

## Tangent Space and Derivative Mapping on Time Scale

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**Abstract:** A *pseudo-Euclidean space*, or *Smarandache space* is a pair  $(\mathbf{R}^n, \omega|_{\vec{\mathcal{O}}})$ . In this paper, considering the time scale concept on Smarandache space with  $\omega|_{\vec{\mathcal{O}}}(\vec{u}) = \vec{0}$  for  $\forall \vec{u} \in \mathbf{E}$ , i.e., the Euclidean space, we introduce the tangent vector and some properties according to directional derivative, the delta differentiable vector fields on regular curve parameterized by time scales and the Jacobian matrix of  $\delta$ -completely delta differentiable two variables function.

**Key Words:** Pseudo-Euclidean space, Smarandache space, time scale, regular curve, derivative mapping.

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### §1. Introduction

A *pseudo-Euclidean space*, or *Smarandache space* is a pair  $(\mathbf{R}^n, \omega|_{\vec{\mathcal{O}}})$ , where  $\omega|_{\vec{\mathcal{O}}} : \mathbf{R}^n \rightarrow \mathcal{O}$  is a continuous function, i.e., a straight line with an orientation  $\vec{\mathcal{O}}$  will has an orientation  $\vec{\mathcal{O}} + \omega|_{\vec{\mathcal{O}}}(\vec{u})$  after it passing through a point  $\vec{u} \in \mathbf{E}$ . It is obvious that  $(\mathbf{E}, \omega|_{\vec{\mathcal{O}}}) = \mathbf{E}$  the Euclidean space if and only if  $\omega|_{\vec{\mathcal{O}}}(\vec{u}) = \vec{0}$  for  $\forall \vec{u} \in \mathbf{E}$ , on which calculus of time scales was introduced by Aulbach and Hilger [1,2]. This theory has proved to be useful in the mathematical modeling of several important dynamic processes [3,4,5]. We know that the directional derivative concept is based on for some geometric and physical investigations. It is used at the motion according to direction of particle at the physics [6]. Then, Bohner and Guseinov has been published a paper about the partial differentiation on time scale [7]. Here, authors introduced partial delta and nabla derivative and the chain rule for multivariable functions on time scale and also the concept of the directional derivative. Then, the directional derivative according to the vector field has defined [8]. The general idea in this paper is to investigate some properties of directional derivative. Then, using the directional derivative, we define tangent vector space and delta derivative on vector fields. Finally, we write Jacobian matrix and the  $\delta$ -derivative mapping of the  $\delta$ -completely delta differentiable two variables functions. So our intention is to use several new concepts, which are defined in differential geometry [7].

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## §2. Partial differentiation on time scale

Let  $n \in \mathbf{N}$  be fixed. Further, for each  $i \in \{1, 2, \dots, n\}$  let  $T_i$  denote a time scale, that is,  $T_i$  is a nonempty closed subset of the real number  $\mathbf{R}$ . Let us set

$$\Lambda^n = T_1 x T_2 x \cdots x T_n = \{t = (t_1, t_2, \dots, t_n) \text{ for } t_i \in T_i, 1 \leq i \leq n\}$$

We call  $\Lambda^n$  an  $n$ -dimensional time scale. The set  $\Lambda^n$  is a complete metric space with the metric  $d$  defined by

$$d(t, s) = \sqrt{\sum_{i=1}^n |t_i - s_i|^2}, \quad \text{for } t, s \in \Lambda^n.$$

Let  $\sigma_i$  and  $\rho_i$  denote, respectively, the forward and backward jump operators in  $T_i$ . Remember that for  $u \in T_i$  the forward jump operator  $\sigma_i : T_i \rightarrow T_i$  is defined by

$$\sigma_i(\mu) = \inf\{\nu \in T_i : \nu > \mu\}$$

and the backward jump operator  $\rho_i : T_i \rightarrow T_i$  by

$$\rho_i(\mu) = \inf\{\nu \in T_i : \nu < \mu\}.$$

In this definition we put  $\sigma_i(\max T_i) = \max T_i$  if  $T_i$  has a finite maximum, and  $\rho_i(\min T_i) = \min T_i$  if  $T_i$  has a finite minimum. If  $\sigma_i(\mu) > \mu$ , then we say that  $\mu$  is right-scattered (in  $T_i$ ), while any  $\mu$  with  $\sigma_i(\mu) = \mu$  is left-scattered (in  $T_i$ ). Also, if  $u < \max T_i$  and  $\sigma_i(\mu) = \mu$ , then  $\mu$  is called right-dense (in  $T_i$ ), and if  $\mu > \min T_i$  and  $\rho_i(\mu) = \mu$ , then  $\mu$  is called left-dense (in  $T_i$ ). If  $T_i$  has a left-scattered minimum  $m$ , then we define  $T_i^k = T_i - \{m\}$ , otherwise  $T_i^k = T_i$ . If  $T_i$  has a right-scattered maximum  $M$ , then we define  $(T_i)_k = T_i - \{M\}$ , otherwise  $(T_i)_k = T_i$ .

Let  $f : \Lambda^n \rightarrow \mathbf{R}$  be a function. The partial delta derivative of  $f$  with respect to  $t_i \in T_i^k$  is defined as the limit

$$\lim_{\substack{s_i \rightarrow t_i \\ s_i \neq \sigma_i(t_i)}} \frac{f(t_1, \dots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \dots, t_n) - f(t_1, \dots, t_{i-1}, s_i, t_{i+1}, \dots, t_n)}{\sigma_i(t_i) - s_i} = \frac{\partial f(t)}{\Delta_i t_i}.$$

**Definition 2.1** We say a function  $f : \Lambda^n \rightarrow \mathbf{R}$  is completely delta differentiable at a point  $t^0 = (t_1^0, t_2^0, \dots, t_n^0) \in T_1^k x T_2^k x \cdots x T_n^k$  if there exist numbers  $A_1, A_2, \dots, A_n$  independent of  $t = (t_1, t_2, \dots, t_n) \in \Lambda^n$  (but in general, dependent on  $t^0$ ) such that for all  $t \in U_\delta(t^0)$ ,

$$f(t_1^0, t_2^0, \dots, t_n^0) - f(t_1, t_2, \dots, t_n) = \sum_{i=1}^n A_i(t_i^0 - t_i) + \sum_{i=1}^n \alpha_i(t_i^0 - t_i)$$

and, for each  $j \in \{1, 2, \dots, n\}$  and all  $t \in U_\delta(t^0)$ ,

$$\begin{aligned}
& f(t_1^0, \dots, t_{j-1}^0, \sigma_j(t_j^0), t_{j+1}^0, \dots, t_n^0) - f(t_1, \dots, t_{j-1}, t_j, t_{j+1}, \dots, t_n) \\
&= A_j(\sigma_j(t_j^0) - t_j) + \sum_{\substack{i=1 \\ i \neq j}}^n A_i(t_i^0 - t_i) + \beta_{jj}(\sigma_j(t_j^0) - t_j) + \sum_{\substack{i=1 \\ i \neq j}}^n \beta_{ij}(t_i^0 - t_i), \quad (2-1)
\end{aligned}$$

where  $\delta$  is a sufficiently small positive number,  $U_\delta(t^0)$  is the  $\delta$ -neighborhood of  $t^0$  in  $\Lambda^n$ ,  $\alpha_i = \alpha_i(t^0, t)$  and  $\beta_{ij} = \beta_{ij}(t^0, t)$  are defined on  $U_\delta(t^0)$  such that they are equal to zero at  $t = t^0$  and such that

$$\lim_{t \rightarrow t^0} \alpha_i(t^0, t) = 0 \text{ and } \lim_{t \rightarrow t^0} \beta_{ij}(t^0, t) = 0 \text{ for all } i, j \in \{1, 2, \dots, n\}.$$

**Definition 2.2** We say that a function  $f : T_1 x T \rightarrow \mathbf{R}$  is  $\sigma_1$ -completely delta differentiable at a point  $(t^0, s^0) \in T_1 x T$  if it is completely delta differentiable at that point in the sense of conditions (2.5)-(2.7) (see in [7]) and moreover, along with the numbers  $A_1$  and  $A_2$  presented in (2.5)-(2.7) (see in [7]) there exists also a number  $B$  independent of  $(t, s) \in T_1 x T$  (but, generally dependent on  $(t^0, s^0)$ ) such that

$$\begin{aligned}
& f(\sigma_1(t^0), \sigma_2(s^0)) - f(t, s) \\
&= A_1(\sigma_1(t^0) - t) + B(\sigma_2(s^0) - s) + \gamma_1(\sigma_1(t^0) - t) + \gamma_2(\sigma_2(s^0) - s) \quad (2-2)
\end{aligned}$$

for all  $(t, s) \in V^{\sigma_1}(t^0, s^0)$ , a neighborhood of the point  $(t^0, s^0)$  containing the point  $(\sigma_1(t^0), s^0)$ , and the functions  $\gamma_1 = \gamma_1(t^0, s^0, t, s)$  and  $\gamma_2 = \gamma_2(t^0, s^0, t, s)$  are equal to zero for  $(t, s) = (t^0, s^0)$  and

$$\lim_{(t,s) \rightarrow (t^0, s^0)} \gamma_1 = \gamma_1(t^0, s^0, t, s) \text{ and } \lim_{s \rightarrow s^0} \gamma_2(t^0, s^0, s) = 0. \quad (2-3)$$

Note that in (2-1) the function  $\gamma_2$  depends only on the variable  $s$ . Setting  $s = \sigma_1(s^0)$  in (2-2) yields

$$B = \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s}.$$

Furthermore, let two functions

$$\varphi : T \rightarrow \mathbf{R} \quad \text{and} \quad \psi : T \rightarrow \mathbf{R}$$

be given and let us set

$$\varphi(T) = T_1 \quad \text{and} \quad \psi(T) = T_2.$$

We will assume that  $T_1$  and  $T_2$  are time scales. Denote by  $\sigma_1, \Delta_1$  and  $\sigma_2, \Delta_2$  the forward jump operators and delta operators for  $T_1$  and  $T_2$ , respectively. Take a point  $\xi^0 \in T^k$  and put  $t^0 = \varphi(\xi^0)$  and  $s^0 = \psi(\xi^0)$ .

We will also assume that

$$\varphi(\sigma(\xi^0)) = \sigma_1(\varphi(\xi^0)) \text{ and } \psi(\sigma(\xi^0)) = \sigma_2(\psi(\xi^0)).$$

Under the above assumptions let a function  $f : T_1 x T_2 \rightarrow \mathbf{R}$  be given.

**Theorem 2.1** *Let the function  $f$  be  $\sigma_1$ -completely delta differentiable at the point  $(t^0, s^0)$ . If the functions  $\varphi$  and  $\psi$  have delta derivative at the point  $\xi^0$ , then the composite function*

$$F(\xi) = f(\varphi(\xi), \psi(\xi)) \text{ for } \xi \in T$$

has a delta derivative at that point which is expressed by the formula

$$F^\Delta(\xi^0) = \frac{\partial f(t^0, s^0)}{\Delta_1 t} \varphi^\Delta(\xi^0) + \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s} \psi^\Delta(\xi^0).$$

*Proof* The proof can be seen in the reference [7]. □

**Theorem 2.2** *Let the function  $f$  be  $\sigma_1$ -completely delta differentiable at the point  $(t^0, s^0)$ . If the functions  $\varphi$  and  $\psi$  have first order partial delta derivative at the point  $(\xi^0, \eta^0)$ , then the composite function*

$$F(\xi, \eta) = f(\varphi(\xi, \eta), \psi(\xi, \eta)) \text{ for } (\xi, \eta) \in T_{(1)} x T_{(2)}$$

has the first order partial delta derivatives at  $(\xi^0, \eta^0)$  which are expressed by the formulas

$$\frac{\partial F(\xi^0, \eta^0)}{\Delta_{(1)} \xi} = \frac{\partial f(t^0, s^0)}{\Delta_1 t} \frac{\partial \varphi(\xi^0, \eta^0)}{\Delta_{(1)} \xi} + \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s} \frac{\partial \psi(\xi^0, \eta^0)}{\Delta_{(1)} \xi}$$

and

$$\frac{\partial F(\xi^0, \eta^0)}{\Delta_{(2)} \xi} = \frac{\partial f(t^0, s^0)}{\Delta_1 t} \frac{\partial \varphi(\xi^0, \eta^0)}{\Delta_{(2)} \eta} + \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s} \frac{\partial \psi(\xi^0, \eta^0)}{\Delta_{(2)} \xi}$$

*Proof* The proof can be also seen in the reference [7]. □

### §3. The directional derivative

Let  $T$  be a time scale with the forward jump operator  $\sigma$  and the delta operator  $\Delta$ . We will assume that  $0 \in T$ . Further, let  $w = (w_1, w_2) \in \mathbf{R}^2$  be a unit vector and let  $(t^0, s^0)$  be a fixed point in  $\mathbf{R}^2$ . Let us set

$$T_1 = \{t = t^0 + \xi w_1 : \xi \in T\} \text{ and } T_2 = \{s = s^0 + \xi w_2 : \xi \in T\}.$$

Then  $T_1$  and  $T_2$  are time scales and  $t^0 \in T_1, s^0 \in T_2$ . Denote the forward jump operators of  $T_1$  and the delta operators by  $\Delta_1$ .

**Definition 3.1** Let a function  $f : T_1xT_2 \rightarrow \mathbf{R}$  be given. The directional delta derivative of the function  $f$  at the point  $(t^0, s^0)$  in the direction of the vector  $w$  (along  $w$ ) is defined as the number

$$\frac{\partial f(t^0, s^0)}{\Delta w} = F^\Delta(0)$$

provided it exists, where

$$F(\xi) = f(t^0 + \xi w_1, s^0 + \xi w_2) \text{ for } \xi \in T.$$

**Theorem 3.1** Suppose that the function  $f$  is  $\sigma_1$ -completely delta differentiable at the point  $(t^0, s^0)$ . Then the directional delta derivative of  $f$  at  $(t^0, s^0)$  in the direction of the  $w$  exists and is expressed by the formula

$$\frac{\partial f(t^0, s^0)}{\Delta w} = \frac{\partial f(t^0, s^0)}{\Delta_1 t} w_1 + \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s} w_2.$$

*Proof* The proof can be found in the reference [7]. □

#### §4. The tangent vector in $\Lambda^n$ and some properties

Let us consider the Cartesian product

$$\Lambda^n = T_1xT_2x \cdots xT_n = \{P = (x_1, x_2, \dots, x_n) \text{ for } x_i \in T_i\}$$

where  $T_i$  are defined time scale for all  $1 \leq i \leq n$ ,  $n \in \mathbf{N}$ . We call  $\Lambda^n$  an  $n$ -dimensional Euclidean space on time scale.

Let  $x_i : \Lambda^n \rightarrow T_i$  be Euclidean coordinate functions on time scale for all  $1 \leq i \leq n$ ,  $n \in \mathbf{N}$ , denoted by the set  $\{x_1, x_2, \dots, x_n\}$ . Let  $f : \Lambda^n \rightarrow \Lambda^m$  be a function described by  $f(P) = (f_1(P), f_2(P), \dots, f_m(P))$  at a point  $P \in \Lambda^n$ . The function  $f$  is called  $\sigma_1$ -completely delta differentiable function at the point  $P$  provided that, all  $f_i$ ,  $i = 1, 2, \dots, m$  functions are  $\sigma_1$ -completely delta differentiable at the point  $P$ . All this kind of functions set will denoted by  $\mathbf{C}_{\sigma_1}^\Delta$ .

Let  $P \in \Lambda^n$  and  $\{(P, v) = v_P, P \in \Lambda^n\}$  be the set of tangent vectors at the point  $P$  denoted by  $V_P(\Lambda^n)$ . Now, we find following properties on this set.

**Theorem 4.1** Let  $a, b \in \mathbf{R}$ ,  $f, g \in \mathbf{C}_{\sigma_1}^\Delta$  and  $v_P, w_P, z_P \in V_P(\Lambda^2)$ . Then, the following properties are proven on the directional derivative.

$$(i) \quad \frac{\partial f(t^0, s^0)}{\Delta(av_P + bw_P)} = a \frac{\partial f(t^0, s^0)}{\Delta v_P} + b \frac{\partial f(t^0, s^0)}{\Delta w_P},$$

$$(ii) \quad \frac{\partial (af + bg)(t^0, s^0)}{\Delta v_P} = a \frac{\partial f(t^0, s^0)}{\Delta v_P} + b \frac{\partial g(t^0, s^0)}{\Delta v_P},$$

$$(iii) \quad \frac{\partial(fg)(t^0, s^0)}{\Delta v_P} = g(\sigma_1(t^0), s^0) \frac{\partial f(t^0, s^0)}{\Delta v_P} + f(\sigma_1(t^0), \sigma_2(s^0)) \frac{\partial g(t^0, s^0)}{\Delta v_P} \\ - \mu_1(t^0) \frac{\partial f(t^0, s^0)}{\Delta_1 t} \frac{\partial g(t^0, s^0)}{\Delta_1 t} v_1 - \mu_2(s^0) \frac{\partial g(t^0, s^0)}{\Delta_1 t} \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s} v_1.$$

*Proof* Considering Definition 3.1 and Theorem 3.1, we get easily (i) and (ii). Then by Theorem 3.1, we have

$$\begin{aligned} \frac{\partial(fg)(t^0, s^0)}{\Delta v_P} &= \frac{\partial(fg)(t^0, s^0)}{\Delta_1 t} v_1 + \frac{\partial(fg)(\sigma_1(t^0), s^0)}{\Delta_2 s} v_2 \\ &= \left( \frac{\partial f(t^0, s^0)}{\Delta_1 t} g(t^0, s^0) + f(\sigma_1(t^0), s^0) \frac{\partial g(t^0, s^0)}{\Delta_1 t} \right) v_1 \\ &\quad + \left( \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s} g(\sigma_1(t^0), s^0) + f(\sigma_1(t^0), \sigma_2(s^0)) \frac{\partial g(\sigma_1(t^0), s^0)}{\Delta_2 s} \right) v_2 \\ &= \frac{\partial f(t^0, s^0)}{\Delta_1 t} g(t^0, s^0) v_1 + \frac{\partial f(t^0, s^0)}{\Delta_1 t} g(\sigma_1(t^0), s^0) v_1 \\ &\quad - \frac{\partial f(t^0, s^0)}{\Delta_1 t} g(\sigma_1(t^0), s^0) v_1 + \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s} g(\sigma_1(t^0), s^0) v_2 \\ &\quad + f(\sigma_1(t^0), s^0) \frac{\partial g(t^0, s^0)}{\Delta_1 t} v_1 + f(\sigma_1(t^0), \sigma_2(s^0)) \frac{\partial g(t^0, s^0)}{\Delta_1 t} v_1 \\ &\quad - f(\sigma_1(t^0), \sigma_2(s^0)) \frac{\partial g(t^0, s^0)}{\Delta_1 t} v_1 + f(\sigma_1(t^0), \sigma_2(s^0)) \frac{\partial g(\sigma_1(t^0), s^0)}{\Delta_2 s} v_2 \\ &= g(\sigma_1(t^0), s^0) \left( \frac{\partial f(t^0, s^0)}{\Delta_1 t} v_1 + \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s} v_2 \right) \\ &\quad + \frac{\partial f(t^0, s^0)}{\Delta_1 t} v_1 (g(t^0, s^0) - g(\sigma_1(t^0), s^0)) \\ &\quad + f(\sigma_1(t^0), \sigma_2(s^0)) \left( \frac{\partial g(t^0, s^0)}{\Delta_1 t} v_1 + \frac{\partial g(\sigma_1(t^0), s^0)}{\Delta_2 s} v_2 \right) \\ &\quad - \frac{\partial g(t^0, s^0)}{\Delta_1 t} v_1 (f(\sigma_1(t^0), \sigma_2(s^0)) - f(\sigma_1(t^0), s^0)) \\ &= g(\sigma_1(t^0), s^0) \left( \frac{\partial f(t^0, s^0)}{\Delta v_P} \right) + f(\sigma_1(t^0), \sigma_2(s^0)) \left( \frac{\partial g(t^0, s^0)}{\Delta v_P} \right) \\ &\quad - \mu_1(t^0) \frac{\partial f(t^0, s^0)}{\Delta_1 t} \frac{\partial g(t^0, s^0)}{\Delta_1 t} v_1 - \mu_2(s^0) \frac{\partial g(t^0, s^0)}{\Delta_1 t} \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s} v_1 \quad \square \end{aligned}$$

## §5. The parameter mapping and delta derivative of vector field

### along a regular curve

**Definition 5.1** A  $\Delta$ -regular curve (or an arc of a  $\Delta$ -regular curve)  $f$  is defined as a mapping

$$x_1 = f_1(t), x_2 = f_2(t), \dots, x_n = f_n(t), t \in [a, b]$$

of the segment  $[a, b] \subset T$ ,  $a < b$ , to the space  $\mathbf{R}^n$ , where  $f_1, f_2, \dots, f_n$  are real-valued functions defined on  $[a, b]$  that are  $\Delta$ -differentiable on  $[a, b]^k$  with rd-continuous  $\Delta$ -derivatives and

$$|f_1^\Delta(t)|^2 + |f_2^\Delta(t)|^2 + \dots + |f_n^\Delta(t)|^2 \neq 0, \quad t \in [a, b]^k.$$

**Definition 5.2** Let  $f : T \rightarrow \Lambda^n$  be a  $\Delta$ -differentiable regular curve on  $[a, b]^k$ . Let  $T$  and  $\bar{T}$  be a time scales. A parameter mapping  $h : \bar{T} \rightarrow T$  of the curve  $f$  is defined as  $t = h(s)$  when  $h$  and  $h^{-1}$  are  $\Delta$ -differentiable functions.

Thus, according to this new parameter,  $f$  can be written as follows:

$$g : \bar{T} \rightarrow \Lambda^n, \quad g(s) = f(h(s)).$$

**Theorem 5.1** Let  $f : T \rightarrow \Lambda^n$  be a  $\Delta$ -differentiable regular curve. Let the function  $h : \bar{T} \rightarrow T$  be parameter map of  $f$  and be  $g = f \circ h$ . The  $\Delta$ -derivative of  $g$  is expressed by the formula

$$g^\Delta(s) = \frac{df(h(s))}{\widetilde{\Delta}t} h^\Delta(s).$$

*Proof* Let  $f(t) = (f_1(t), f_2(t), \dots, f_n(t))$  be a regular curve given by the vectorial form in Euclidean space  $\Lambda^n$ . Let  $h$  be a parameter mapping of  $f$ . Considering chain rule for all  $f_i$ , we get

$$\begin{aligned} g^\Delta(s) &= \left( \frac{d(f_1 \circ h)}{\Delta(s)}, \frac{d(f_2 \circ h)}{\Delta(s)}, \dots, \frac{d(f_n \circ h)}{\Delta(s)} \right) \\ &= \left( \frac{df_1(h(s))}{\widetilde{\Delta}t} h^\Delta(s), \frac{df_2(h(s))}{\widetilde{\Delta}t} h^\Delta(s), \dots, \frac{df_n(h(s))}{\widetilde{\Delta}t} h^\Delta(s) \right) \\ &= \left( \frac{df_1(h(s))}{\widetilde{\Delta}t}, \frac{df_2(h(s))}{\widetilde{\Delta}t}, \dots, \frac{df_n(h(s))}{\widetilde{\Delta}t} \right) h^\Delta(s) \\ &= \frac{df(h(s))}{\widetilde{\Delta}t} h^\Delta(s). \quad \square \end{aligned}$$

**Definition 5.3** A vector field  $Z$  is a function which is associated a tangent vector to each point of  $\Lambda^n$  and so  $Z(P)$  belongs to the set of tangent vector space  $V_P(\Lambda^n)$  at the point  $P$ . Generally, a vector field is denoted by

$$Z(P) = \sum_{i=1}^n g_i(t) \frac{\partial}{\partial x_i} \Big|_P$$

where  $g_i(t)$  are real valued functions defined on  $T = [a, b]$ , that are  $\Delta$ -differentiable on  $[a, b]^k$  with rd-continuous  $\Delta$ -derivative.

Let a function  $f : T \rightarrow \Lambda^n$ ,  $f(t) = (f_1(t), f_2(t), \dots, f_n(t))$  be a  $\Delta$ -differentiable curve and  $Z(P)$  be a vector field along it. Thus, we define  $\Delta$ -derivative of vector fields as follows:

**Definition 5.4** Let  $Z(P) = \sum_{i=1}^n g_i(t) \frac{\partial}{\partial x_i} |_P$  be a vector field given along the curve  $f$ . The  $\Delta$ -derivative of the function with respect to the parameter is defined as

$$Z^\Delta(P) = \frac{dZ}{\Delta t} |_P = \sum_{i=1}^n \lim_{s \rightarrow t} \frac{g_i(\sigma(t)) - g_i(s)}{\sigma(t) - s} \frac{\partial}{\partial x_i} |_P = \sum_{i=1}^n g_i^\Delta \frac{\partial}{\partial x_i} |_P.$$

**Theorem 5.2** Let  $f$  be a curve given by  $f : T \rightarrow \Lambda^n$  and  $h : \bar{T} \rightarrow T$  be a parameter mapping of  $f$ . Let  $Z(P) = \sum_{i=1}^n g_i(t) \frac{\partial}{\partial x_i} |_{P \in f(t)}$  be a vector field given along the curve  $f$ . The  $\Delta$ -derivative of the function  $Z(P)$  according to the new parameter  $s$  can be written as:

$$\left( \frac{dZ}{\Delta s} \right) = \left( \frac{dZ}{\widetilde{\Delta s}} \right) \left( \frac{dh}{\Delta s} \right).$$

*Proof* When we consider the chain rule on the real valued function for all the  $\Delta$ -differentiable functions  $g_i(t)$ , we prove Theorem 5.2.  $\square$

**Definition 5.5** Let a vector field  $Z(P) = \sum_{i=1}^n g_i(t) \frac{\partial}{\partial x_i} |_{P \in f(t)}$  be given along a curve  $f$ . If  $\frac{dZ}{\Delta t} = 0$ ,  $Z(P)$  is called constant vector field along the curve  $f$ .

**Theorem 5.3** Let  $Y$  and  $Z$  be two vector fields along the curve  $f(t)$  and  $h : T \rightarrow \mathbf{R}$  be a  $\Delta$ -differentiable function. Then,

- (i)  $(Y + Z)^\Delta(t) = (Y)^\Delta(t) + (Z)^\Delta(t)$ ;
- (ii)  $(hZ)^\Delta(t) = (h(\sigma(t)))(Z)^\Delta(t) + h^\Delta(t)Z(t)$ ;
- (iii)  $\langle Y, Z \rangle^\Delta(t) = \langle Y^\Delta(t), Z(\sigma(t)) \rangle + \langle Y(t), Z^\Delta(t) \rangle$ .

where  $\langle , \rangle$  is the inner product between the vector fields  $Y$  and  $Z$ .

*Proof* The (i) is obvious. Let  $Y(t) = \sum_{i=1}^n k_i(t) \frac{\partial}{\partial x_i} |_{f(t)}$  and  $Z(t) = \sum_{i=1}^n g_i(t) \frac{\partial}{\partial x_i} |_{f(t)}$  be two vector fields. Then

$$\begin{aligned} (hZ)^\Delta(t) &= \sum_{i=1}^n \lim_{s \rightarrow t} \frac{(hg_i)(\sigma(t)) - (hg_i)(s)}{\sigma(t) - s} \frac{\partial}{\partial x_i} |_{f(t)} \\ &= \sum_{i=1}^n \lim_{s \rightarrow t} \frac{h(\sigma(t))g_i(\sigma(t)) - h(s)g_i(s)}{\sigma(t) - s} \frac{\partial}{\partial x_i} |_{f(t)} \\ &= \sum_{i=1}^n \lim_{s \rightarrow t} \frac{h(\sigma(t))g_i(\sigma(t)) - h(\sigma(t))g_i(s) + h(\sigma(t))g_i(s) - h(s)g_i(s)}{\sigma(t) - s} \frac{\partial}{\partial x_i} |_{f(t)} \\ &= \sum_{i=1}^n \lim_{s \rightarrow t} \frac{h(\sigma(t))[g_i(\sigma(t)) - g_i(s)] + [h(\sigma(t)) - h(s)]g_i(s)}{\sigma(t) - s} \frac{\partial}{\partial x_i} |_{f(t)} \\ &= h(\sigma(t)) \sum_{i=1}^n \lim_{s \rightarrow t} \frac{g_i(\sigma(t)) - g_i(s)}{\sigma(t) - s} \frac{\partial}{\partial x_i} |_{f(t)} + \lim_{s \rightarrow t} \frac{h(\sigma(t)) - h(s)}{\sigma(t) - s} \sum_{i=1}^n g_i(s) \frac{\partial}{\partial x_i} |_{f(t)} \\ &= (h(\sigma(t)))(Z)^\Delta(t) + h^\Delta(t)Z(t). \end{aligned}$$



That is the formula (ii). For (iii), we have

$$\begin{aligned}
\langle Y, Z \rangle^\Delta(t) &= \sum_{i=1}^n \lim_{s \rightarrow t} \frac{(k_i g_i)(\sigma(t)) - (k_i g_i)(s)}{\sigma(t) - s} \frac{\partial}{\partial x_i} \Big|_{f(t)} \\
&= \sum_{i=1}^n \lim_{s \rightarrow t} \frac{k_i(\sigma(t))g_i(\sigma(t)) - k_i(s)g_i(s)}{\sigma(t) - s} \frac{\partial}{\partial x_i} \Big|_{f(t)} \\
&= \sum_{i=1}^n \lim_{s \rightarrow t} \frac{k_i(\sigma(t))g_i(\sigma(t)) - k_i(s)g_i(\sigma(t)) + k_i(s)g_i(\sigma(t)) - k_i(s)g_i(s)}{\sigma(t) - s} \frac{\partial}{\partial x_i} \Big|_{f(t)} \\
&= \sum_{i=1}^n \lim_{s \rightarrow t} \frac{k_i(\sigma(t))g_i(\sigma(t)) - k_i(s)g_i(\sigma(t))}{\sigma(t) - s} \frac{\partial}{\partial x_i} \Big|_{f(t)} \\
&\quad + \sum_{i=1}^n \lim_{s \rightarrow t} \frac{k_i(s)g_i(\sigma(t)) - k_i(s)g_i(s)}{\sigma(t) - s} \frac{\partial}{\partial x_i} \Big|_{f(t)} \\
&= \sum_{i=1}^n \lim_{s \rightarrow t} \frac{k_i(\sigma(t)) - k_i(s)}{\sigma(t) - s} g_i(\sigma(t)) + \sum_{i=1}^n \lim_{s \rightarrow t} k_i(s) \lim_{s \rightarrow t} \frac{g_i(\sigma(t)) - g_i(s)}{\sigma(t) - s} \\
&= \langle Y^\Delta(t), Z(\sigma(t)) \rangle + \langle Y(t), Z^\Delta(t) \rangle.
\end{aligned}$$

This completes the proof.  $\square$

**Definition 5.6** Let  $f : \Lambda^2 \rightarrow \Lambda^m$ ,  $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2), \dots, f_m(x_1, x_2))$  be a  $\sigma_1$ -completely delta differentiable function at the point  $P(t^0, s^0) \in \Lambda^2$ . For any tangent vector  $v_P \in V_P(\Lambda^2)$ , the  $\Delta$ -derivative mapping at the point  $P(t^0, s^0)$  of  $f$  is defined by

$$f_{*P}^\Delta(v_P) = \left( \frac{\partial f_1(x_1, x_2)}{\Delta v_P}, \frac{\partial f_2(x_1, x_2)}{\Delta v_P}, \dots, \frac{\partial f_m(x_1, x_2)}{\Delta v_P} \right)_{f(x_1, x_2)}.$$

$f_{*P}^\Delta(v_P)$  is a function from the tangent vector space  $V_P(\Lambda^2)$  to tangent vector space  $V_P(\Lambda^m)$ .

**Theorem 5.4** The function  $f_{*P}$  is linear mapping.

*Proof* Let us prove the linearity for any  $a \in \mathbf{R}$  and for any two tangent vectors  $v_P, w_P \in V_P(\Lambda^2)$ . In fact,

$$\begin{aligned}
f_{*P}^\Delta(av_P + bw_P) &= \left( \frac{\partial f_1(x_1, x_2)}{\Delta(av_P + bw_P)}, \frac{\partial f_2(x_1, x_2)}{\Delta(av_P + bw_P)}, \dots, \frac{\partial f_m(x_1, x_2)}{\Delta(av_P + bw_P)} \right)_{f(x_1, x_2)} \\
&= \left( a \frac{\partial f_1(x_1, x_2)}{\Delta v_P} + b \frac{\partial f_1(x_1, x_2)}{\Delta w_P}, \dots, a \frac{\partial f_m(x_1, x_2)}{\Delta v_P} + b \frac{\partial f_m(x_1, x_2)}{\Delta w_P} \right)_{f(x_1, x_2)} \\
&= a \left( \frac{\partial f_1(x_1, x_2)}{\Delta v_P}, \dots, \frac{\partial f_m(x_1, x_2)}{\Delta v_P} \right)_{f(x_1, x_2)} \\
&\quad + b \left( \frac{\partial f_1(x_1, x_2)}{\Delta w_P}, \dots, \frac{\partial f_m(x_1, x_2)}{\Delta w_P} \right)_{f(x_1, x_2)} \\
&= af_{*P}^\Delta(v_P) + bf_{*P}^\Delta(w_P).
\end{aligned}$$

Thus the proof is completed.  $\square$

**Definition 5.7** Let  $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2), \dots, f_n(x_1, x_2))$  be a  $\sigma_1$ -completely delta differentiable function at the point  $P(x_1, x_2) \in \Lambda^2$ . The Jacobian matrix of  $f$  is defined by

$$J(f, P) = \begin{bmatrix} \frac{\partial f_1(x_1, x_2)}{\partial \Delta_1 x_1} & \frac{\partial f_1(\sigma_1(x_1), x_2)}{\partial \Delta_1 x_1} \\ \frac{\partial f_2(x_1, x_2)}{\partial \Delta_1 x_1} & \frac{\partial f_2(\sigma_1(x_1), x_2)}{\partial \Delta_1 x_1} \\ \dots & \dots \\ \frac{\partial f_m(x_1, x_2)}{\partial \Delta_1 x_1} & \frac{\partial f_m(\sigma_1(x_1), x_2)}{\partial \Delta_1 x_1} \end{bmatrix}_{m \times 2}$$

**Theorem 5.5** Let  $f : \Lambda^2 \rightarrow \Lambda^n$  be a  $\sigma_1$ -completely delta differentiable function. The  $\Delta$ -derivative mapping for any  $P \in \Lambda^2$  and  $w_P = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in V_P(\Lambda^2)$  is expressed by the formula

$$f_{*P}^\Delta(w_P) = (J(f, P), w_P)^T.$$

*Proof* Theorem 5.5 is proven considering Definitions 5.6, 5.7 and Theorem 3.1.  $\square$

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