

A NUMBER THEORETIC FUNCTION AND ITS MEAN VALUE PROPERTY*

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ABSTRACT. Let p be a prime, n be any positive integer, $\alpha(n, p)$ denotes the power of p in the factorization of $n!$. In this paper, we study the asymptotic properties of the mean value $\sum_{p \leq n} \alpha(n, p)$, and give an interesting asymptotic formula for it.

1. INTRODUCTION

Let p be a prime, $e_p(n)$ denotes the largest exponent (of power p) which divides n , $\alpha(n, p) = \sum_{k \leq n} e_p(k)$. In problem 68 of [1], Professor F.Smarandach asked us to study the properties of the sequences $e_p(n)$. This problem is interesting because there are close relations between $e_p(n)$ and the factorization of $n!$. In fact, $\alpha(n, p)$ is the power of p in the factorization of $n!$. In this paper, we use the elementary methods to study the asymptotic properties of the mean value $\sum_{p \leq n} \alpha(n, p)$, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. *For any prime p and any fixed positive integer n , we have the asymptotic formula*

$$\sum_{p \leq n} \alpha(n, p) = n \ln \ln n + cn + c_1 \frac{n}{\ln n} + c_2 \frac{n}{\ln^2 n} + \cdots + c_k \frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n}\right).$$

where k is any fixed positive integer, c_i ($i = 1, 2, \dots$) are some computable constants.

2. PROOF OF THE THEOREM

In this section, we complete the proof of the Theorem. First for any prime p and any fixed positive integer n , we let $a(n, p)$ denote the sum of the base p digits of n . That is, if $n = a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 0$, where $1 \leq a_i \leq p-1$, $i = 1, 2, \dots, s$, then $a(n, p) = \sum_{i=1}^s a_i$, and for this number theoretic function, we have the following two simple Lemmas:

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Lemma 1. For any integer $n \geq 1$, we have the identity

$$\alpha_p(n) \equiv \alpha(n) \equiv \sum_{i=1}^{+\infty} \left[\frac{n}{p^i} \right] = \frac{1}{p-1} (n - a(n, p)),$$

where $[x]$ denotes the greatest integer not exceeding x .

Proof. From the properties of $[x]$ we know that

$$\begin{aligned} \left[\frac{n}{p^i} \right] &= \left[\frac{a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s}}{p^i} \right] \\ &= \begin{cases} \sum_{j=k}^s a_j p^{\alpha_j - i}, & \text{if } \alpha_{k-1} < i \leq \alpha_k \\ 0, & \text{if } i > \alpha_s. \end{cases} \end{aligned}$$

So from this formula we have

$$\begin{aligned} \alpha(n) &\equiv \sum_{i=1}^{+\infty} \left[\frac{n}{p^i} \right] = \sum_{i=1}^{+\infty} \left[\frac{a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s}}{p^i} \right] \\ &= \sum_{j=1}^s \sum_{k=1}^{\alpha_j} a_j p^{\alpha_j - k} = \sum_{j=1}^s a_j (1 + p + p^2 + \cdots + p^{\alpha_j - 1}) \\ &= \sum_{j=1}^s a_j \cdot \frac{p^{\alpha_j} - 1}{p - 1} = \frac{1}{p - 1} \sum_{j=1}^s (a_j p^{\alpha_j} - a_j) \\ &= \frac{1}{p - 1} (n - a(n, p)). \end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2. For any positive integer n , we have the estimate

$$a(n, p) \leq \frac{p-1}{\ln p} \ln n.$$

Proof. Let $n = a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 0$, where $1 \leq a_i \leq p-1$, $i = 1, 2, \dots, s$. Then from the definition of $a(n, p)$ we have

$$(1) \quad a(n, p) = \sum_{i=1}^s a_i \leq \sum_{i=1}^s (p-1) = (p-1)s.$$

On the other hand, using the mathematical induction we can easily get the inequality

$$n = a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s} \geq a_s p^s,$$

or

$$(2) \quad s \leq \frac{\ln(n/a_s)}{\ln p} \leq \frac{\ln n}{\ln p}.$$

Combining (1) and (2) we immediately get the estimate

$$a(n, p) \leq \frac{p-1}{\ln p} \ln n.$$

This proves Lemma 2.

Now we use Lemma 1 and Lemma 2 to complete the proof of the Theorem. First, we separate the summation in the Theorem into two parts.

$$(3) \quad \sum_{p \leq n} \alpha(n, p) = \sum_{p \leq \sqrt{n}} \alpha(n, p) + \sum_{\sqrt{n} < p \leq n} \alpha(n, p).$$

For the first part, from Lemma 1 we have

$$\begin{aligned} \sum_{p \leq \sqrt{n}} \alpha(n, p) &= \sum_{p \leq \sqrt{n}} \frac{1}{p-1} (n - a(n, p)) \\ &= n \sum_{p \leq \sqrt{n}} \left(\frac{1}{p} + \frac{1}{p(p-1)} \right) - \sum_{p \leq \sqrt{n}} \frac{a(n, p)}{p-1} \\ &= n \left(\sum_{p \leq \sqrt{n}} \frac{1}{p} + \sum_p \frac{1}{p(p-1)} + O \left(\sum_{m > \sqrt{n}} \frac{1}{m^2} \right) \right) - \sum_{p \leq \sqrt{n}} \frac{a(n, p)}{p-1} \\ (4) \quad &= n \left(\int_{\frac{3}{2}}^{\sqrt{n}} \frac{1}{x} d\pi(x) + A + O \left(\frac{1}{\sqrt{n}} \right) \right) - \sum_{p \leq \sqrt{n}} \frac{a(n, p)}{p-1}. \end{aligned}$$

where $\pi(x)$ denotes the number of all prime not exceeding x . For $\pi(x)$, we have the asymptotic formula

$$(5) \quad \pi(x) = \frac{x}{\ln x} + a_2 \frac{x}{\ln^2 x} + \cdots + a_k \frac{x}{\ln^k x} + O \left(\frac{x}{\ln^{k+1} x} \right)$$

and

$$\begin{aligned} \int_{\frac{3}{2}}^{\sqrt{n}} \frac{1}{x} d\pi(x) &= \frac{\pi(\sqrt{n})}{\sqrt{n}} + \int_{\frac{3}{2}}^{\sqrt{n}} \frac{\pi(x)}{x^2} dx \\ &= \frac{1}{\ln \sqrt{n}} + \frac{a_2}{\ln^2 \sqrt{n}} + \cdots + \frac{a_k}{\ln^k \sqrt{n}} + O \left(\frac{1}{\ln^{k+1} \sqrt{n}} \right) + \int_{\frac{3}{2}}^{\sqrt{n}} \frac{1}{x \ln x} dx \\ &\quad + a_2 \int_{\frac{3}{2}}^{\sqrt{n}} \frac{1}{x \ln^2 x} dx + \cdots + a_{k+1} \int_{\frac{3}{2}}^{\sqrt{n}} \frac{1}{x \ln^{k+1} x} dx + O \left(\frac{1}{\ln^{k+1} n} \right) \\ &= \frac{a_{11}}{\ln n} + \frac{a_{12}}{\ln^2 n} + \cdots + \frac{a_{1k}}{\ln^k n} + \ln \ln n + B + \frac{a_{21}}{\ln n} + \frac{a_{22}}{\ln^2 n} \\ &\quad + \cdots + \frac{a_{2k}}{\ln^k n} + O \left(\frac{1}{\ln^{k+1} n} \right) \\ (6) \quad &= \ln \ln n + B + \frac{a_{31}}{\ln n} + \frac{a_{32}}{\ln^2 n} + \cdots + \frac{a_{3k}}{\ln^k n} + O \left(\frac{1}{\ln^{k+1} n} \right). \end{aligned}$$

From Lemma 2 we have

$$(7) \quad \sum_{p \leq \sqrt{n}} \frac{a(n, p)}{p-1} \leq \sum_{p \leq \sqrt{n}} \frac{\ln n}{\ln p} = \ln n \sum_{p \leq \sqrt{n}} \frac{1}{\ln p} \leq \ln n \sum_{p \leq \sqrt{n}} 1 \leq \sqrt{n} \ln n.$$

Combining (4), (6) and (7) we obtain

$$(8) \quad \begin{aligned} \sum_{p \leq \sqrt{n}} \alpha(n, p) &= n \ln \ln n + c_0 n + a_{31} \frac{n}{\ln n} + a_{32} \frac{n}{\ln^2 n} \\ &+ \cdots + a_{3k} \frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n}\right). \end{aligned}$$

For the second part, we have

$$(9) \quad \begin{aligned} \sum_{\sqrt{n} < p \leq n} \alpha(n, p) &= \sum_{\sqrt{n} < p \leq n} \sum_{i=1}^{+\infty} \left[\frac{n}{p^i} \right] = \sum_{\sqrt{n} < p \leq n} \left[\frac{n}{p} \right] = \sum_{\sqrt{n} < p \leq n} \sum_{m \leq \frac{n}{p}} 1 \\ &= \sum_{m \leq \sqrt{n}} \sum_{\sqrt{n} < p \leq \frac{n}{m}} 1 = \sum_{m \leq \sqrt{n}} \left(\pi\left(\frac{n}{m}\right) - \pi(\sqrt{n}) \right) \\ &= \sum_{m \leq \sqrt{n}} \pi\left(\frac{n}{m}\right) - [\sqrt{n}] \pi(\sqrt{n}). \end{aligned}$$

Applying Euler's summation formula (see [2] Theorem 3.1) and the expansion into power-series we have

$$\begin{aligned} \sum_{m \leq \sqrt{n}} \frac{1}{m(\ln n - \ln m)^r} &= \sum_{m \leq \sqrt{n}} \frac{1}{m \ln^r n \left(1 - \frac{\ln m}{\ln n}\right)^r} \\ &= \sum_{s=0}^{+\infty} \sum_{m \leq \sqrt{n}} \frac{\binom{r-1+s}{r-1} \ln^s m}{m \ln^{s+r} n} \\ &= \sum_{s=0}^{+\infty} \binom{r-1+s}{r-1} \left(\sum_{m \leq \sqrt{n}} \frac{\ln^s m}{m \ln^{s+r} n} \right) \\ &= \sum_{s=0}^{+\infty} \frac{\binom{r-1+s}{r-1}}{\ln^{s+r} n} \left(\frac{\ln^{s+1} n}{(s+1)2^{s+1}} + d_{s+1} + O\left(\frac{\ln^s n}{2^s \sqrt{n}}\right) \right) \\ &= \sum_{i=r-1}^k \frac{d_{1i}}{\ln^i n} + O\left(\frac{\ln^s n}{\sqrt{n}}\right). \end{aligned}$$

From this and (5) we get

$$\begin{aligned}
 & \sum_{m \leq \sqrt{n}} \pi\left(\frac{n}{m}\right) \\
 &= \sum_{m \leq \sqrt{n}} \left(\frac{\frac{n}{m}}{\ln\left(\frac{n}{m}\right)} + a_2 \frac{\frac{n}{m}}{\ln^2\left(\frac{n}{m}\right)} + \cdots + a_{k+1} \frac{\frac{n}{m}}{\ln^{k+1}\left(\frac{n}{m}\right)} + O\left(\frac{\frac{n}{m}}{\ln^{k+2}\left(\frac{n}{m}\right)}\right) \right) \\
 &= n \sum_{m \leq \sqrt{n}} \left(\frac{1}{m(\ln n - \ln m)} + a_2 \frac{1}{m(\ln n - \ln m)^2} + \cdots \right. \\
 &\quad \left. + a_{k+1} \frac{1}{m(\ln n - \ln m)^{k+1}} + O\left(\frac{1}{m(\ln n - \ln m)^{k+2}}\right) \right) \\
 &= n \left(b_0 + \frac{b_1}{\ln n} + \frac{b_2}{\ln^2 n} + \cdots + \frac{b_k}{\ln^k n} + O\left(\frac{1}{\ln^{k+1} n}\right) \right) \\
 (10) \quad &= b_0 n + b_1 \frac{n}{\ln n} + b_2 \frac{n}{\ln^2 n} + \cdots + b_k \frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n}\right)
 \end{aligned}$$

and

$$\begin{aligned}
 (\sqrt{n})\pi(\sqrt{n}) &= \frac{n}{\ln \sqrt{n}} + a_2 \frac{n}{\ln^2 \sqrt{n}} + \cdots + a_k \frac{n}{\ln^k \sqrt{n}} + O\left(\frac{n}{\ln^{k+1} \sqrt{n}}\right) \\
 (11) \quad &= a_{41} \frac{n}{\ln n} + a_{42} \frac{n}{\ln^2 n} + \cdots + a_{4k} \frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n}\right).
 \end{aligned}$$

Combining (9), (10) and (11) we have

$$(12) \quad \sum_{\sqrt{n} < p \leq n} \alpha(n, p) = b_0 n + a_{51} \frac{n}{\ln n} + a_{52} \frac{n}{\ln^2 n} + \cdots + a_{5k} \frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n}\right).$$

From (3), (8) and (12) we obtain the asymptotic formula

$$\sum_{p \leq n} \alpha(n, p) = n \ln \ln n + cn + c_1 \frac{n}{\ln n} + c_2 \frac{n}{\ln^2 n} + \cdots + c_k \frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n}\right).$$

This completes the proof of the Theorem.

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