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The Toroidal Crossing Number of $K_{4,n}$

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Abstract: In this paper, we study the crossing number of the complete bipartite graph $K_{4,n}$ in torus and obtain

$$cr_T(K_{4,n}) = \lfloor \frac{n}{4} \rfloor (2n - 4(1 + \lfloor \frac{n}{4} \rfloor)).$$

Key Words: Smarandache $\mathscr{P}$-drawing, crossing number, complete bipartite graph, torus.

AMS(2010): 05C10

§1. Introduction

A *complete bipartite graph* $K_{m,n}$ is a graph with vertex set $V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$, $|V_1| = m$ and $|V_2| = n$; and with edge set of all pairs of vertices with one element in V_1 and the other in V_2. The vertices in V_1 will be denoted by $b_i, b_j, b_k, \cdots$ and the vertices in V_2 will be denoted by $a_i, a_j, a_k, \cdots$.

A *drawing* is a mapping of a graph G into a surface. A *Smarandache $\mathscr{P}$-drawing* of a graph G for a graphical property $\mathscr{P}$ is such a good drawing of G on the plane with minimal intersections for its each subgraph $H \in \mathscr{P}$. A Smarandache $\mathscr{P}$-drawing is said to be *optimal* if $\mathscr{P} = G$ and it minimizes the number of crossings. Particularly, a drawing is *good* if it satisfies: (1) no two arcs which are incident with a common node have a common point; (2) no arc has a self-intersection; (3) no two arcs have more than one point in common; (4) no three arcs have a point in common. A common point of two arcs is called as a *crossing*. An *optimal drawing* in a given surface is a good drawing which has the smallest possible number of crossings. This number is the *crossing number* of the graph in the surface. We denote the crossing number of G in T, the torus, by $cr_T(G)$, a drawing of G in T by D. In this paper, we often speak of the nodes as vertices and the arcs as edges. For more graph terminologies and notations not mentioned here, you can refer to [1,3].

Garey and Johnson [2] stated that determining the crossing number of an arbitrary graph

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is NP-complete. In 1969, Guy and Jenkyns [4] proved that the crossing number of the complete bipartite graph $K_{3,n}$ in torus is $\lfloor\frac{(n-3)^2}{12}\rfloor$, and obtained the bounds on the crossing number of the complete bipartite graph $K_{m,n}$ in torus. In 1971, Kleitman [6] proved that the crossing number of the complete bipartite graph $K_{5,n}$ in plane is $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ and the crossing number of the complete bipartite graph $K_{6,n}$ in plane is $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$. Later, Richter and Širáň [7] obtained the crossing number of the complete bipartite graph $K_{3,n}$ in an arbitrary surface. Recently, Ho [5] proved that the crossing number of the complete bipartite graph $K_{4,n}$ in real projective plane is $\lfloor\frac{n}{3}\rfloor(2n-3(1+\lfloor\frac{n}{3}\rfloor))$. In this paper, we obtain the crossing number of the complete bipartite graph $K_{4,n}$ in torus following.

Theorem 1 *The crossing number of the complete bipartite graph $K_{4,n}$ in torus is*

$$\mathrm{cr}_T(K_{4,n}) = \lfloor\frac{n}{4}\rfloor(2n-4(1+\lfloor\frac{n}{4}\rfloor)).$$

For convenience, let $f(n) = \lfloor\frac{n}{4}\rfloor(2n-4(1+\lfloor\frac{n}{4}\rfloor))$.

§2. Some Lemmas

In a drawing D of the complete bipartite $K_{m,n}$ in T, we denote by $cr_D(a_i,a_j)$ the number of crossings on edges one of which is incident with a vertex a_i and the other incident with a_j, and by $cr_D(a_i)$ the number of crossings on edges incident with a_i. Obviously,

$$cr_D(a_i) = \sum_{k=1}^{n} cr_D(a_i,a_k).$$

In every good drawing D, the *crossing number in D, $cr_T(D)$*, is

$$cr_T(D) = \sum_{i=1}^{n}\sum_{k=i+1}^{n} cr_D(a_i,a_k).$$

As $cr_D(a_i,a_i)=0$ for all i, hence

$$cr_T(D) = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} cr_D(a_i,a_k) = \frac{1}{2}\sum_{i=1}^{n} cr_D(a_i). \tag{1}$$

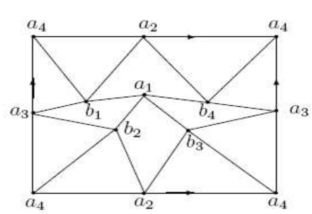

Fig.1

Fig.1. An optimal drawing of $K_{4,4}$ in T

Note that, in a crossing-free drawing of a connected subgraph of the complete bipartite graph $K_{m,n}$, every circuit has an even number of vertices, and in particular, every region into which the edges divide the surface is bounded by an even circuit. So, if F is the number of regions, E the number of edges and V the number of vertices, by the Eular's formula for T,

$$V - E + F \geq 0$$

$$F \geq E - V, \tag{2}$$

$$4F \leq 2E. \tag{3}$$

Suppose we have an optimal drawing of the complete bipartite graph $K_{m,n}$ in T, i.e., one with exactly $cr_T(K_{m,n})$ crossings. Then by deleting $cr_T(K_{m,n})$ edges, a crossing-free drawing will be obtained. From equations (2) and (3),

$$E - V = (mn - cr_T(K_{m,n})) - (m+n) \leq F \leq \frac{1}{2}E = \frac{1}{2}((mn - cr_T(K_{m,n})),$$

this implies

$$cr_T(K_{m,n}) \geq mn - 2(m+n). \tag{4}$$

In particular,

$$cr_T(K_{4,n}) \geq 2n - 8. \tag{5}$$

In Fig.1, it is a crossing-free drawing of the complete bipartite graph $K_{4,4}$ in T, hence

$$cr_T(K_{4,4}) = 0. \tag{6}$$

In paper [4], the following two lemmas can be find.

Lemma 1 *Let m, n, h be positive integers such that the complete bipartite graph $K_{m,h}$ embeds in T, then*

$$cr_T(K_{m,n}) \leq \frac{1}{2}\lfloor\frac{n}{h}\rfloor[2n - h(1 + \lfloor\frac{n}{h}\rfloor)]\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor.$$

Lemma 2 *If D is a good drawing of the complete bipartite graph $K_{m,n}$ in a surface Σ such that, for some $k < n$, some $K_{m,k}$ is optimally drawn in Σ, then*

$$cr_\Sigma(D) \geq cr_\Sigma(K_{m,k}) + (n-k)(cr_\Sigma(K_{m,k+1}) - cr_\Sigma(K_{m,k})) + cr_\Sigma(K_{m,n-k}).$$

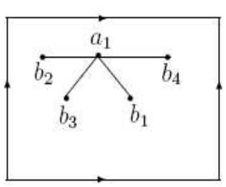

Fig.2

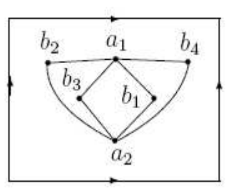

Fig.3

Lemma 3 *For $n \geq 4$, $cr_T(K_{4,n}) \leq f(n)$; especially, when $4 \leq n \leq 8$, $cr_T(K_{4,n}) = f(n)$.*

Proof As $cr_T(K_{4,4}) = 0$, by applying Lemma 1 with $m = h = 4$, then $cr_T(K_{4,n}) \leq f(n)$, $n \geq 4$. Especially, as $f(n) = 2n - 8$ for $4 \leq n \leq 8$, combining with equation (5), then $cr_T(K_{4,n}) = f(n)$ for $4 \leq n \leq 8$. □

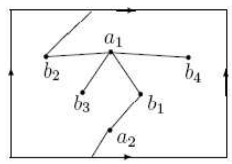

Fig.4

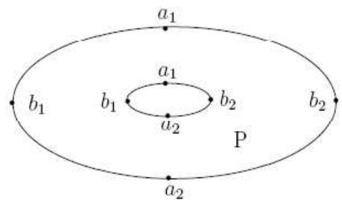

Fig.5

Lemma 4 *There is no good drawing D of $K_{4,5}$ in T such that*

(1) $cr_D(a_1, a_2) = cr_D(a_1, a_i) = cr_D(a_2, a_i) = 0$ *for* $3 \leq i \leq 5$;

(2) $cr_D(a_3, a_4) = cr_D(a_3, a_5) = cr_D(a_4, a_5) = 1$.

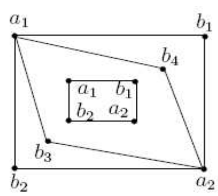

Fig.6(1)

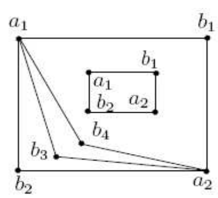

Fig.6(2)

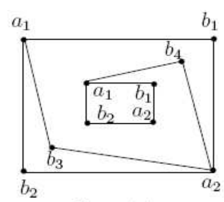

Fig.6(3)

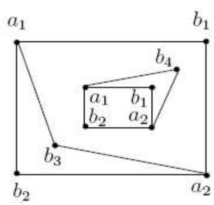

Fig.6(4)

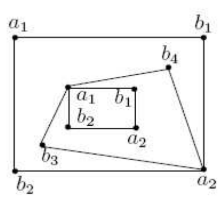

Fig.6(5)

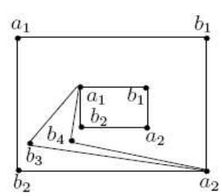

Fig.6(6)

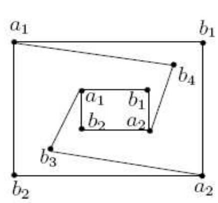

Fig.6(7)

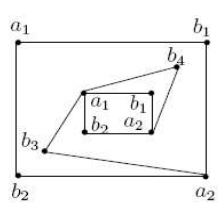

Fig.6(8)

Proof Note that T can be viewed as a rectangle with its opposite sides identified. As D is a good drawing, by deformation of the edges without changing the crossings and renaming the vertices if necessary, we can assume that the edges incident with a_1 are drawn as in Fig.2. Since $cr_D(a_1, a_2) = 0$, by deformation of edges without changing the crossings, we also assume that the edge a_2b_1 is drawn as in Fig.3. If the other three edges incident with a_2 are drawn without passing the sides of the rectangle (see Fig.3), then no matter which region a_3 is located, we have $cr_D(a_1, a_3) \geq 1$ or $cr_D(a_2, a_3) \geq 1$.

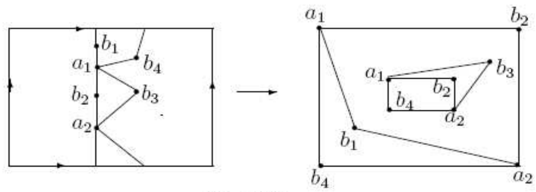

Fig.7(1)

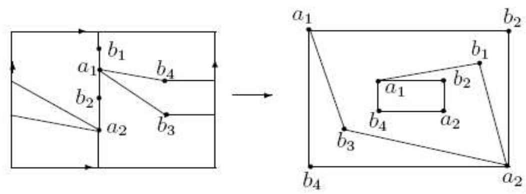

Fig.7(2)

So, there is at least one edge incident with a_2 which passes the sides of the rectangle. By deformation without changing the crossings and renaming the vertices if necessary, we assume that edge a_2b_2 passes the top and bottom sides of the rectangle only one time and is drawn as in Fig.4. Then we cut T along the circuit $a_1b_1a_2b_2a_1$ and obtain a surface which is homeomorphic to a ring in plane, denote by P, see Fig.5. Now, we put the vertices b_3, b_4 in P and use two rectangles to represent the outer and inner boundary which are both the circuit $a_1b_1a_2b_2a_1$.

As the vertices b_3 and b_4 are connected to a_1 and a_2 either in the outer or in the inner rectangle, which together presents 16 possibilities. In some cases, the four edges can either separate the two rectangles or not, implying up to 32 cases. Using symmetry, several cases are eliminated: without loss of generality, the vertex b_3 is connected to a_2 in the outer rectangle.

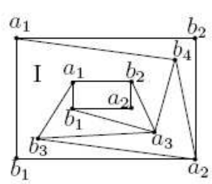

Fig.8(1)

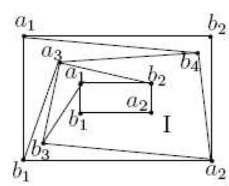

Fig.8(2)

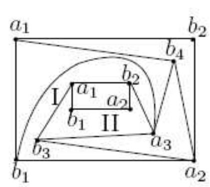

Fig.8(3)

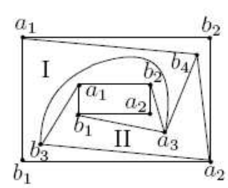

Fig.8(4)

First, assume that b_3 is also connected to a_1 in the outer rectangle. If b_4 is connected to both a_1 and a_2 in the outer rectangle, we obtain Fig.6(1) if the four edges separate the two rectangles, and Fig.6(2) if they do not. If b_4 is connected to a_1 in the inner rectangle and a_2 in the outer rectangle, we obtain Fig.6(3). If it is connected to a_1 in the outer rectangle and a_2 in the inner rectangle, then by relabeling a_1 and a_2, we obtain Fig.6(3). If b_4 is connected to both a_1 and a_2 in the inner rectangle, we obtain Fig.6(4).

Second, assume that b_3 is connected to a_1 in the inner rectangle. If b_4 is connected to both a_1 and a_2 in the outer rectangle, then by relabeling of b_3 and b_4, we obtain Fig.6(3). If b_4 is connected to a_1 in the inner rectangle and a_2 in the outer rectangle, we obtain Fig.6(5) if the four edges separate the two rectangles, and Fig.6(6) if they do not. If b_4 is connected to a_2 in the inner rectangle and a_1 in the outer rectangle, we obtain Fig.6(7). Finally, if b_4 is connected to both a_1 and a_2 in the inner rectangle, we obtain Fig.6(8).

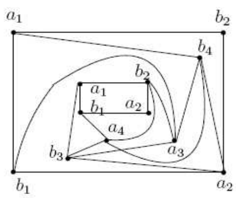

Fig.9(1)

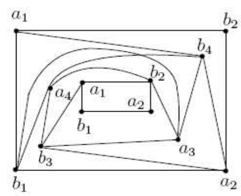

Fig.9(2)

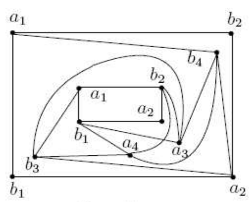

Fig.9(3)

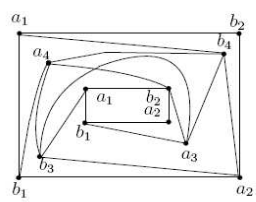

Fig.9(4)

Now, by drawing Fig.6(1) back into T and cut T along the circuit $a_1b_2a_2b_4a_1$, we obtain Fig.7(1); by drawing Fig.6(6) back into T and cut T along the circuit $a_1b_4a_2b_2a_1$, we obtain Fig.7(2). It is easy to find out that Fig.7(1) and Fig.6(4), Fig.7(2) and Fig.6(3) have the same structure if ignoring the labels of b. In Fig.6(8), by exchanging the inner and outer rectangles and the labels of b_3, b_4, we obtain Fig.6(3). In Fig.6(2), as each region has at most 3 vertices of $\{b_1, b_2, b_3, b_4\}$ on its boundary, we will have $cr_D(a_1, a_i) \geq 1$ or $cr_D(a_2, a_i) \geq 1$ for $i = 3, 4, 5$. So, we only need to consider the cases in Fig.6(3-5,7).

In Fig.6(3), since $cr_D(a_1, a_3) = cr_D(a_2, a_3) = 0$, we can draw the edges incident with a_3 in four different ways, see Fig.8(1-4). Furthermore, as $cr_D(a_1, a_4) = cr_D(a_2, a_4) = 0$ and $cr_D(a_3, a_4) = 1$, a_4 can only be putted in region I or II. In Fig.8(3-4), we can draw the edges incident with a_4 in four different ways, see Fig.9(1-4). In Fig.8(1-2), there are also four different ways to draw the edges incident with a_4, but they can be obtained by relabeling a_3 and a_4 in Fig.9((1-4). Then, we can see that no matter which region a_5 lies, we cannot have $cr_D(a_3, a_5) = cr_D(a_4, a_5) = 1$.

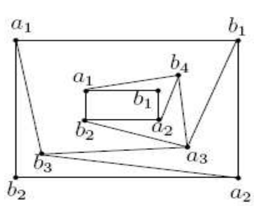

Fig.10(1)

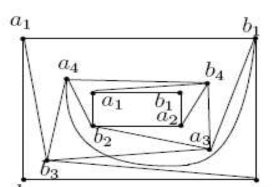

Fig.10(2)

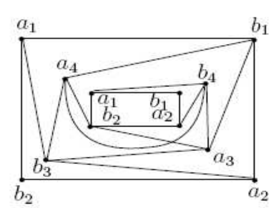

Fig.10(3)

In Fig.6(4), we have only one way to draw the edges incident with a_3, see Fig.10(1). Furthermore, we have two drawings of a_4 in Fig.10(1), see Fig.10(2-3). But, by observation, we cannot have $cr_D(a_3, a_5) = cr_D(a_4, a_5) = 1$.

In Fig.6(5,7), no matter which regions a_3, a_4 locate, we will have $cr_D(a_3, a_4) \geq 2$ or $cr_D(a_3, a_4) = 0$. Now, the proof completes. □

§3. The proof of the Main Theorem

The proof of Theorem 1 is by induction on n. The base of the induction is $n \leq 8$ and has been obtained from Lemma 3. For $n \geq 9$, by Lemma 3, we only need to prove that $cr_T(K_{4,n}) \geq f(n)$. Let $n = 4q + r$ where $0 \leq r \leq 3$, and D be an optimal drawing of $K_{4,n}$ in T.

First, we assume that there exists a $K_{4,4}$ in D which is drawn without crossings. From Lemma 3, $cr_T(K_{4,5}) = 2$, and by the inductive assumption, $cr_T(K_{4,n-4}) = f(n-4)$. Hence, by applying Lemma 2 with $m = k = 4$,

$$\begin{aligned} cr_T(D) &\geq 2(n-4) + f(n-4) = 2(n-4) + \lfloor \frac{n-4}{4} \rfloor (2(n-4) - 4(1 + \lfloor \frac{n-4}{4} \rfloor)) \\ &= 8q + 2r - 8 + (q-1)(4q + 2r - 8) = 4q^2 + 2qr - 4q, \end{aligned}$$

which is $f(n)$, since

$$f(n) = \lfloor \frac{n}{4} \rfloor (2n - 4(1 + \lfloor \frac{n}{4} \rfloor)) = q(8q + 2r - 4(1+q)) = 4q^2 + 2qr - 4q. \tag{7}$$

Second, we assume that every $K_{4,4}$ in D is drawn with at least one crossings. Clearly, $K_{4,n}$ contains n subgraphs $K_{4,n-1}$, each contains at least $f(n-1)$ crossings by the inductive hypothesis. As each crossing will be counted $n-2$ times, hence

$$cr_T(D) \geq \frac{n}{n-2} cr_T(K_{4,n-1}) = \frac{n}{n-2} f(n-1). \tag{8}$$

From equation (7),

$$f(n) = \begin{cases} q(4q-4), & \text{for } n = 4q, \\ q(4q-2), & \text{for } n = 4q+1, \\ 4q^2, & \text{for } n = 4q+2, \\ q(4q+2), & \text{for } n = 4q+3. \end{cases}$$

Combining this with equation (8),

$$cr_T(D) \geq \begin{cases} q(4q-4), & \text{for } n = 4q, \\ q(4q-2) - 1 - \frac{2q+1}{4q-1}, & \text{for } n = 4q+1, \\ 4q^2 - 1, & \text{for } n = 4q+2, \\ q(4q+2) - \frac{2q}{4q+1}, & \text{for } n = 4q+3. \end{cases}$$

As $n \geq 9$, namely $q \geq 2$, and the crossing number is an integer, thus, when $n = 4q$ or $4q + 3$,

$$cr_T(K_{4,n}) = cr_T(D) \geq f(n);$$

when $n = 4q + 1$ or $4q + 2$,

$$cr_T(K_{4,n}) = cr_T(D) \geq f(n) - 1.$$

Therefore, only the two cases $n = 4q+1$ and $n = 4q+2$ are needed considering. In the following, we assume that $cr_T(K_{4,n}) = cr_T(D) = f(n)-1$ for $n = 4q+1$ or $4q+2$, and denote the drawing of $K_{4,n-1}$ obtained by deleting the vertex a_i of $K_{4,n}$ in D by $D - \{a_i\}$.

Case 1. $n = 4q + 1$.

By the inductive assumption,

$$cr_T(D - \{a_i\}) \geq f(4q),\ 1 \leq i \leq 4q + 1.$$

As $cr_T(D) = f(4q + 1) - 1 = 4q^2 - 2q - 1$, then

$$cr_D(a_i) = cr_T(D) - cr_T(D - \{a_i\}) \leq f(4q + 1) - 1 - f(4q) = 2q - 1,\ 1 \leq i \leq 4q + 1.$$

Let x be the number of a_i such that $cr_D(a_i) = 2q - 1$, y be the number of a_i such that $cr_D(a_i) = 2q - 2$, thus, the number of a_i such that $cr_D(a_i) \leq 2q - 3$ is $4q + 1 - (x + y)$. By equation(1), it holds

$$\begin{aligned}(2q-1)x &+ (2q-2)y + (4q+1-x-y)(2q-3) \geq 2cr_T(D) = 8q^2 - 4q - 2\\ 2x + y &\geq 6q + 1.\end{aligned}$$

As $x + y \leq 4q + 1$, then $x \geq 2q$. Without loss of generality, by renaming the vertices, suppose that $cr_D(a_i) = 2q - 1$ for $i \leq x$.

Case 1.1 There exists a pair of (i, j), $1 \leq i < j \leq x$, such that $cr_D(a_i, a_j) = 0$. Denote the drawing of the graph $K_{4,4q-1}$ obtained by deleting the vertices a_i, a_j of the graph $K_{4,4q+1}$ in D by $D - \{a_i, a_j\}$. Then,

$$cr_T(D - \{a_i, a_j\}) = f(4q + 1) - 1 - 2(2q - 1) = 4q^2 - 6q + 1.$$

But this contradicts the inductive assumption that $cr_T(K_{4,4q-1}) = f(4q - 1) = 4q^2 - 6q + 2$.

Case 1.2 For every (i, j), $1 \leq i < j \leq x$, $cr_D(a_i, a_j) \geq 1$. As $cr_D(a_i) = 2q - 1$, obviously, $x = 2q$ and

$$cr_D(a_i, a_j) = 1,\ 1 \leq i < j \leq 2q,\ cr_D(a_i, a_h) = 0,\ 1 \leq i \leq 2q < h \leq 4q + 1.$$

Furthermore, as $x + y \leq 4q + 1$ and $2x + y \geq 6q + 1$, then $y = 2q + 1$. By the definition of y, there exist a_h, a_k, where $2q + 1 \leq h < k \leq 4q + 1$, such that $cr_D(a_h, a_k) = 0$. Now, we obtain a drawing of $K_{4,5}$ in T with vertices a_h, a_k, a_1, a_2, a_3 such that $cr_D(a_h, a_k) = cr_D(a_h, a_i) = cr_D(a_k, a_i) = 0$ $(1 \leq i \leq 3)$ and $cr_D(a_1, a_2) = cr_D(a_1, a_3) = cr_D(a_2, a_3) = 1$. Contradicts to Lemma 4.

Combining the above two subcases, we have $cr_T(K_{4,4q+1}) = f(4q + 1) = q(4q - 2)$.

Case 2. $n = 4q + 2$.

By the inductive assumption,

$$cr_T(D - \{a_i\}) \geq f(4q + 1) = q(4q - 2),\ 1 \leq i \leq 4q + 2.$$

As $cr_T(D) = f(4q+2) - 1 = 4q^2 - 1$, thus

$$cr_D(a_i) = cr_T(D) - cr_T(D - \{a_i\}) \leq (f(4q+2) - 1) - f(4q+1) = 2q - 1.$$

Let t be the number of a_i such that $cr_D(a_i) = 2q - 1$, then there are $(4q + 2 - t)$ vertices a_i such that $cr_D(a_i) \leq 2q - 2$. From equation (1),

$$\begin{aligned} (2q-1)t &+ (2q-2)(4q+2-t) \geq 2cr_T(D) = 8q^2 - 2 \\ t &\geq 4q + 2. \end{aligned}$$

As $t \leq n = 4q + 2$, hence, $t = 4q + 2$, this implies that $cr_D(a_i) = 2q - 1$ $(1 \leq i \leq 4q + 2)$.

If there exists a pair of (i, j), $1 \leq i < j \leq 4q + 2$, such that $cr_D(a_i, a_j) \geq 3$, then,

$$cr_T(D - \{a_i\}) = cr_T(D) - cr_D(a_i) = 4q^2 - 1 - (2q - 1) = 4q^2 - 2q,$$

and

$$cr_{(D-\{a_i\})}(a_j) = cr_D(a_j) - cr_D(a_i, a_j) \leq 2q - 1 - 3 = 2q - 4.$$

Now, by putting a new vertex $a_i^{'}$ near the vertex a_j in $D - \{a_i\}$ and drawing the edges $a_i^{'} b_k (1 \leq k \leq 4)$ nearly to $a_j b_k$, a new drawing of $K_{4,4q+2}$ in T is obtained, denoted by $D^{'}$. Clearly,

$$cr_{D'}(a_i^{'}, a_j) = 2 \text{ and } cr_{D'}(a_i^{'}, a_h) = cr_{D-\{a_i\}}(a_j, a_h),\ h \neq j.$$

Thus,

$$cr_T(D^{'}) = cr_T(D - \{a_i\}) + 2 + cr_{(D-\{a_i\})}(a_j) \leq 4q^2 - 2.$$

But, this contradicts to the hypothesis that $cr_T(K_{4,4q+2}) \geq 4q^2 - 1$.

Therefore, for $1 \leq i < j \leq 4q + 2$, $cr_D(a_i, a_j) \leq 2$. For each a_i, $1 \leq i \leq 4q + 2$, let

$$S_0^{(i)} = \{a_j \,|\, cr_D(a_i, a_j) = 0,\ j \neq i\}, \quad S_{\geq 1}^{(i)} = \{a_j \,|\, cr_D(a_i, a_j) \geq 1\},$$

$$S_1^{(i)} = \{a_j \,|\, cr_D(a_i, a_j) = 1\}, \qquad S_2^{(i)} = \{a_j \,|\, cr_D(a_i, a_j) = 2\}.$$

As $cr_D(a_i, a_j) \leq 2$, $cr_D(a_i) = 2q - 1$ is odd, then, for $1 \leq i \leq 4q + 2$,

$$\emptyset \neq S_1^{(i)} \subseteq S_{\geq 1}^{(i)}, \quad |S_1^{(i)}| + |S_2^{(i)}| = |S_{\geq 1}^{(i)}|, \quad |S_{\geq 1}^{(i)}| = 2q - 1 - |S_2^{(i)}|. \tag{9}$$

Furthermore, since $q \geq 2$,

$$|S_0^{(i)}| = 4q + 2 - 1 - |S_{\geq 1}^{(i)}| = 2q + 2 + |S_2^{(i)}| \geq 6.$$

For $1 \leq i < j \leq 4q + 2$, clearly,

$$S_0^{(i)} \cup S_{\geq 1}^{(i)} \cup \{a_i\} = S_0^{(j)} \cup S_{\geq 1}^{(j)} \cup \{a_j\}.$$

If $cr_D(a_i, a_j) = 0$ and $S_{\geq 1}^{(i)} \cap S_{\geq 1}^{(j)} = \emptyset$, then, the above equation implies that

$$S_{\geq 1}^{(i)} \subseteq S_0^{(j)} \quad \text{and} \quad S_{\geq 1}^{(j)} \subseteq S_0^{(i)}. \tag{10}$$

Without loss of generality, let

$$|S_2^{(1)}| = \max\{|S_2^{(i)}| \mid 1 \le i \le 4q+2\}, \quad |S_2^{(2)}| = \max\{|S_2^{(j)}| \mid a_j \in S_0^{(1)}\}.$$

For $3 \le i \le 4q+2$, if $a_i \notin S_{\ge 1}^{(1)} \cup S_{\ge 1}^{(2)}$, then $a_i \in S_0^{(1)} \cap S_0^{(2)}$. This means that

$$|S_0^{(1)} \cap S_0^{(2)}| = 4q - |S_{\ge 1}^{(1)} \cup S_{\ge 1}^{(2)}| = 4q - |S_{\ge 1}^{(1)}| - |S_{\ge 1}^{(2)}| + |S_{\ge 1}^{(1)} \cap S_{\ge 1}^{(2)}|.$$

From equation (9), then

$$|S_0^{(1)} \cap S_0^{(2)}| = 2 + |S_2^{(1)}| + |S_2^{(2)}| + |S_{\ge 1}^{(1)} \cap S_{\ge 1}^{(2)}|. \tag{11}$$

With these notations, it is obvious that $|S_2^{(1)}| \ge |S_2^{(2)}|$ and $cr_D(a_1, a_2) = 0$. In the following, the discussions are divided into two subcases according to $S_{\ge 1}^{(1)} \cap S_{\ge 1}^{(2)} = \emptyset$ or not.

Case 2.1 $S_{\ge 1}^{(1)} \cap S_{\ge 1}^{(2)} \ne \emptyset$. Let $|S_{\ge 1}^{(1)} \cap S_{\ge 1}^{(2)}| = \alpha \ge 1$, from equation (11),

$$|S_0^{(1)} \cap S_0^{(2)}| = 2 + |S_2^{(1)}| + |S_2^{(2)}| + \alpha.$$

First, we choose a vertex from $S_0^{(1)} \cap S_0^{(2)}$, without loss of generality, denoted by a_3. By the assumption that every $K_{4,4}$ in D is drawn with at least one crossings, hence $cr_D(a_3, a_i) \ge 1$ for all $a_i \in S_0^{(1)} \cap S_0^{(2)}$, $a_i \ne a_3$. Let $U = \{a_i \mid cr_D(a_3, a_i) = 1,\ a_i \in S_0^{(1)} \cap S_0^{(2)}\}$. Since $a_3 \in S_0^{(1)}$ and $|S_2^{(2)}| = \max\{|S_2^{(j)}| \mid a_j \in S_0^{(1)}\}$, then $|S_2^{(3)}| \le |S_2^{(2)}|$ and

$$|U| \ge |S_0^{(1)} \cap S_0^{(2)}| - 1 - |S_2^{(3)}| \ge 1 + |S_2^{(1)}| + \alpha.$$

Second, we choose a vertex from U, denoted by a_4. By the assumption that every $K_{4,4}$ in D is drawn with at least one crossings, $cr_D(a_4, a_i) \ge 1$ for all $a_i \in U$, $a_i \ne a_4$. As $|S_2^{(4)}| \le |S_2^{(1)}|$(for $|S_2^{(1)}| = \max\{|S_2^{(i)}| \mid 1 \le i \le 4q+2\}$), thus $|U \setminus S_2^{(4)}| \ge \alpha \ge 1$ and there exists one vertex in U, denoted by a_5, such that $cr_D(a_4, a_5) = 1$. Now, we have a drawing of $K_{4,5}$ in T with vertices a_1, a_2, a_3, a_4, a_5 such that $cr_D(a_1, a_2) = cr_D(a_1, a_k) = cr_D(a_2, a_k) = 0$ for $3 \le k \le 5$ and $cr_D(a_3, a_4) = cr_D(a_3, a_5) = cr_D(a_4, a_5) = 1$. But, this contradicts to Lemma 4.

Case 2.2 $S_{\ge 1}^{(1)} \cap S_{\ge 1}^{(2)} = \emptyset$. From equation (11),

$$|S_0^{(1)} \cap S_0^{(2)}| = 2 + |S_2^{(1)}| + |S_2^{(2)}|.$$

We choose a vertex from $S_0^{(1)} \cap S_0^{(2)}$, also denoted by a_3. By the same discussion as in case 2.1, we have $cr_D(a_3, a_i) \ge 1$ for all $a_i \in S_0^{(1)} \cap S_0^{(2)}$, $a_i \ne a_3$. Let $\Lambda = \{a_i \mid cr_D(a_3, a_i) = 2,\ a_i \in S_0^{(1)} \cap S_0^{(2)}\}$, $\Phi = \{a_i \mid cr_D(a_3, a_i) = 1,\ a_i \in S_0^{(1)} \cap S_0^{(2)}\}$. As $a_3 \in S_0^{(1)}$, $|S_2^{(2)}| = \max\{|S_2^{(j)}| \mid a_j \in S_0^{(1)}\}$ and $|S_2^{(1)}| = \max\{|S_2^{(i)}| \mid 1 \le i \le 4q+2\}$, then

$$\Lambda \subseteq S_2^{(3)}, \quad |\Lambda| \le |S_2^{(3)}| \le |S_2^{(2)}| \le |S_2^{(1)}|, \tag{12}$$

and

$$|\Phi| = |S_0^{(1)} \cap S_0^{(2)}| - 1 - |\Lambda| = 1 + |S_2^{(1)}| + |S_2^{(2)}| - |\Lambda| \tag{13}$$

If there are two vertices in Φ, denoted by a_4, a_5, such that $cr_D(a_4, a_5) = 1$. Then we also have a drawing of $K_{4,5}$ with vertices a_1, a_2, a_3, a_4, a_5 which will contradict to Lemma 4. Hence, for all $a_i, a_j \in \Phi(a_i \neq a_j)$, $cr_D(a_i,a_j) \neq 1$, this implies that $cr_D(a_i,a_j)=2$ since $cr_D(a_i,a_j)$ cannot be zero (otherwise there exists $K_{4,4}$ in D drawn with no crossings), and

$$|S_2^{(i)}| \geq |\Phi| - 1.$$

Furthermore, if $|\Lambda| < |S_2^{(2)}|$, by equation(13), $|\Phi| > 1 + |S_2^{(1)}|$, and for each $a_i \in \Phi$,

$$|S_2^{(i)}| \geq |\Phi| - 1 > |S_2^{(1)}|.$$

This contradicts the maximum of $|S_2^{(1)}|$. Thus,

$$|\Lambda| = |S_2^{(2)}|,\ |\Phi| = 1 + |S_2^{(1)}|,$$

and for each $a_i \in \Phi$,

$$|S_2^{(i)}| \geq |\Phi| - 1 = |S_2^{(1)}|.$$

As $|S_2^{(i)}| \leq |S_2^{(2)}| \leq |S_2^{(1)}|$, combining equation (12),

$$|S_2^{(1)}| = |S_2^{(2)}| = |S_2^{(3)}| = |S_2^{(i)}|, \tag{14}$$

and

$$S_2^{(3)} = \Lambda \subseteq S_0^{(1)} \cap S_0^{(2)}.$$

Combining equations (14) and (9), for each $a_i \in \Phi$,

$$|S_{\geq 1}^{(1)}| = |S_{\geq 1}^{(2)}| = |S_{\geq 1}^{(3)}| = |S_{\geq 1}^{(i)}|,$$

and

$$|S_1^{(1)}| = |S_1^{(2)}| = |S_1^{(3)}| = |S_1^{(i)}|.$$

As $|\Phi| = 1 + |S_2^{(1)}| + |S_2^{(2)}| - |\Lambda| \geq 1$, we choose a vertex from Φ and denote it by a_4.

If there exists a pair of (i,j), $i \in \{1,2\}$ and $j \in \{3,4\}$, such that $S_{\geq 1}^{(i)} \cap S_{\geq 1}^{(j)} \neq \emptyset$, by replacing $S_{\geq 1}^{(1)} \cap S_{\geq 1}^{(2)} \neq \emptyset$ with $S_{\geq 1}^{(i)} \cap S_{\geq 1}^{(j)} \neq \emptyset$ in case 2.1, as $a_j \in S_0^{(1)} \cap S_0^{(2)} (j=3,4)$ and $|S_2^{(i)}| = |S_2^{(j)}| = \max\{|S_2^{(k)}| \mid 1 \leq k \leq 4q+2\}$, we also can obtain a contradiction to Lemma 4.

So, for every (i,j), $i \in \{1,2\}$ and $j \in \{3,4\}$, $S_{\geq 1}^{(i)} \cap S_{\geq 1}^{(j)} = \emptyset$. As $S_{\geq 1}^{(1)} \cap S_{\geq 1}^{(2)} = \emptyset$ and $cr_D(a_i,a_j) = cr_D(a_1,a_2) = 0$, combining equations (9) and (10), then

$$\emptyset \neq S_1^{(1)} \subseteq S_{\geq 1}^{(1)} \subseteq S_0^{(2)} \cap S_0^{(3)} \cap S_0^{(4)} \quad \text{and} \quad \emptyset \neq S_1^{(2)} \subseteq S_{\geq 1}^{(2)} \subseteq S_0^{(1)} \cap S_0^{(3)} \cap S_0^{(4)}.$$

Since $S_1^{(1)} \neq \emptyset$, there exists a vertex, denoted by a_5, such that $a_5 \in S_1^{(1)} \subseteq S_0^{(2)} \cap S_0^{(3)} \cap S_0^{(4)}$. This implies that

$$cr_D(a_1,a_5) = 1 \text{ and } cr_D(a_2,a_5) = cr_D(a_3,a_5) = cr_D(a_4,a_5) = 0.$$

As $S_{\geq 1}^{(2)} \cap S_{\geq 1}^{(3)} = \emptyset$, $|S_2^{(1)}| = |S_2^{(2)}| = |S_2^{(3)}|$, $cr_D(a_2,a_3) = 0$ and $a_5 \subseteq S_0^{(2)} \cap S_0^{(3)}$, by replacing $S_{\geq 1}^{(1)} \cap S_{\geq 1}^{(2)} = \emptyset$ with $S_{\geq 1}^{(2)} \cap S_{\geq 1}^{(3)} = \emptyset$ and replacing a_3 with a_5 in the beginning part of Case 2.2, we also can obtain that $|S_1^{(5)}| = |S_1^{(2)}| = |S_1^{(3)}|$ and $S_2^{(5)} \subseteq S_0^{(2)} \cap S_0^{(3)}$. This means that, for any vertex $a_k \in S_1^{(2)}$,

$$cr_D(a_5,a_k) \leq 1. \tag{15}$$

As $S_1^{(2)} \neq \emptyset$, there exists one vertex in $S_1^{(2)}$, denoted by a_6, such that $cr_D(a_5, a_6) = 0$. Otherwise, from equation (15) and $cr_D(a_1, a_5) = 1$, $S_1^{(2)} \cup \{a_1\} \subseteq S_1^{(5)}$. As $a_1 \notin S_1^{(2)}$, then $|S_1^{(5)}| \geq |S_1^{(2)}| + 1$, which contradicts to $|S_1^{(5)}| = |S_1^{(2)}| = |S_1^{(3)}|$. Furthermore, as $a_6 \in S_1^{(2)} \subseteq S_0^{(1)} \cap S_0^{(3)} \cap S_0^{(4)}$, we also have

$$cr_D(a_2, a_6) = 1 \text{ and } cr_D(a_1, a_6) = cr_D(a_3, a_6) = cr_D(a_4, a_6) = 0.$$

Hence, we obtain a good drawing of $K_{4,6}$ in T, denoted by D', with

$$cr_{D'}(a_i) = \sum_{j=1}^{6} cr_D(a_i, a_j) = 1,\ 1 \leq i \leq 6,$$

and

$$cr_T(K_{4,6}) \leq cr_T(D') = \frac{1}{2}\sum_{i=1}^{6} cr_{D'}(a_i) = 3.$$

This contradicts to Lemma 3. Thus, $cr_T(K_{4,4q+2}) = cr_T(D) = f(4q+2) = 4q^2$. □

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