Super Mean Labeling of Some Classes of Graphs

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Abstract: Let G be a (p,q) graph and $f:V(G) \to \{1,2,3,\ldots,p+q\}$ be an injection. For each edge e=uv, let $f^*(e)=(f(u)+f(v))/2$ if f(u)+f(v) is even and $f^*(e)=(f(u)+f(v)+1)/2$ if f(u)+f(v) is odd. Then f is called a super mean labeling if $f(V)\cup\{f^*(e):e\in E(G)\}=\{1,2,3,\ldots,p+q\}$. A graph that admits a super mean labeling is called a super mean graph. In this paper we prove that $S(P_n \odot K_1), S(P_2 \times P_4), S(B_{n,n}), \langle B_{n,n}: P_m \rangle, C_n \odot \overline{K_2}, n \geq 3$, generalized antiprism \mathcal{A}_n^m and the double triangular snake $D(T_n)$ are super mean graphs.

Key Words: Smarandachely super m-mean labeling, Smarandachely super m-mean graph, super mean labeling, super mean graph.

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§1. Introduction

By a graph we mean a finite, simple and undirected one. The vertex set and the edge set of a graph G are denoted by V(G) and E(G) respectively. The disjoint union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The disjoint union of m copies of the graph G is denoted by mG. The corona $G_1 \odot G_2$ of the graphs G_1 and G_2 is obtained by taking one copy of G_1 (with p vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2 . Armed crown $C_n\Theta P_m$ is a graph obtained from a cycle C_n by identifying the pendant vertices of two vertex disjoint paths of equal length m-1 at each vertex of the cycle. We denote a bi-armed crown by $C_n\Theta 2P_m$, where P_m is a path of length m-1. The double triangular snake $D(T_n)$ is the graph obtained from the path $v_1, v_2, v_3, \ldots, v_n$ by joining v_i and v_{i+1} with two new vertices i_i and w_i for $1 \le i \le m-1$. The bistar $B_{m,n}$ is a graph obtained from

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 K_2 by joining m pendant edges to one end of K_2 and n pendant edges to the other end of K_2 . The generalized prism graph $C_n \times P_m$ has the vertex set $V = \{v_i^j : 1 \le i \le n \text{ and } 1 \le j \le m\}$ and the edge set $E = \{v_i^j v_{i+1}^j, v_n^j v_1^j : 1 \le i \le n-1 \text{ and } 1 \le j \le m\} \cup \{v_i^j v_{i-1}^{j+1}, v_1^j v_n^{j+1} : 2 \le i \le n \text{ and } 1 \le j \le m-1\}$. The generalized antiprism \mathcal{A}_n^m is obtained by completing the generalized prism $C_n \times P_m$ by adding the edges $v_i^j v_i^{j+1}$ for $1 \le i \le n$ and $1 \le j \le m-1$. Terms and notations not defined here are used in the sense of Harary [1].

§2. Preliminary Results

Let G be a graph and $f: V(G) \to \{1, 2, 3, \dots, |V| + |E(G)|\}$ be an injection. For each edge e = uv and an integer $m \ge 2$, the induced Smarandachely edge m-labeling f_S^* is defined by

$$f_S^*(e) = \left\lceil \frac{f(u) + f(v)}{m} \right\rceil.$$

Then f is called a Smarandachely super m-mean labeling if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \dots, |V| + |E(G)|\}$. A graph that admits a Smarandachely super mean m-labeling is called Smarandachely super m-mean graph. Particularly, if m = 2, we know that

$$f^*(e) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even;} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}$$

Such a labeling f is called a super mean labeling of G if $f(V(G)) \cup (f^*(e) : e \in E(G)) = \{1, 2, 3, \ldots, p + q\}$. A graph that admits a super mean labeling is called a super mean graph. The concept of super mean labeling was introduced in [7] and further discussed in [2-6].

We use the following results in the subsequent theorems.

Theorem 2.1([7]) The bistar $B_{m,n}$ is a super mean graph for m = n or n + 1.

Theorem 2.2([2]) The graph $\langle B_{n,n} : w \rangle$, obtained by the subdivision of the central edge of $B_{n,n}$ with a vertex w, is a super mean graph.

Theorem 2.3([2]) The bi-armed crown $C_n\Theta 2P_m$ is a super mean graph for odd $n \geq 3$ and $m \geq 2$.

Theorem 2.4([7]) Let $G_1 = (p_1, q_1)$ and $G_2 = (p_2, q_2)$ be two super mean graphs with super mean labeling f and g respectively. Let $f(u) = p_1 + q_1$ and g(v) = 1. Then the graph $(G_1)_f * (G_2)_g$ obtained from G_1 and G_2 by identifying the vertices u and v is also a super mean graph.

§3. Super Mean Graphs

If G is a graph, then S(G) is a graph obtained by subdividing each edge of G by a vertex.

Theorem 3.1 The graph $S(P_n \odot K_1)$ is a super mean graph.

Proof Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \le i \le n\}$. Let $x_i (1 \le i \le n)$ be the vertex which divides the edge $u_i v_i (1 \le i \le n)$ and $y_i (1 \le i \le n-1)$ be the vertex which divides the edge $u_i u_{i+1} (1 \le i \le n-1)$. Then $V(S(P_n \odot K_1)) = \{u_i, v_i, x_i, y_j : 1 \le i \le n, 1 \le j \le n-1\}$.

Define
$$f: V(S(P_n \odot K_1)) \to \{1, 2, 3, \dots, p + q = 8n - 3\}$$
 by

$$f(v_1) = 1; f(v_2) = 14; f(v_{2+i}) = 14 + 8i \text{ for } 1 \le i \le n - 4;$$

$$f(v_{n-1}) = 8n - 11; f(v_n) = 8n - 10; f(x_1) = 3;$$

$$f(x_{1+i}) = 3 + 8i \text{ for } 1 \le i \le n - 2; f(x_n) = 8n - 7;$$

$$f(u_1) = 5; f(u_2) = 9; f(u_{2+i}) = 9 + 8i \text{ for } 1 \le i \le n - 3;$$

$$f(u_n) = 8n - 5; f(y_i) = 8i - 1 \text{ for } 1 \le i \le n - 2; f(y_{n-1}) = 8n - 3.$$

It can be verified that f is a super mean labeling of $S(P_n \odot K_1)$. Hence $S(P_n \odot K_1)$ is a super mean graph.

Example 3.2 The super mean labeling of $S(P_5 \odot K_1)$ is given in Fig.1.

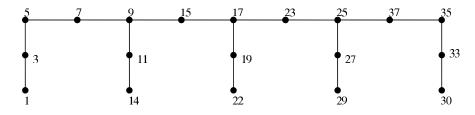


Fig.1

Theorem 3.2 The graph $S(P_2 \times P_n)$ is a super mean graph.

Proof Let $V(P_2 \times P_n) = \{u_i, v_i : 1 \le i \le n\}$. Let $u_i^1, v_i^1 (1 \le i \le n-1)$ be the vertices which divide the edges $u_i u_{i+1}, v_i v_{i+1} (1 \le i \le n-1)$ respectively. Let $w_i (1 \le i \le n)$ be the vertex which divides the edge $u_i v_i$. That is $V(S(P_2 \times P_n)) = \{u_i, v_i, w_i : 1 \le i \le n\} \cup \{u_i^1, v_i^1 : 1 \le i \le n-1\}$.

Define
$$f: V(S(P_2 \times P_n)) \to \{1, 2, 3, \dots, p+q = 11n-6\}$$
 by

$$f(u_1) = 1$$
; $f(u_2) = 9$; $f(u_3) = 27$;
 $f(u_i) = f(u_{i-1}) + 5$ for $4 \le i \le n$ and i is even
 $f(u_i) = f(u_{i-1}) + 17$ for $4 \le i \le n$ and i is odd
 $f(v_1) = 7$; $f(v_2) = 16$;
 $f(v_i) = f(v_{i-1}) + 5$ for $3 \le i \le n$ and i is odd
 $f(v_i) = f(v_{i-1}) + 17$ for $3 \le i \le n$ and i is even
 $f(w_1) = 3$; $f(w_2) = 12$;
 $f(w_{2+i}) = 12 + 11i$ for $1 \le i \le n - 2$;
 $f(u_1^1) = 6$; $f(u_2^1) = 24$;

$$\begin{split} f(u_i^1) &= f(u_{i-1}^1) + 6 \text{ for } 3 \leq i \leq n-1 \text{ and } i \text{ isodd} \\ f(u_i^1) &= f(u_{i-1}^1) + 16 \text{ for } 3 \leq i \leq n-1 \text{ and } i \text{ iseven} \\ f(v_1^1) &= 13; f(v_i^1) = f(v_{i-1}^1) + 6 \text{ for } 2 \leq i \leq n-1 \text{ and } i \text{ iseven} \\ f(v_i^1) &= f(v_{i-1}^1) + 16 \text{ for } 2 \leq i \leq n-1 \text{ and } i \text{ isodd.} \end{split}$$

It is easy to check that f is a super mean labeling of $S(P_2 \times P_n)$. Hence $S(P_2 \times P_n)$ is a super mean graph.

Example 3.4 The super mean labeling of $S(P_2 \times P_6)$ is given in Fig.2.

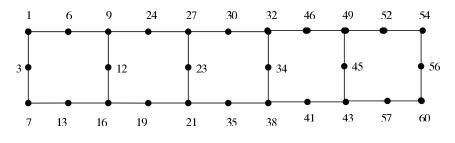


Fig.2

Theorem 3.5 The graph $S(B_{n,n})$ is a super mean graph.

Proof Let $V(B_{n,n}) = \{u, u_i, v, v_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uu_i, vv_i, uv : 1 \leq i \leq n\}$. Let $w, x_i, y_i, (1 \leq i \leq n)$ be the vertices which divide the edges $uv, uu_i, vv_i (1 \leq i \leq n)$ respectively. Then $V(S(B_{n,n})) = \{u, u_i, v, v_i, x_i, y_i, w : 1 \leq i \leq n\}$ and $E(S(B_{n,n})) = \{ux_i, x_iu_i, uw, wv, vy_i, y_iv_i : 1 \leq i \leq n\}$.

Define $f: V(S(B_{n,n})) \to \{1, 2, 3, \dots, p+q = 8n+5\}$ by

 $f(u)=1; f(x_i)=8i-5$ for $1 \leq i \leq n; f(u_i)=8i-3$ for $1 \leq i \leq n; f(w)=8n+3;$ $f(v)=8n+5; f(y_i)=8i-1$ for $1 \leq i \leq n; f(v_i)=8i+1$ for $1 \leq i \leq n$. It can be verified that f is a super mean labeling of $S(B_{n,n})$. Hence $S(B_{n,n})$ is a super mean graph. \square

Example 3.6 The super mean labeling of $S(B_{n,n})$ is given in Fig.3.

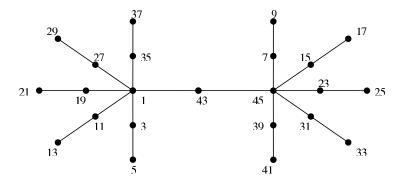


Fig.3

Next we prove that the graph $\langle B_{n,n} : P_m \rangle$ is a super mean graph. $\langle B_{m,n} : P_k \rangle$ is a graph obtained by joining the central vertices of the stars $K_{1,m}$ and $K_{1,n}$ by a path P_k of length k-1.

Theorem 3.7 The graph $\langle B_{n,n}: P_m \rangle$ is a super mean graph for all $n \geq 1$ and m > 1.

Proof Let $V(\langle B_{n,n}: P_m \rangle) = \{u_i, v_i, u, v, w_j : 1 \le i \le n, 1 \le j \le m \text{ with } u = w_1, v = w_m\}$ and $E(\langle B_{n,n}: P_m \rangle) = \{uu_i, vv_i, w_jw_{j+1} : 1 \le i \le n, 1 \le j \le m-1\}.$

Case 1 n is even.

Subcase 1 m is odd.

By Theorem 2.2, $\langle B_{n,n}: P_3 \rangle$ is a super mean graph. For m > 3, define $f: V(\langle B_{n,n}: P_m \rangle) \to \{1, 2, 3, \dots, p + q = 4n + 2m - 1\}$ by

$$f(u) = 1; f(u_i) = 4i - 1 \text{ for } 1 \le i \le n \text{ and for } i \ne \frac{n}{2} + 1;$$

$$f(u_{\frac{n}{2}+1}) = 2n + 2; f(v_i) = 4i + 1 \text{ for } 1 \le i \le n; f(v) = 4n + 3;$$

$$f(w_2) = 4n + 4; f(w_3) = 4n + 9;$$

$$f(w_{3+i}) = 4n + 9 + 4i \text{ for } 1 \le i \le \frac{m-5}{2}; f(w_{\frac{m+3}{2}}) = 4n + 2m - 4$$

$$f(w_{\frac{m+3}{2}+i}) = 4n + 2m - 4 - 4i \text{ for } 1 \le i \le \frac{m-5}{2}.$$

It can be verified that f is a super mean labeling of $\langle B_{n,n} : P_m \rangle$.

Subcase 2 m is even.

By Theorem 2.1, $\langle B_{n,n}: P_2 \rangle$ is a super mean graph. For m > 2, define $f: V(\langle B_{n,n}: P_m \rangle) \to \{1, 2, 3, \dots, p+q=4n+2m-1\}$ by

$$f(u) = 1; f(u_i) = 4i - 1 \text{ for } 1 \le i \le n \text{ and for } i \ne \frac{n}{2} + 1;$$

$$f(u_{\frac{n}{2}+1}) = 2n + 2; f(v_i) = 4i + 1 \text{ for } 1 \le i \le n; f(v) = 4n + 3;$$

$$f(w_2) = 4n + 4; f(w_{2+i}) = 4n + 4 + 2i \text{ for } 1 \le i \le \frac{m-4}{2};$$

$$f(w_{\frac{m+2}{2}}) = 4n + m + 3;$$

$$f(w_{\frac{m+2}{2}+i}) = 4n + m + 3 + 2i \text{ for } 1 \le i \le \frac{m-4}{2}.$$

It can be verified that f is a super mean labeling of $\langle B_{n,n} : P_m \rangle$.

Case 2 n is odd.

Subcase 1 m is odd.

By Theorem 2.1, $\langle B_{n,n}: P_2 \rangle$ is a super mean graph. For m > 2, define $f: V(\langle B_{n,n}: P_m \rangle) \to$

$$\{1, 2, 3, \dots, p + q = 4n + 2m - 1\} \text{ by}$$

$$f(u) = 1; f(v) = 4n + 3; f(u_i) = 4i - 1 \text{ for } 1 \le i \le n;$$

$$f(v_i) = 4i + 1 \text{ for } 1 \le i \le n \text{ and for } i \ne \frac{n+1}{2};$$

$$f(v_{\frac{n+1}{2}}) = 2n + 2; f(w_2) = 4n + 4;$$

$$f(w_{2+i}) = 4n + 4 + 2i \text{ for } 1 \le i \le \frac{m-4}{2}; f(w_{\frac{m+2}{2}}) = 4n + m + 3;$$

$$f(w_{\frac{m+2}{2}+i}) = 4n + m + 3 + 2i \text{ for } 1 \le i \le \frac{m-4}{2}.$$

It can be verified that f is a super mean labeling of $\langle B_{n,n}:P_m\rangle$.

Subcase 2 m is even.

By Theorem 2.2, $\langle B_{n,n}: P_3 \rangle$ is a super mean graph. For m>3, define $f:V(\langle B_{n,n}: P_m \rangle) \to \{1,2,3,\ldots,p+q=4n+2m-1\}$ by

$$f(u) = 1; f(v) = 4n + 3; f(u_i) = 4i - 1 \text{ for } 1 \le i \le n;$$

$$f(v_i) = 4i + 1 \text{ for } 1 \le i \le n \text{ and for } i \ne \frac{n+1}{2}; f(v_{\frac{n+1}{2}}) = 2n + 2;$$

$$f(w_2) = 4n + 4; f(w_3) = 4n + 9;$$

$$f(w_{3+i}) = 4n + 9 + 4i \text{ for } 1 \le i \le \frac{m-5}{2}; f(w_{\frac{m+3}{2}}) = 4n + 2m - 4;$$

$$f(w_{\frac{m+3}{2}+i}) = 4n + 2m - 4 - 2i \text{ for } 1 \le i \le \frac{m-5}{2}.$$

It can be verified that f is a super mean labeling of $\langle B_{n,n}:P_m\rangle$. Hence $\langle B_{n,n}:P_m\rangle$ is a super mean graph for all $n\geq 1$ and m>1.

Example 3.8 The super mean labeling of $\langle B_{4,4} : P_5 \rangle$ is given in Fig.4.

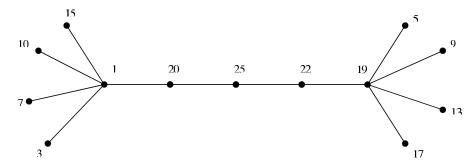


Fig.4

Theorem 3.9 The corona graph $C_n \odot \overline{K_2}$ is a super mean graph for all $n \geq 3$.

Proof Let
$$V(C_n) = \{u_1, u_2, \dots, u_n\}$$
 and $V(C_n \odot \overline{K_2}) = \{u_i, v_i, w_i : 1 \le i \le n\}$. Then $E(C_n \odot \overline{K_2}) = \{u_i u_{i+1}, u_n u_1, u_j v_j, u_j w_j : 1 \le i \le n-1 \text{ and } 1 \le j \le n\}$.

Case 1 n is odd.

The proof follows from Theorem 2.3 by taking m=2.

Case 2 n is even.

Take
$$n = 2k$$
 for some k . Define $f: V(C_n \odot \overline{K_2}) \to \{1, 2, 3, \dots, p + q = 6n\}$ by
$$f(u_i) = 6i - 3 \text{ for } 1 \le i \le k - 1; f(u_k) = 6k - 2;$$
$$f(u_{k+i}) = 6k - 2 + 6i \text{ for } 1 \le i \le k - 2; f(u_{2k-1}) = 12k - 2;$$
$$f(u_{2k}) = 12k - 9; f(v_i) = 6i - 5 \text{ for } 1 \le i \le k - 1; f(v_k) = 6k - 6;$$
$$f(v_{k+1}) = 6k + 2; f(v_{k+1+i}) = 6k + 2 + 6i \text{ for } 1 \le i \le k - 3; f(v_{2k-1}) = 12k;$$
$$f(v_{2k}) = 12k - 6; f(w_i) = 6i - 1 \text{ for } 1 \le i \le k - 1; f(w_k) = 6k;$$
$$f(w_{k+i}) = 6k + 6i \text{ for } 1 \le i \le k - 2; f(w_{2k-1}) = 12k - 4;$$
$$f(w_{2k}) = 12k - 11.$$

It can be verified that $f(V) \cup (f^*(e) : e \in E) = \{1, 2, 3, \dots, 6n\}$. Hence $C_n \odot \overline{K_2}$ is a super mean graph.

Example 3.10 The super mean labeling of $C_8 \odot \overline{K_2}$ is given in Fig.5.

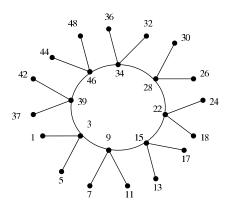


Fig.5

Theorem 3.11 The double triangular snake $D(T_n)$ is a super mean graph.

Proof We prove this result by induction on n. A super mean labeling of $G_1 = D(T_2)$ is given in Fig.6.

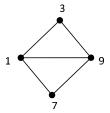


Fig.6

Therefore the result is true for n = 2. Let f be the super mean labeling of G_1 as in the above figure. Now $D(T_3) = (G_1)_f * (G_1)_f$, by Theorem 2.4, $D(T_3)$ is a super mean graph. Therefore the result is true for n = 3. Assume that $D(T_{n-1})$ is a super mean graph with the super mean labeling g. Now by Theorem 2.4, $(D(T_{n-1}))_g * (G_1)_f = D(T_n)$ is a super mean graph. Therefore the result is true for n. Hence by induction principle the result is true for all n. Thus $D(T_n)$ is a super mean graph.

Example 3.12 The super mean labeling of $D(T_6)$ is given in Fig.7.

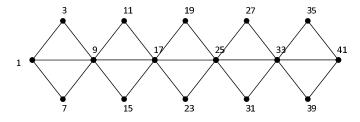


Fig.7

Theorem 3.13 The generalized antiprism \mathcal{A}_n^m is a super mean graph for all $m \geq 2, n \geq 3$ except for n = 4.

Proof Let $V(\mathcal{A}_n^m) = \{v_i^j : 1 \le i \le n, 1 \le j \le m\}$ and $E(\mathcal{A}_n^m) = \{v_i^j v_{i+1}^j, v_n^j v_1^j : 1 \le i \le n-1, 1 \le j \le m\} \cup \{v_i^j v_{i-1}^{j+1}, v_1^j v_n^{j+1} : 2 \le i \le n, 1 \le j \le m-1\} \cup \{v_i^j v_i^{j+1} : 1 \le i \le n \text{ and } 1 \le j \le m-1\}.$

Case 1 n is odd.

Define
$$f: V(\mathcal{A}_n^m) \to \{1, 2, 3, \dots, p+q=4mn-2n\}$$
 by
$$f(v_i^j) = 4(j-1)n + 2i - 1 \text{ for } 1 \le i \le \frac{n+1}{2} \text{ and } 1 \le j \le m;$$

$$f(v_{\frac{n+3}{2}}^j) = 4(j-1)n + n + 3 \text{ for } 1 \le j \le m;$$

$$f(v_{\frac{n+3}{2}+i}^j) = 4(j-1)n + n + 3 + 2i \text{ for } 1 \le i \le \frac{n-3}{2} \text{ and } 1 \le j \le m.$$

Then f is a super mean labeling of \mathcal{A}_n^m . Hence \mathcal{A}_n^m is a super mean graph.

Case 2 n is even and $n \neq 4$.

Define
$$f: V(\mathcal{A}_n^m) \to \{1, 2, 3, \dots, p+q=4mn-2n\}$$
 by
$$f(v_1^j) = 4(j-1)n+1 \text{ for } 1 \leq j \leq m; f(v_2^j) = 4(j-1)n+3 \text{ for } 1 \leq j \leq m;$$

$$f(v_3^j) = 4(j-1)n+7 \text{ for } 1 \leq j \leq m; f(v_4^j) = 4(j-1)n+12 \text{ for } 1 \leq j \leq m;$$

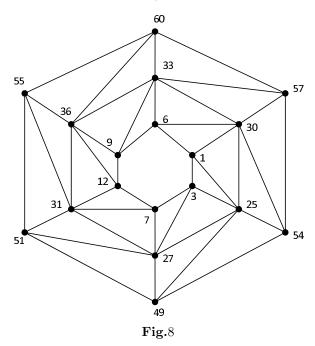
$$f(v_{4+i}^j) = 4(j-1)n+12+4i \text{ for } 1 \leq j \leq m \text{ and } 1 \leq i \leq \frac{n-6}{2};$$

$$f(v_{n-1}^j) = 4(j-1)n+2n+1-4i \text{ for } 1 \leq j \leq m \text{ and } 1 \leq i \leq \frac{n-6}{2};$$

$$f(v_{n-1}^j) = 4(j-1)n+9 \text{ for } 1 \leq j \leq m; f(v_n^j) = 4(j-1)n+6 \text{ for } 1 \leq j \leq m.$$

Then f is a super mean labeling of \mathcal{A}_n^m . Hence \mathcal{A}_n^m is a super mean graph.

Example 3.14 The super mean labeling of \mathcal{A}_6^3 is given in Fig.8.



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