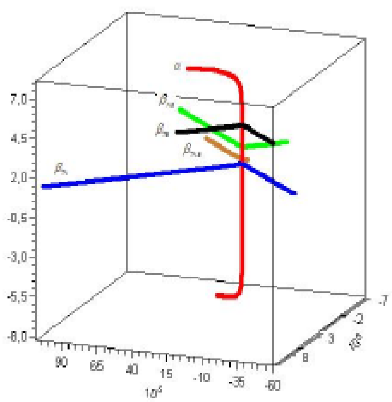

Figure 3. Smarandache curves of a nonplanar spacelike curve with timelike principal normal

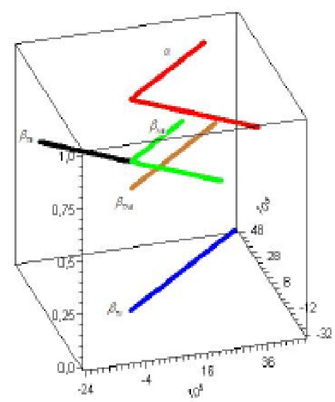

Figure 4. Smarandache curves of a planar spacelike curve with timelike principal normal

Example 4. *Let*

$$\alpha(s) = (\cosh s, \sinh s, 1)$$

be a planar spacelike curve parametrized by arc length with timelike principal normal.

Frenet invariants of this curve can be calculated namely: $\kappa = 1 \neq 0$ and $\tau = 0$. Hence, it follows that $\kappa \neq -\tau$ and $\kappa\tau = 0$. From Table 6.1 **TB**-Smarandache curve is timelike curve, **NB**-Smarandache curve is spacelike curve and **TN**, **TNB**-Smarandache curves are null curves such that:

$$\begin{aligned}
\beta_{TN}(s) &= \tfrac{1}{\sqrt{2}}(\cosh s + \sinh s, \cosh s + \sinh s, 0) \\
\beta_{TB}(s) &= \tfrac{1}{\sqrt{2}}(\sinh s, \cosh s, 1) \\
\beta_{NB}(s) &= \tfrac{1}{\sqrt{2}}(\cosh s, \sinh s, 1) \\
\beta_{TNB}(s) &= \tfrac{1}{\sqrt{3}}(\cosh s + \sinh s, \cosh s + \sinh s, 1)
\end{aligned}$$

They are illustrated in Figure 4.

Example 5. *Let*

$$\alpha(s) = (\sinh s, s, \cosh s)$$

be a null curve.

Frenet invariants of the above null curve can be obtained such that: $\kappa = 1 \neq 0$ and $\tau = 1/2$. Thus, we have $\kappa \neq \tau$, $\kappa^2 - 2\kappa\tau = 0$, $\tau^2 - 2\kappa\tau < 0$ and $\kappa^2 - 4\kappa\tau + \tau^2 < 0$. By using these conditions, it is obvious from Table 6.1, **NB**, **TNB**-Smarandache curves are timelike, **TN**-Smarandache curve is spacelike curve and **TB**-Smarandache curve is null curve such that:

$$\begin{aligned}
\beta_{TN}(s) &= \tfrac{1}{2\sqrt{2}}(\cosh s, 3, \sinh s) \\
\beta_{TB}(s) &= \tfrac{1}{\sqrt{2}}(\cosh s + \sinh s, 1, \cosh s + \sinh s) \\
\beta_{NB}(s) &= \tfrac{1}{2\sqrt{2}}(-\cosh s + 2\sinh s, 1, -\sinh s + 2\cosh s) \\
\beta_{TNB}(s) &= \tfrac{1}{2\sqrt{3}}(\cosh s + 2\sinh s, 3, \sinh s + 2\cosh s)
\end{aligned}$$

They are indicated in Figure 5.

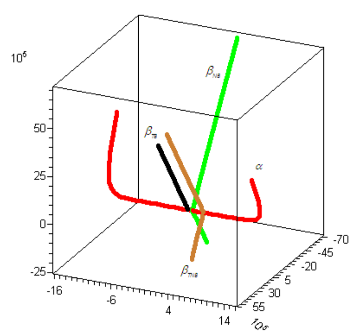

Figure 5. Smarandache curves of a null curve.

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