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On two inequalities for the composition of arithmetic functions

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Abstract Let f, g be arithmetic functions satisfying certain conditions. We prove the inequalities $f(g(n)) \leq 2n - \omega(n) \leq 2n - 1$ and $f(g(n)) \leq n + \omega(n) \leq 2n - 1$ for any $n \geq 1$, where $\omega(n)$ is the number of distinct prime factors of n. Particular cases include $f(n) =$ Smarandache function, $g(n) = \sigma(n)$ or $g(n) = \sigma^*(n)$.

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§1. Introduction

Let $S(n)$ be the Smarandache (or Kempner-Smarandache) function, i.e., the function that associates to each positive integer n the smallest positive integer k such that $n|k!$. Let $\sigma(n)$ denote the sum of distinct positive divisors of n, while $\sigma^*(n)$ the sum of distinct unitary divisors of n (introduced for the first time by E. Cohen, see e.g. [7] for references and many informations on this and related functions). Put $\omega(n) =$ number of distinct prime divisors of n, where $n > 1$. In paper [4] we have proved the inequality

$$S(\sigma(n)) \leq 2n - \omega(n), \tag{1}$$

for any $n > 1$, with equality if and only if $\omega(n) = 1$ and $2n - 1$ is a Mersenne prime.

In what follows we shall prove the similar inequality

$$S(\sigma^*(n)) \leq n + \omega(n), \tag{2}$$

for $n > 1$. Remark that $n + \omega(n) \leq 2n - \omega(n)$, as $2\omega(n) \leq n$ follows easily for any $n > 1$. On the other hand $2n - \omega(n) \leq 2n - 1$, so both inequalities (1) and (2) are improvements of

$$S(g(n)) \leq 2n - 1, \tag{3}$$

where $g(n) = \sigma(n)$ or $g(n) = \sigma^*(n)$.

We will consider more general inequalities, for the composite functions $f(g(n))$, where f, g are arithmetical functions satisfying certain conditions.

§2. Main results

Lemma 2.1. For any real numbers $a \geq 0$ and $p \geq 2$ one has the inequality

$$\frac{p^{a+1}-1}{p-1} \leq 2p^a - 1, \tag{4}$$

with equality only for $a = 0$ or $p = 2$.

Proof. It is easy to see that (4) is equivalent to

$$(p^a - 1)(p - 2) \geq 0,$$

which is true by $p \geq 2$ and $a \geq 0$, as $p^a \geq 2^a \geq 1$ and $p - 2 \geq 0$.

Lemma 2.2. For any real numbers $y_i \geq 2$ $(1 \leq i \leq r)$ one has

$$y_1 + \ldots + y_r \leq y_1 \ldots y_r \tag{5}$$

with equality only for $r = 1$.

Proof. For $r = 2$ the inequality follows by $(y_1 - 1)(y_2 - 1) \geq 1$, which is true, as $y_1 - 1 \geq 1$, $y_2 - 1 \geq 1$. Now, relation (5) follows by mathematical induction, the induction step $y_1 \ldots y_r + y_{r+1} \leq (y_1 \ldots y_r)y_{r+1}$ being an application of the above proved inequality for the numbers $y_1' = y_1 \ldots y_r$, $y_2' = y_{r+1}$.

Now we can state the main results of this paper.

Theorem 2.1. Let $f,\ g : \mathbb{N} \to \mathbb{R}$ be two arithmetic functions satisfying the following conditions:

(i) $f(xy) \leq f(x) + f(y)$ for any $x, y \in \mathbb{N}$.

(ii) $f(x) \leq x$ for any $x \in \mathbb{N}$.

(iii) $g(p^\alpha) \leq 2p^\alpha - 1$, for any prime powers p^α (p prime, $\alpha \geq 1$).

(iv) g is multiplicative function.

Then one has the inequality

$$f(g(n)) \leq 2n - \omega(n),\ n > 1. \tag{6}$$

Theorem 2.2. Assume that the arithmetical functions f and g of Theorem 2.1 satisfy conditions (i), (ii), (iv) and

(iii)' $g(p^\alpha) \leq p^\alpha + 1$ for any prime powers p^α.

Then one has the inequality

$$f(g(n)) \leq n + \omega(n),\ n > 1. \tag{7}$$

Proof of Theorem 2.1. As $f(x_1) \leq f(x_1)$ and

$$f(x_1x_2) \leq f(x_1) + f(x_2),$$

it follows by mathematical induction, that for any integers $r \geq 1$ and $x_1, \ldots, x_r \geq 1$ one has

$$f(x_1 \ldots x_r) \leq f(x_1) + \ldots + f(x_r). \tag{8}$$

Let now $n = p_1^{\alpha_1} \dots p_r^{\alpha_r} > 1$ be the prime factorization of n, where p_i are distinct primes and $\alpha_i \geq 1$ $(i = 1, \dots, r)$. Since g is multiplicative, by inequality (8) one has

$$f(g(n)) = f(g(p_1^{\alpha_1}) \dots g(p_r^{\alpha_r})) \leq f(g(p_1^{\alpha_1})) + \dots + f(g(p_r^{\alpha_r})).$$

By using conditions (ii) and (iii), we get

$$f(g(n)) \leq g(p_1^{\alpha_1}) + \dots + g(p_r^{\alpha_r}) \leq 2(p_1^{\alpha_1} + \dots + p_r^{\alpha_r}) - r.$$

As $p_i^{\alpha_i} \geq 2$, by Lemma 2.2 we get inequality (6), as $r = \omega(n)$.

Proof of Theorem 2.2. Use the same argument as in the proof of Theorem 2.1, by remarking that by (iii)'

$$f(g(n)) \leq (p_1^{\alpha+1} + \dots + p_r^{\alpha_r}) + r \leq p_1^{\alpha_1} \dots p_r^{\alpha_r} + r = n + \omega(n).$$

Remark 2.1. By introducing the arithmetical function $B^1(n)$ (see [7], Ch.IV.28)

$$B^1(n) = \sum_{p^\alpha \| n} p^\alpha = p_1^{\alpha_1} + \dots + p_r^{\alpha_r}.$$

(i.e., the sum of greatest prime power divisors of n), the following stronger inequalities can be stated:

$$f(g(n)) \leq 2B^1(n) - \omega(n), \tag{6'}$$

(in place of (6)); as well as:

$$f(g(n)) \leq B^1(n) + \omega(n), \tag{7'}$$

(in place of (7)).

For the average order of $B^1(n)$, as well as connected functions, see e.g. [2], [3], [8], [7].

§3. Applications

1. First we prove inequality (1).

Let $f(n) = S(n)$. Then inequalities (i), (ii) are well-known (see e.g. [1], [6], [4]). Put $g(n) = \sigma(n)$. As $\sigma(p^\alpha) = \frac{p^{\alpha+1}-1}{p-1}$, inequality (iii) follows by Lemma 2.1. Theorem 2.1 may be applied.

2. Inequality (2) holds true.

Let $f(n) = S(n)$, $g(n) = \sigma^*(n)$. As $\sigma^*(n)$ is a multiplicative function, with $\sigma^*(p^\alpha) = p^\alpha + 1$, inequality (iii)' holds true. Thus (2) follows by Theorem 2.2.

3. Let $g(n) = \psi(n)$ be the Dedekind arithmetical function, i.e., the multiplicative function whose value of the prime power p^α is

$$\psi(p^\alpha) = p^{\alpha-1}(p+1).$$

Then $\psi(p^\alpha) \leq 2p^\alpha - 1$ since

$$p^\alpha + p^{\alpha-1} \leq 2p^\alpha - 1;\ p^{\alpha-1} + 1 \leq p^\alpha;\ p^{\alpha-1}(p-1) \geq 0,$$

which is true, with strict inequality.

Thus Theorem 2.1 may be applied for any function f satisfying (i) and (ii).

4. There are many functions satisfying inequalities (i) and (ii) of Theorems 2.1 and 2.2.

Let $f(n) = \log \sigma(n)$.

As $\sigma(mn) \leq \sigma(m)\sigma(n)$ for any $m,\ n \geq 1$, relation (i) follows. The inequality $f(n) \leq n$ follows by $\sigma(n) \leq e^n$, which is a consequence of e.g. $\sigma(n) \leq n^2 < e^n$ (the last inequality may be proved e.g. by induction).

Remark 3.1. More generally, assume that $F(n)$ is a submultiplicative function, i.e., satisfying

$$F(mn) \leq F(m)F(n) \text{ for } m, n \geq 1. \tag{i'}$$

Assume also that

$$F(n) \leq e^n. \tag{ii'}$$

Then $f(n) = \log F(n)$ satisfies relations (i) and (ii).

5. Another nontrivial function, which satisfies conditions (i) and (ii) is the following

$$f(n) = \begin{cases} p, & \text{if } \ n = p \text{ (prime)}, \\ 1, & \text{if } \ n = \text{composite or } n = 1. \end{cases} \tag{9}$$

Clearly, $f(n) \leq n$, with equality only if $n = 1$ or $n =$ prime. For $y = 1$ we get $f(x) \leq f(x) + 1 = f(x) + f(1)$, when $x,\ y \geq 2$ one has

$$f(xy) = 1 \leq f(x) + f(y).$$

6. Another example is

$$f(n) = \Omega(n) = \alpha_1 + \ldots + \alpha_r, \tag{10}$$

for $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, i.e., the total number of prime factors of n. Then $f(mn) = f(m) + f(n)$, as $\Omega(mn) = \Omega(m) + \Omega(n)$ for all $m,\ n \geq 1$. The inequality $\Omega(n) < n$ follows by $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r} \geq 2^{\alpha_1 + \ldots + \alpha_r} > \alpha_1 + \ldots + \alpha_r$.

7. Define the additive analogue of the sum of divisors function σ, as follows: If $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ is the prime factorization of n, put

$$\Sigma(n) = \Sigma\left(\frac{p^{\alpha+1} - 1}{p - 1}\right) = \sum_{i=1}^{r} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}. \tag{11}$$

As $\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{\alpha_i+1}-1}{p-1}$, and $\frac{p^{\alpha+1}-1}{p-1} > 2$, clearly by Lemma 2.2 one has

$$\Sigma(n) \leq \sigma(n). \tag{12}$$

Let $f(n)$ be any arithmetic function satisfying condition (ii), i.e., $f(n) \leq n$ for any $n \geq 1$. Then one has the inequality:

$$f(\Sigma(n)) \leq 2B^1(n) - \omega(n) \leq 2n - \omega(n) \leq 2n - 1 \tag{13}$$

for any $n > 1$.

Indeed, by Lemma 2.1, and Remark 2.1, the first inequality of (13) follows. Since $B^1(n) \leq n$ (by Lemma 2.2), the other inequalities of (13) will follow. An example:

$$S(\Sigma(n)) \leq 2n - 1, \tag{14}$$

which is the first and last term inequality in (13).

It is interesting to study the cases of equality in (14). As $S(m) = m$ if and only if $m = 1$, 4 or p (prime) (see e.g. [1], [6], [4]) and in Lemma 2.2 there is equality if $\omega(n) = 1$, while in Lemma 2.1, as $p = 2$, we get that n must have the form $n = 2^\alpha$. Then $\Sigma(n) = 2^{\alpha+1} - 1$ and $2^{\alpha+1} - 1 \neq 1$, $2^{\alpha+1} - 1 \neq 4$, $2^{\alpha+1} - 1 =$ prime, we get the following theorem:

There is equality in (14) iff $n = 2^\alpha$, where $2^{\alpha+1} - 1$ is a prime.

In paper [5] we called a number n almost f-perfect, if $f(n) = 2n - 1$ holds true. Thus, we have proved that n is almost $S \circ \Sigma$-perfect number, iff $n = 2^\alpha$, with $2^{\alpha+1} - 1$ a prime (where "$\circ$" denotes composition of functions).

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