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Total Semirelib Graph

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Abstract: In this paper, the concept of Total semirelib graph of a planar graph is introduced. We present a characterization of those graphs whose total semirelib graphs are planar, outer planar, Eulerian, hamiltonian with crossing number one.

Key Words: Blocks, edge degree, inner vertex number, line graph, regions Smarandachely semirelib M-graph.

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§1. Introduction

The concept of block edge cut vertex graph was introduced by Venkanagouda M Goudar [4]. For the graph G(p,q), if $B = u_1, u_2, \cdots, u_r : r \geq 2$ is a block of G, then we say that the vertex u_i and the block B are incident with each other. If two blocks B_1 and B_2 are incident with a common cutvertex, then they are adjacent blocks.

All undefined terminology will conform with that in Harary [1]. All graphs considered here are finite, undirected, planar and without loops or multiple edges.

The semirelib graph of a planar graph G is introduced by Venkanagouda M Goudar and Manjunath Prasad K B [5] denoted by $R_s(G)$ is the graph whose vertex set is the union of set of edges, set of blocks and set of regions of G in which two vertices are adjacent if and only if the corresponding edges of G are adjacent, the corresponding edges lies on the blocks and the corresponding edges lies on the region. Now we define the total semirelib graph.

Let M be a maximal planar graph of a graph G. A *Smarandachely semirelib M-graph* $T_s^M(G)$ of M is the graph whose vertex set is the union of set of edges, set of blocks and set of regions of M in which two vertices are adjacent if and only if the corresponding edges of M are adjacent, the corresponding edges lies on the blocks, the corresponding edges lies on the region, the corresponding blocks are adjacent and the graph $G \setminus M$. Particularly, if G is a planar graph, such a $T_s^M(G)$ is called the *total semirelib graph* of G denoted, denoted by $T_s(G)$.

The *edge degree* of an edge uv is the sum of the degree of the vertices of u and v. For the planar graph G, the inner vertex number i(G) of a graph G is the minimum number of vertices

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not belonging to the boundary of the exterior region in any embedding of G in the plane. A graph G is said to be minimally nonouterplanar if i(G)=1 as was given by Kulli [4].

§2. Preliminary Notes

We need the following results to prove further results.

Theorem 2.1([1]) *If G is a (p,q) graph whose vertices have degree d_i then the line graph L(G) has q vertices and q_L edges, where $q_L = -q + \frac{1}{2}\sum d_i^2$ edges.*

Theorem 2.2([1]) *The line graph L(G) of a graph is planar if and only if G is planar, $\Delta(G) \leq 4$ and if $degv = 4$, for a vertex v of G, then v is a cutvertex.*

Theorem 2.3([2]) *A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.*

Theorem 2.4([3]) *A graph is outerplanar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$.*

§3. Main Results

We start with few preliminary results.

Lemma 3.1 *For any planar graph G, $L(G) \subseteq R_s(G) \subseteq T_s(G)$.*

Lemma 3.2 *For any graph with block degree n_i, the block graph has $\binom{n_i}{2}$ edges.*

Definition 3.3 *For the graph G the block degree of a cutvertex v_i is the number of blocks incident to the cutvertex v_i and is denoted by n_i.*

In the following theorem we obtain the number of vertices and edges of a Total semirelib graph of a graph.

Theorem 3.4 *For any planar graph G, the total semirelib graph $T_s(G)$ whose vertices have degree d_i, has $q + r + b$ vertices and $\frac{1}{2}\sum d_i^2 + \sum q_j$ edges where r and b be the number of regions and blocks respectively.*

Proof By the definition of $T_s(G)$, the number of vertices is the union of edges, regions and blocks of G. Hence $T_s(G)$ has (q + r + b) vertices. Further by the Theorem 2.1, number of edges in $L(G)$ is $q_L = -q + \frac{1}{2}\sum d_i^2$. Thus the number of edges in $T_s(G)$ is the sum of the number of edges in $L(G)$, the number of edges bounded by the regions which is q, the number of edges lies on the blocks is $\sum q_j$ and the number the sum of the block degree of cutvertices which is $\sum\binom{n_i}{2}$ by the Lemma 3.2. Hence

$$E[T_s(G)] = -q + \frac{1}{2}\sum d_i^2 + q + \sum q_j + \sum\binom{n_i}{2} = \frac{1}{2}\sum d_i^2 + \sum q_j + \sum\binom{n_i}{2}. \qquad \square$$

Theorem 3.5 *For any edge in a plane graph G with edge degree e_i is n, the degree of the corresponding vertex in $T_s(G)$ is i). n if e_i is incident to a cutvertex and ii). n+1 if e_i is not incident to a cutvertex.*

Proof Suppose an edge $e_i \in E(G)$ have degree n. By the definition of total semirelib graph, the corresponding vertex in $T_s(G)$ has n-1. Since edge lies on a block, we have the degree of the vertex is $n-1+1=n$. Further, if $e_i \neq b_i \in E(G)$ then by the definition of total semirelib graph, $\forall e_i \in E(G)$, e_i is adjacent to all vertices e_j of $T_s(G)$ which are adjacent edges of e_i of G. Also the block vertex of $T_s(G)$ is adjacent to e_i. Clearly degree of e_i is $n+1$. $\square$

Theorem 3.6 *For any planar graph G with n blocks which are K_2 then $T_s(G)$ contains n pendent vertices.*

Theorem 3.7 *For any graph G, $T_s(G)$ is nonseparable.*

Proof Let $e_1, e_2, \cdots, e_n \in E(G)$, $b_1 = e_1, b_2 = e_2, \cdots, b_n = e_n$ be the blocks and $r_1, r_2, \cdots, r_k$ be the regions of G. By the definition of line graph L(G), $e_1, e_2, \cdots, e_n$ form a subgraph without isolated vertex. By the definition of $T_s(G)$, the region vertices are adjacent to these vertices to form a graph without isolated vertex. Since there are n blocks which are K_2, we have each $b_1 = e_1, b_2 = e_2, \cdots, b_n = e_n$ are adjacent to $e_1, e_2, \cdots, e_n$. Hence semirelib graph $R_s(G)$ contains n pendent vertices. By the definition of total semirelib graph, the block vertices are also adjacent. Hence $T_s(G)$ is nonseparable. $\square$

In the following theorem we obtain the condition for the planarity on total semirelib graph of a graph.

Theorem 3.8 *For any planar graph G, the $T_s(G)$ is planar if and only if G is a tree such that $\Delta(G) \leq 3$.*

Proof Suppose $R_s(G)$ is planar. Assume that $\exists v_i \in G$ such that $degv_i \geq 4$. Suppose $degv_i = 4$ and e_1, e_2, e_3, e_4 are the edges incident to v_i. By the definition of line graph, e_1, e_2, e_3, e_4 form K_4 as an induced subgraph. In $T_s(G)$, the region vertex r_i is adjacent with all vertices of $L(G)$ to form K_5 as an induced subgraph. Further the corresponding block vertices $b_1, b_2, b_3, \cdots, b_{n-1}$ of of blocks $B_1, B_2, B_3, \cdots, B_n$ in G are adjacent to vertices of K_4 and the corresponding blocks are adjacent. Clearly $T_s(G)$ forms graph homeomorphic to K_5. By the Theorem 2.3, it is non planar, a contradiction.

Conversely, Suppose $degv \leq 3$ and let e_1, e_2, e_3 be the edges of G incident to v. By the definition of line graph e_1, e_2, e_3 form K_3 as a subgraph. By the definition of $T_s(G)$, the region vertex r_i is adjacent to e_1, e_2, e_3 to form K_4 as a subgraph. Further, by the Lemma 3.2, the blocks $b_1, b_2, b_3, \cdots, b_n$ of T with n vertices such that $b_1 = e_1, b_2 = e_2, \cdots, b_{n-1} = e_{n-1}$ becomes $p-1$ pendant vertices.By the definition of $T_s(G)$, these block vertices are adjacent. Hence $T_s(G)$ is planar. $\square$

In the following theorem we obtain the condition for the outer planarity on total semirelib graph of a graph.

Theorem 3.9 *For any planar graph G, $T_s(G)$ is outer planar if and only if G is a path P_3.*

Proof Suppose $T_s(G)$ is outer planar. Assume that G is a tree with at least one vertex v such that $degv = 3$. Let e_1, e_2, e_3 be the edges of G incident to v. By the definition of line graph e_1, e_2, e_3 form K_3 as a subgraph.In $T_s(G)$, the region vertex r_i is adjacent to e_1, e_2, e_3 to form K_4 as induced subgraph. Further by the lemma 3.2, $b_1 = e_1, b_2 = e_2, \cdots, b_{n-1} = e_{n-1}$ becomes n-1 pendant vertices in $R_s(G)$. By the definition of $T_s(G)$, $i[R_s(G) \geq 1]$, which is non-outer planar ,a contradiction.

Conversely, Suppose G is a path P_3. Let $e_1, e_2 \in E(G)$. By the definition of line graph $L[P_3](G) = P_2$. Further by definition of $T_s(G), b_1 = e_1, b_2 = e_2$ forms and the vertices of line graph form C_4 . Further the region vertex r_1 is adjacent to all the vertices of $T_s(G)$ which is outer planar. □

In the following theorem we obtain the condition for the minimally non outer planar on total semirelib graph of a graph.

Theorem 3.10 *For any planar graph G, $T_s(G)$ is minimally non-outer planar if and only if G is P_4.*

Proof Suppose $T_s(G)$ is minimally non-outer planar. Assume that $G \neq P_4$.Consider the following cases.

Case 1 Assume that $G = K_{1,n}$ for $n \geq 3$. Then there exist at least one vertex of degree at least 3. Suppose $degv = 3$ for any $v \in G$. By the definition of line graph, $L[K_{1,3}] = K_3$. By the definition of $T_s(G)$,these vertices are adjacent to a region vertex r_1, which form K_4. Further the block vertices form K_3 and it has e_1, e_2, e_3 as its internal vertices. Clearly, T_s is not minimally non-outer planar, a contradiction.

Case 2 Suppose $G \neq K_{1,n}$. By the Theorem 3.9, $T_s(G)$ is non-outer planar,a contradiction.

Case 3 Assume that $G = P_n$,for $n \geq 5$. Suppose $n = 5$. By the definition of line graph,$L[P_5](G) = P_4$ and e_2, e_3 are the internal vertices of $L(G)$. By the definition of T_s, the region vertex r_1 is adjacent to all vertices of $L(G)$ to form connected graph. Further the block vertices are adjacent to all vertices of $L(G)$. Clearly the vertices e_2, e_3 becomes the internal vertices of P_s. Clearly $i[T_s] = 2$, which is not minimally nonouterplanar, a contradiction.

Conversely, suppose $G = P_4$ and let $e_1, e_2, e_3 \in E(G)$. By the definition of line graph, $L[P_4] = P_3$. Let r_1 be the region vertex in $T_s(G)$ such that r_1 is adjacent to all vertices of $L(G)$. Further the blocks b_i are adjacent to the vertices e_j for $i = j$. Clearly $i[T_s(G)] = 1$. Hence G is minimally non-outer planar. □

In the following theorem we obtain the condition for the non Eulerian on total semirelib graph of a graph.

Theorem 3.11 *For any planar graph G, $T_s(G)$ is always non Eulerian.*

Proof We consider the following cases.

Case 1 Assume that G is a tree. In a tree each edge is a block and hence $b_1 = e_1, b_2 = e_2, \cdots, b_{n-1} = e_{n-1} \forall e_{n-1} \in E(G)$ and $\forall b_{n-1} \in V[T_s(G)]$. In $T_s(G)$, the degree of a block vertex b_i is always even, but the pendent edges of G becomes the odd degree vertex in $T_s(G)$, which is non Eulerian.

Case 2 Assume that G is K_2 -free graph. We have the following subcases of Case 2.

Subcase 1 Suppose G itself is a block with even number of edges. Clearly each edge of G is of even degree. By the definition of $T_s(G)$, both the region vertices and blocks have even degree. By the Theorem 2.3, $e_i = b_i \in V[T_s(G)]$ is of odd degree, which is non Eulerian. Further if G is a block with odd number of edges, then by the Theorem 3.3, each $e_i = b_i \in V[T_s(G)]$ is of even degree. Also the block vertex and region vertex b_i, r_i are adjacent to these vertices. Clearly degree of b_i and r_i is odd, which is non Eulerian.

Subcase 2 Suppose G is a graph such that it contains at least one cutvertex. If each edge is even degree then by the sub case 1, it is non Eulerian. Assume that G contains at least one edge with odd edge degree. Clearly for any $e_j \in E(G)$, degree of $e_j \in V[T_s(G)]$ is odd, which is non Eulerian. Hence for any graph G $T_s(G)$ is always non Eulerian. □

In the following theorem we obtain the condition for the hamiltonian on total semirelib graph of a graph.

Theorem 3.12 *For any graph G, $T_s(G)$ is always hamiltonian.*

Proof Suppose G is any graph. We have the following cases.

Case 1 Consider a graph G is a tree. In a tree, each edge is a block and hence $b_1 = e_1, b_2 = e_2, \cdots, b_{n-1} = e_{n-1} \forall e_{n-1} \in E(G)$ and $\forall b_{n-1} \in V[T_s(G)]$. Since a tree T contains only ne region r_1 which is adjacent to all vertices $e_1, e_2, \cdots, e_{n-1}$ of $T_s(G)$. Also the block vertices are adjacent to each vertex e_i which corresponds to the edge of G and it is a block in G. Clearly $r_1, e_1, b_1, b_2, e_2, e_3, b_3, \cdots, r_1$ form a hamiltonian cycle. Hence$T_s(G)$ is hamiltonian graph.

Case 2 Suppose G is not a tree. Let $e_1, e_2, \cdots, e_{n-1} \in E(G)$, $b_1, b_2, \cdots, b_i$ be the blocks and $r_1, r_2, \cdots, r_k$ be the regions of G such that $e_1, e_2, \cdots, e_l \in V(b_1)$, $e_{l+1}, e_{l+2}, \cdots, e_m \in V(b_2), \cdots$, $e_{m+1}, e_{m+2}, \cdots, e_{n-1} \in V(b_i)$. By the Theorem 3.3, $V[T_s(G)] = e_1, e_2, \cdots, e_{n-1} \cup b_1, b_2, \cdots, b_i \cup r_1, r_2, \cdots, r_k$. By theorem 3.7, $T_s(G)$ is non separable. By the definition, $b_1 e_1, e_2, \cdots, e_{l-1} r_1 b_2 \cdots r_2 e_m b_3 \cdots e_{k+1}, e_{k+2}, \cdots, e_{n-1} b_k r_k e_l b_1$ form a cycle which contains all the vertices of $T_s(G)$. Hence $T_s(G)$ is hamiltonian. □

References

[1] Harary F., *Graph Theory*, Addition - Weseley Reading Mass, 1969.

[2] Harary F., *Annals of New York Academy of Science*, 1975,175 :198.

[3] Kulli V. R., On minimally non- outer planar graphs, *Proceeding of the Indian National Science Academy*, Vol.41, Part A, No.3 (1975), pp 275 -280.

[4] Y.B. Maralabhavi and Venkanagouda M. Goudar, On block edge cutvertex graph, *ACTA CIENCIA INDICA*, Vol.XXXIII M.No.2, (2007), pp. 493-49.

[5] Venkanagouda M. Goudar and K B Manjunatha Prasad, Semirelib graph of a planar graph, *Applied Mathematical Sciences*,vol.7, No.39(2013), pp1909 - 1915.