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Surface Embeddability of Graphs via Reductions

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Abstract: On the basis of reductions, polyhedral forms of Jordan axiom on closed curve in the plane are extended to establish characterizations for the surface embeddability of a graph.

Key Words: Surface, graph, Smarandache λ^S-drawing, embedding, Jordan closed cure axiom, forbidden minor.

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§1. Introduction

A drawing of a graph G on a surface S is such a drawing with no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. A *Smarandache* λ^S*-drawing* of G on S is a drawing of G on S with minimal intersections λ^S. Particularly, a Smarandache 0-drawing of G on S, if existing, is called an embedding of G on S.

The classical version of Jordan curve theorem in topology states that a single closed curve C separates the sphere into two connected components of which C is their common boundary. In this section, we investigate the polyhedral statements and proofs of the Jordan curve theorem.

Let $\Sigma = \Sigma(G; F)$ be a polyhedron whose underlying graph $G = (V, E)$ with F as the set of faces. If any circuit C of G not a face boundary of Σ has the property that there exist two proper subgraphs In and Ou of G such that

$$In \bigcup Ou = G;\ In \bigcap Ou = C, \tag{A}$$

then Σ is said to have the *first Jordan curve property*, or simply write as 1-JCP. For a graph G, if there is a polyhedron $\Sigma = \Sigma(G; F)$ which has the 1-JCP, then G is said to have the 1-JCP as well.

Of course, in order to make sense for the problems discussed in this section, we always suppose that all the members of F in the polyhedron $\Sigma = \Sigma(G; F)$ are circuits of G.

Theorem A(First Jordan curve theorem) *G has the 1-JCP If, and only if, G is planar.*

Proof Because of $\mathcal{H}_1(\Sigma) = 0, \Sigma = \Sigma(G; F)$, from Theorem 4.2.5 in [1], we know that

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Im ∂_2 = Ker $\partial_1 = \mathcal{C}$, the cycle space of G and hence Im $\partial_2 \supseteq F$ which contains a basis of $\mathcal{C}$. Thus, for any circuit $C \notin F$, there exists a subset D of F such that

$$C = \sum_{f \in D} \partial_2 f;\ C = \sum_{f \in F \setminus D} \partial_2 f. \qquad (B)$$

Moreover, if we write

$$Ou = G[\bigcup_{f \in D} f];\ In = G[\bigcup_{f \in F \setminus D} f],$$

then Ou and In satisfy the relations in (A) since any edge of G appears exactly twice in the members of F. This is the sufficiency.

Conversely, if G is not planar, then G only have embedding on surfaces of genus not 0. Because of the existence of non contractible circuit, such a circuit does not satisfy the 1-JCP and hence G is without 1-JCP. This is the necessity. □

Let $\Sigma^* = \Sigma(G^*; F^*)$ be a dual polyhedron of $\Sigma = \Sigma(G; F)$. For a circuit C in G, let $C^* = \{e^* | \ \forall e \in C\}$, or say the corresponding vector in $\mathcal{G}_1^*$, of $C \in \mathcal{G}_1$.

Lemma 1 *Let C be a circuit in Σ. Then, $G^* \setminus C^*$ has at most two connected components.*

Proof Suppose H^* be a connected component of $G^* \setminus C^*$ but not the only one. Let D be the subset of F corresponding to $V(H^*)$. Then,

$$C' = \sum_{f \in D} \partial_2 f \subseteq C.$$

However, if $\emptyset \neq C' \subset C$, then C itself is not a circuit. This is a contradiction to the condition of the lemma. From that any edge appears twice in the members of F, there is only one possibility that

$$C = \sum_{f \in F \setminus D} \partial_2 f.$$

Hence, $F \setminus D$ determines the other connected component of $G^* \setminus C^*$ when $C' = C$. □

Any circuit C in G which is the underlying graph of a polyhedron $\Sigma = \Sigma(G; F)$ is said to have the *second Jordan curve property,* or simply write 2-JCP for Σ with its dual $\Sigma^* = \Sigma(G^*; F^*)$ if $G^* \setminus C^*$ has exactly two connected components. A graph G is said to have the 2-JCP if all the circuits in G have the property.

Theorem B(Second Jordan curve theorem) *A graph G has the 2-JCP if, and only if, G is planar.*

Proof To prove the necessity. Because for any circuit C in G, $G^* \setminus C^*$ has exactly two connected components, any C^* which corresponds to a circuit C in G is a cocircuit. Since any edge in G^* appears exactly twice in the elements of V^*, which are all cocircuits, from Lemma 1, V^* contains a basis of Ker δ_1^*. Moreover, V^* is a subset of Im δ_0^*. Hence, Ker $\delta_1 \subseteq$ Im δ_0. From Lemma 4.3.2 in [1], Im $\delta_0^* \subseteq$ Ker δ_1^*. Then, we have Ker δ_1^* =Im δ_0^*, *i.e.*, $\widetilde{\mathcal{H}}_1(\Sigma^*) = 0$. From the dual case of Theorem 4.3.2 in [1], G^* is planar and hence so is G. Conversely, to prove the sufficiency. From the planar duality, for any circuit C in G, C^* is a cocircuit in G^*. Then, $G^*\backslash C^*$ has two connected components and hence C has the 2- JCP. $\square$

For a graph G, of course connected without loop, associated with a polyhedron $\Sigma = \Sigma(G;F)$, let C be a circuit and E_C, the set of edges incident to, but not on C. We may define an equivalence on E_C, denoted by $\sim_C$ as the transitive closure of that $\forall a, b \in E_C$,

$$\begin{aligned} a \sim_C b \Leftrightarrow \quad & \exists f \in F,\ (a^\alpha C(a,b)b^\beta \subset f) \\ & \vee (b^{-\beta}C(b,a)a^{-\alpha} \subset f), \end{aligned} \tag{C}$$

where $C(a,b)$, or $C(b,a)$ is the common path from a to b, or from b to a in $C \cap f$ respectively. It can be seen that $|E_C/\sim_C| \leqslant 2$ and the equality holds for any C not in F only if Σ is orientable.

In this case, the two equivalent classes are denoted by $E_{\mathcal{L}} = E_{\mathcal{L}}(C)$ and $E_{\mathcal{R}} = E_{\mathcal{R}}(C)$. Further, let $V_{\mathcal{L}}$ and $V_{\mathcal{R}}$ be the subsets of vertices by which a path between the two ends of two edges in $E_{\mathcal{L}}$ and $E_{\mathcal{R}}$ without common vertex with C passes respectively.

From the connectedness of G, it is clear that $V_{\mathcal{L}} \cup V_{\mathcal{R}} = V \backslash V(C)$. If $V_{\mathcal{L}} \cap V_{\mathcal{R}} = \emptyset$, then C is said to have the *third Jordan curve property,* or simply write 3-JCP. In particular, if C has the 3-JCP, then every path from $V_{\mathcal{L}}$ to $V_{\mathcal{R}}$ (or vice versa) crosses C and hence C has the 1-JCP. If every circuit which is not the boundary of a face f of $\Sigma(G)$, one of the underlain polyhedra of G has the 3-JCP, then G is said to have the 3-JCP as well.

Lemma 2 Let C be a circuit of G which is associated with an orientable polyhedron $\Sigma = \Sigma(G;F)$. If C has the 2-JCP, then C has the 3-JCP. Conversely, if $V_{\mathcal{L}}(C) \neq \emptyset,\ V_{\mathcal{R}}(C) \neq \emptyset$ and C has the 3-JCP, then C has the 2-JCP.

Proof For a vertex $v^* \in V^* = V(G^*)$, let $f(v^*) \in F$ be the corresponding face of Σ. Suppose In^* and Ou^* are the two connected components of $G^*\backslash C^*$ by the 2-JCP of C. Then,

$$In = \bigcup_{v^* \in In^*} f(v^*) \text{ and } Ou = \bigcup_{v^* \in Ou^*} f(v^*)$$

are subgraphs of G such that $In \cup Ou = G$ and $In \cap Ou = C$. Also, $E_{\mathcal{L}} \subset In$ and $E_{\mathcal{R}} \subset Ou$ (or vice versa). The only thing remained is to show $V_{\mathcal{L}} \cap V_{\mathcal{R}} = \emptyset$. By contradiction, if $V_{\mathcal{L}} \cap V_{\mathcal{R}} \neq \emptyset$, then In and Ou have a vertex which is not on C in common and hence have an edge incident with the vertex, which is not on C, in common. This is a contradiction to $In \cap Ou = C$.

Conversely, from Lemma 1, we may assume that $G^*\backslash C^*$ is connected by contradiction. Then there exists a path P^* from v_1^* to v_2^* in $G^*\backslash C^*$ such that $V(f(v_1^*)) \cap V_{\mathcal{L}} \neq \emptyset$ and $V(f(v_2^*)) \cap V_{\mathcal{R}} \neq \emptyset$. Consider

$$H = \bigcup_{v^* \in P^*} f(v^*) \subseteq G.$$

Suppose $P = v_1 v_2 \cdots v_l$ is the shortest path in H from $V_{\mathcal{L}}$ to $V_{\mathcal{R}}$.

To show that P does not cross C. By contradiction, assume that v_{i+1} is the first vertex of P crosses C. From the shortestness, v_i is not in $V_{\mathcal{R}}$. Suppose that subpath $v_{i+1} \cdots v_{j-1},\ i+2 \leqslant j < l$, lies on C and that v_j does not lie on C. By the definition of $E_{\mathcal{L}}$, $(v_{j-1}, v_j) \in E_{\mathcal{L}}$ and hence $v_j \in V_{\mathcal{L}}$. This is a contradiction to the shortestness. However, from that P does not cross C, $V_{\mathcal{L}} \cap V_{\mathcal{R}} \neq \emptyset$. This is a contradiction to the 3-JCP. $\Box$

Theorem C(Third Jordan curve theorem) *Let $G = (V, E)$ be with an orientable polyhedron $\Sigma = \Sigma(G; F)$. Then, G has the 3-JCP if, and only if, G is planar.*

Proof From Theorem B and Lemma 2, the sufficiency is obvious. Conversely, assume that G is not planar. By Lemma 4.2.6 in [1], $\mathrm{Im}\partial_2 \subseteq \mathrm{Ker}\partial_1 = \mathcal{C}$, the cycle space of G. By Theorem 4.2.5 in [1], $\mathrm{Im}\partial_2 \subset \mathrm{Ker}\partial_1$. Then, from Theorem B, there exists a circuit $C \in \mathcal{C} \setminus \mathrm{Im}\partial_2$ without the 2-JCP. Moreover, we also have that $V_{\mathcal{L}} \neq \emptyset$ and $V_{\mathcal{R}} \neq \emptyset$. If otherwise $V_{\mathcal{L}} = \emptyset$, let

$$D = \{f | \exists e \in E_{\mathcal{L}}, e \in f\} \subset F.$$

Because $V_{\mathcal{L}} = \emptyset$, any $f \in D$ contains only edges and chords of C, we have

$$C = \sum_{f \in D} \partial_2 f$$

that contradicts to $C \notin \mathrm{Im}\partial_2$. Therefore, from Lemma 2, C does not have the 3-JCP. The necessity holds. $\Box$

§2 Reducibilities

For S_g as a surface(orientable, or nonorientable) of genus g, If a graph H is not embedded on a surface S_g but what obtained by deleting an edge from H is embeddable on S_g, then H is said to be *reducible* for S_g. In a graph G, the subgraphs of G homeomorphic to H are called a type of *reducible configuration* of G, or shortly a *reduction*. Robertson and Seymour in [2] has been shown that graphs have their types of reductions for a surface of genus given finite. However, even for projective plane the simplest nonorientable surface, the types of reductions are more than 100 [3,7].

For a surface S_g, $g \geqslant 1$, let $\mathcal{H}_{g-1}$ be the set of all reductions of surface S_{g-1}. For $H \in \mathcal{H}_{g-1}$, assume the embeddings of H on S_g have ϕ faces. If a graph G has a decomposition of ϕ subgraphs H_i, $1 \leqslant i \leqslant \phi$, such that

$$\bigcup_{i=1}^{\phi} H_i = G; \ \bigcup_{i \neq j}^{\phi} (H_i \bigcap H_j) = H; \tag{1}$$

all H_i, $1 \leqslant i \leqslant \phi$, are planar and the common vertices of each H_i with H in the boundary of a face, then G is said to be with the *reducibility* 1 for the surface S_g.

Let $\Sigma^* = (G^*; F^*)$ be a polyhedron which is the dual of the embedding $\Sigma = (G; F)$ of G on surface S_g. For surface S_{g-1}, a reduction $H \subseteq G$ is given. Denote $H^* = [e^* | \forall e \in E(H)]$. Naturally, $G^* - E(H^*)$ has at least $\phi = |F|$ connected components. If exact ϕ components and each component planar with all boundary vertices are successively on the boundary of a face, then Σ is said to be with the *reducibility* 2.

A graph G which has an embedding with reducibility 2 then G is said to be with *reducibility* 2 as well.

Given $\Sigma = (G; F)$ as a polyhedron with under graph $G = (V, E)$ and face set F. Let H be a reduction of surface S_{p-1} and, $H \subseteq G$. Denote by C the set of edges on the boundary of H in G and E_C, the set of all edges of G incident to but not in H. Let us extend the relation $\sim_C$: $\forall a, b \in E_C$,

$$a \sim_C b \Leftrightarrow \exists f \in F_H,\ a, b \in \partial_2 f \tag{2}$$

by transitive law as a equivalence. Naturally, $|E_C / \sim_C| \leqslant \phi_H$. Denote by $\{E_i | 1 \leqslant i \leqslant \phi_C\}$ the set of equivalent classes on E_C. Notice that $E_i = \emptyset$ can be missed without loss of generality. Let V_i, $1 \leqslant i \leqslant \phi_C$, be the set of vertices on a path between two edges of E_i in G avoiding boundary vertices. When $E_i = \emptyset$, $V_i = \emptyset$ is missed as well. By the connectedness of G , it is seen that

$$\bigcup_{i=1}^{\phi_C} V_i = V - V_H. \tag{3}$$

If for any $1 \leqslant i < j \leqslant \phi_C$, $V_i \cap V_j = \emptyset$, and all $[V_i]$ planar with all vertices incident to E_i on the boundary of a face, then H, G as well, is said to be with *reducibility* 3.

§3. Reducibility Theorems

Theorem 1 *A graph G can be embedded on a surface $S_g (g \geqslant 1)$ if, and only if, G is with the reducibility* 1.

Proof Necessity. Let $\mu(G)$ be an embedding of G on surface $S_g (g \geqslant 1)$. If $H \in \mathcal{H}_{g-1}$, then $\mu(H)$ is an embedding on $S_g (g \geqslant 1)$ as well. Assume $\{f_i | 1 \leqslant i \leqslant \phi\}$ is the face set of $\mu(H)$, then $G_i = [\partial f_i + E([f_i]_{in})]$, $1 \leqslant i \leqslant \phi$, provide a decomposition satisfied by (1). Easy to show that all G_i, $1 \leqslant i \leqslant \phi$, are planar. And, all the common edges of G_i and H are successively in a face boundary. Thus, G is with reducibility 1.

Sufficiency. Because of G with reducibility 1, let $H \in \mathcal{H}_{g-1}$, assume the embedding $\mu(H)$ of H on surface S_g has ϕ faces. Let G have ϕ subgraphs H_i, $1 \leqslant i \leqslant \phi$, satisfied by (1), and all H_i planar with all common edges of H_i and H in a face boundary. Denote by $\mu_i(H_i)$ a planar embedding of H_i with one face whose boundary is in a face boundary of $\mu(H)$, $1 \leqslant i \leqslant \phi$. Put each $\mu_i(H_i)$ in the corresponding face of $\mu(H)$, an embedding of G on surface $S_g (g \geqslant 1)$ is then obtained. □

Theorem 2 *A graph G can be embedded on a surface $S_g (g \geqslant 1)$ if, and only if, G is with the reducibility* 2.

Proof Necessity. Let $\mu(G) = \Sigma = (G; F)$ be an embedding of G on surface $S_g (g \geqslant 1)$ and $\mu^*(G) = \mu(G^*) = (G^*, F^*)(= \Sigma^*)$, its dual. Given $H \subseteq G$ as a reduction. From the duality between the two polyhedra $\mu(H)$ and $\mu^*(H)$, the interior domain of a face in $\mu(H)$ has at least a vertex of G^*, $G^* - E(H^*)$ has exactly $\phi = |F_{\mu(H)}|$ connected components. Because of each component on a planar disc with all boundary vertices successively on the boundary of the disc, H is with the reducibility 2. Hence, G has the reducibility 2.

Sufficiency. By employing the embedding $\mu(H)$ of reduction H of G on surface $S_g (g \geqslant 1)$ with reducibility 2, put the planar embedding of the dual of each component of $G^* - E(H^*)$ in the corresponding face of $\mu(H)$ in agreement with common boundary, an embedding of $\mu(G)$ on surface $S_g(g \geqslant 1)$ is soon done. □

Theorem 3 *A 3-connected graph G can be embedded on a surface $S_g(g \geqslant 1)$ if, and only if, G is with reducibility 3.*

Proof Necessity. Assume $\mu(G) = (G, F)$ is an embedding of G on surface $S_g(g \geqslant 1)$. Given $H \subseteq G$ as a reduction of surface S_{p-1}. Because of $H \subseteq G$, the restriction $\mu(H)$ of $\mu(G)$ on H is also an embedding of H on surface $S_g(g \geqslant 1)$. From the 3-connectedness of G, edges incident to a face of $\mu(H)$ are as an equivalent class in E_C. Moreover, the subgraph determined by a class is planar with boundary in coincidence, *i.e.*, H has the reducibility 3. Hence, G has the reducibility 3.

Sufficiency. By employing the embedding $\mu(H)$ of the reduction H in G on surface $S_g(g \geqslant 1)$ with the reducibility 3, put each planar embedding of $[V_i]$ in the interior domain of the corresponding face of $\mu(H)$ in agreement with the boundary condition, an embedding $\mu(G)$ of G on $S_g(g \geqslant 1)$ is extended from $\mu(H)$. □

§4. Research Notes

A. On the basis of Theorems 1–3, the surface embeddability of a graph on a surface(orientabl or nonorientable) of genus smaller can be easily found with better efficiency.

For an example, the sphere S_0 has its reductions in two class described as $K_{3,3}$ and K_5. Based on these, the characterizations for the embeddability of a graph on the torus and the projective plane has been established in [4]. Because of the number of distinct embeddings of K_5 and $K_{3,3}$ on torus and projective plane much smaller as shown in the Appendix of [5], the characterizations can be realized by computers with an algorithm much efficiency compared with the existences, *e.g.*, in [7].

B. The three polyhedral forms of Jordan closed planar curve axiom as shown in section 2 initiated from Chapter 4 of [6] are firstly used for surface embeddings of a graph in [4]. However, characterizations in that paper are with a mistake of missing the boundary conditions as shown in this paper.

C. The condition of 3-connectedness in Theorem 3 is not essential. It is only for the simplicity in description.

D. In all of Theorem 1–3, the conditions on planarity can be replaced by the corresponding Jordan curve property as shown in section 2 as in [4] with the attention of the boundary conditions.

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