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On the Time-like Curves of Constant Breadth in Minkowski 3-Space

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Abstract: A regular curve with more than 2 breadths in Minkowski 3-space is called a *Smarandache breadth curve*. In this paper, we study a special case of Smarandache breadth curves. Some characterizations of the time-like curves of constant breadth in Minkowski 3-Space are presented.

Key Words: Smarandache breadth curves, curves of constant breadth, Minkowski 3-Space, time-like curves.

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§1. Introduction

Curves of constant breadth were introduced by L. Euler [3]. In [8], some geometric properties of plane curves of constant breadth are given. And, in another work [9], these properties are studied in the Euclidean 3-Space E^3. Moreover, M. Fujivara [5] had obtained a problem to determine whether there exist space curve of constant breadth or not, and he defined *breadth* for space curves and obtained these curves on a surface of constant breadth. In [1], this kind curves are studied in four dimensional Euclidean space E^4.

A regular curve with more than 2 breadths in Minkowski 3-space is called a *Smarandache breadth curve*. In this paper, we study a special case of Smarandache breadth curves. We investigate position vector of simple closed time-like curves and some characterizations in the case of constant breadth. Thus, we extended this classical topic to the space E_1^3, which is related with Smarandache geometries, see [4] for details. We used the method of [9].

§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^3 are briefly presented. A more complete elementary treatment can be found in the reference [2].

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by

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$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since $\langle , \rangle$ is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian characters: it can be space-like if $\langle v, v \rangle > 0$ or $v = 0$, time-like if $\langle v, v \rangle < 0$ and null if $\langle v, v \rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\varphi = \varphi(s)$ in E_1^3 can locally be space-like, time-like or null (light-like), if all of its velocity vectors φ' are respectively space-like, time-like or null (light-like), for every $s \in I \subset R$. The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by $\|a\| = \sqrt{|\langle a, a \rangle|}$. φ is called an unit speed curve if velocity vector v of φ satisfies $\|v\| = \pm 1$. For vectors $v, w \in E_1^3$ it is said to be orthogonal if and only if $\langle v, w \rangle = 0$.

Denote by $\{T, N, B\}$ the moving Frenet frame along the curve φ in the space E_1^3. For an arbitrary curve φ with first and second curvature, κ and τ in the space E_1^3, the following Frenet formulae are given in [6]:

Let φ be a time-like curve, then the Frenet formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1}$$

where

$$\langle T, T \rangle = -1, \ \langle N, N \rangle = \langle B, B \rangle = 1,$$
$$\langle T, N \rangle = \langle T, B \rangle = \langle T, N \rangle = \langle N, B \rangle = 0$$

Let a and b be two time-like vectors in E_1^3. If a and b aren't in the same time cone then there is unique real number $\delta \geq 0$ called the hyperbolic angle between a and b, such that $g(a, b) = \|a\| \, \|b\| \cosh \delta$. Let $\vartheta = \vartheta(s)$ be a time-like curve in E_1^3. If tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called an inclined curve.

In [7], the author wrote a characterization for the inclined time-like curves with the following theorem.

Theorem 2.1 *Let $\varphi = \varphi(s)$ be an unit speed time-like curve in E_1^3. φ is an inclined curve if and only if*

$$\frac{\kappa}{\tau} = \text{constant}. \tag{2}$$

§3. The Time-like Curves of Constant Breadth in $\mathbf{E}_1^3$

Definition 3.1 *A regular curve with more than 2 breadths in Minkowski 3-space is called a Smarandache breadth curve.*

Let $\varphi = \varphi(s)$ be a Smarandache breadth curve. Moreover, let us suppose $\varphi = \varphi(s)$ simple closed time-like curve in the space E_1^3. These curves will be denoted by (C). The normal plane at every point P on the curve meets the curve at a single point Q other than P. We call the point Q the opposite point of P. We consider a curve in the class Γ as in [?] having parallel tangents T and T^* in opposite directions at the opposite points φ and φ^* of the curve. A simple closed curve having parallel tangents in opposite directions at opposite points can be represented with respect to Frenet frame by the equation

$$\varphi^*(s) = \varphi(s) + m_1 T + m_2 N + m_3 B, \tag{3}$$

where $m_i(s)$, $1 \leq i \leq 3$ are arbitrary functions and φ and φ^* are opposite points. Differentiating both sides of (3) and considering Frenet equations, we have

$$\left\{ \begin{array}{c} \dfrac{d\varphi^*}{ds} = T^* \dfrac{ds^*}{ds} = (\dfrac{dm_1}{ds} + m_2\kappa + 1)T + \\ (\dfrac{dm_2}{ds} + m_1\kappa - m_3\tau)N + (\dfrac{dm_3}{ds} + m_2\tau)B \end{array} \right\}. \tag{4}$$

Since $T^* = -T$. Rewriting (4), we have respectively,

$$\left\{ \begin{array}{c} \dfrac{dm_1}{ds} = -m_2\kappa - 1 - \dfrac{ds^*}{ds} \\ \dfrac{dm_2}{ds} = -m_1\kappa + m_3\tau \\ \dfrac{dm_3}{ds} = -m_2\tau \end{array} \right\}. \tag{5}$$

If we call ϕ as the angle between the tangent of the curve (C) at point $\varphi(s)$ with a given fixed direction and consider $\dfrac{d\phi}{ds} = \kappa$, we have (5) as follow:

$$\left\{ \begin{array}{c} \dfrac{dm_1}{d\phi} = -m_2 - f(\phi) \\ \dfrac{dm_2}{d\phi} = -m_1 + m_3\rho\tau \\ \dfrac{dm_3}{d\phi} = -m_2\rho\tau \end{array} \right\}, \tag{6}$$

where $f(\phi) = \rho + \rho^*$, $\rho = \dfrac{1}{\kappa}$ and $\rho^* = \dfrac{1}{\kappa^*}$ denote the radius of curvatures at φ and φ^*, respectively. And using system (6), we have the following differential equation with respect to m_1 as

$$\frac{\kappa}{\tau}\left[\frac{d^3 m_1}{d\phi^3} + \frac{d^2 f}{d\phi^2}\right] + \frac{d}{d\phi}(\frac{\kappa}{\tau})\left[\frac{d^2 m_1}{d\phi^2} - m_1 + \frac{df}{d\phi}\right] + (\frac{\tau^2 - \kappa^2}{\tau\kappa})\frac{dm_1}{d\phi} + \frac{\tau}{\kappa} f = 0. \tag{7}$$

Equation (7) is a characterization for φ^*. If the distance between opposite points of (C) and (C^*) is constant, then, we can write that

$$\|\varphi^* - \varphi\| = -m_1^2 + m_2^2 + m_3^2 = l^2 = constant. \tag{8}$$

Hence, we write

$$-m_1 \frac{dm_1}{d\phi} + m_2 \frac{dm_2}{d\phi} + m_3 \frac{dm_3}{d\phi} = 0. \tag{9}$$

Considering system (6), we obtain

$$m_1(\frac{dm_1}{d\phi} + m_2) = 0. \tag{10}$$

We write $m_1 = 0$ or $\dfrac{dm_1}{d\phi} = -m_2$. Thus, we shall study in the following subcases.

Case 1. $\dfrac{dm_1}{d\phi} = -m_2$. Then $f(\phi) = 0$. In this case, (C^*) is translated by the constant vector

$$u = m_1 T + m_2 N + m_3 B \tag{11}$$

of (C). Now, let us to investigate solution of the equation (7), in some special cases.

Case 1.1 Suppose that φ is an inclined curve. If we rewrite (7), we have the following differential equation:

$$\frac{d^3 m_1}{d\phi^3} + (\frac{\tau^2}{\kappa^2} - 1)\frac{dm_1}{d\phi} = 0. \tag{12}$$

General solution of (12) depends on character of $\dfrac{\tau}{\kappa}$. Due to this, we distinguish following subcases.

Case 1.1.1 $\tau > \kappa$. Then the solution above differential equation is:

$$m_1 = C_1 \cos\sqrt{\frac{\tau^2}{\kappa^2} - 1}\phi + C_2 \sin\sqrt{\frac{\tau^2}{\kappa^2} - 1}\phi. \tag{13}$$

And therefore, we have m_2 and m_3, respectively,

$$m_2 = \sqrt{\frac{\tau^2}{\kappa^2} - 1}\left\{C_1 \sin\sqrt{\frac{\tau^2}{\kappa^2} - 1}\phi - C_2 \cos\sqrt{\frac{\tau^2}{\kappa^2} - 1}\phi\right\}, \tag{14}$$

$$m_3 = \frac{\tau}{\kappa}\left[C_1 \cos\sqrt{\frac{\tau^2}{\kappa^2} - 1}\phi + C_2 \sin\sqrt{\frac{\tau^2}{\kappa^2} - 1}\phi\right]. \tag{15}$$

where C_1 and C_2 are real numbers.

Case 1.1.2 $\tau < \kappa$. Then the solution has the form

$$m_1 = A_1 e^{\sqrt{1-\frac{\tau^2}{\kappa^2}}\phi} + A_2 e^{-\sqrt{1-\frac{\tau^2}{\kappa^2}}\phi.} \tag{16}$$

Hence, we have m_2 and m_3 as follows:

$$m_2 = \sqrt{1 - \frac{\tau^2}{\kappa^2}}\left\{-A_1 e^{\sqrt{1-\frac{\tau^2}{\kappa^2}}\phi} + A_2 e A_2 e^{-\sqrt{1-\frac{\tau^2}{\kappa^2}}\phi}\right\}, \tag{17}$$

$$m_3 = \frac{\tau}{\kappa}\left[A_1 e^{\sqrt{1-\frac{\tau^2}{\kappa^2}}\phi} + A_2 e^{-\sqrt{1-\frac{\tau^2}{\kappa^2}}\phi}\right]. \tag{18}$$

where A_1 and A_2 are real numbers.

Corollary 3.1 *Position vector of φ^* can be formed by the equations* (13), (14) *and* (15) *or* (16), (17) *and* (18) *according to ratio of* $\dfrac{\tau}{\kappa}$.

Case 1.2 Let us suppose $m_1 = c_1 = constant \neq 0$. Thus $m_2 = 0$. From $(6)_3$ we easily have $m_3 = c_3 = constant$. And using $(6)_2$ we get

$$\frac{\kappa}{\tau} = \frac{c_3}{c_1} = constant. \tag{19}$$

Equation (19) shows that φ is an inclined curve. Therefore, **Case 1.2** is a characterization for the inclined time-like curves of constant breadth in E_1^3. Then the position vector of φ^* can be written as follow:

$$\varphi^* = \varphi + c_1 T + c_3 B. \tag{20}$$

And curvature of φ^* is obtained as

$$\kappa^* = \kappa. \tag{21}$$

Case 2 $m_1 = 0$. Then $m_2 = -f(\phi)$. And, here, let us suppose that φ is an inclined curve. Thus, the equation (7) has the form

$$\frac{d^2 f}{d\phi^2} + \frac{\tau^2}{\kappa^2} f = 0. \tag{22}$$

The solution of (22) is

$$f(\phi) = L_1 \cos\frac{\tau}{\kappa}\phi + L_2 \sin\frac{\tau}{\kappa}\phi. \tag{23}$$

where L_1 and L_2 are real numbers. Using equation (23), we have m_2 and m_3

$$m_2 = -L_1 \cos\frac{\tau}{\kappa}\phi - L_2 \sin\frac{\tau}{\kappa}\phi = -\rho - \rho^*, \tag{24}$$

$$m_3 = L_1 \sin\frac{\tau}{\kappa}\phi - L_2 \sin\frac{\tau}{\kappa}\phi. \tag{25}$$

And therefore, we write the position vector and the curvature of φ^*

$$\varphi^* = \varphi + (-\rho - \rho^*)N + (L_1 \sin\frac{\tau}{\kappa}\phi - L_2 \sin\frac{\tau}{\kappa}\phi)B, \tag{26}$$

$$\kappa^* = \frac{1}{L_1 \cos\frac{\tau}{\kappa}\phi + L_2 \sin\frac{\tau}{\kappa}\phi - \frac{1}{\kappa}}. \tag{27}$$

And the distance between the opposite points of (C) and (C^*) is

$$\|\varphi^* - \varphi\| = L_1^2 + L_2^2 = constant. \tag{28}$$

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