Smarandache Zero Divisors

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ABSTRACT

In this paper, we study the notion of Smarandache zero divisor in semigroups and rings. We illustrate them with examples and prove some interesting results about them.

Keywords: Zero divisor, Smarandache zero divisor

Throughout this paper, S denotes a semigroup and R a ring. They need not in general be Smarandache semigroups or Smarandache rings respectively. Smarandache zero divisors are defined for any general ring and semigroup.

Definition 1 Let S be any semigroup with zero under multiplication (or any ring R). We say that a non-zero element $a \in S$ (or R) is a Smarandache zero divisor if there exists a non-zero element b in S (or in R) such that $a.b = 0$ and there exist $x, y \in S \setminus \{a, b, 0\}$ (or $x, y \in R \setminus \{a, b, 0\}$), $x \neq y$, with

1. $ax = 0$ or $xa = 0$
2. $by = 0$ or $yb = 0$ and
3. $xy \neq 0$ or $yx \neq 0$

Remark If S is a commutative semigroup then we will have $ax = 0$ and $xa = 0$, $yb = 0$ and $by = 0$; so what we need is at least one of xa or ax is zero 'or' not in the mutually exclusive sense.

Example 1 Let $Z_{12} = \{0,1,2,...,11\}$ be the semigroup under multiplication. Clearly, Z_{12} is a commutative semigroup with zero. We have $6 \in Z_{12}$ is a zero divisor as $6.8 \equiv 0 (\text{mod } 12)$. Now 6 is a Smarandache zero divisor as $6.2 \equiv 0(\text{mod } 12)$, $8.3 \equiv 0(\text{mod } 12)$ and $2.3 \not\equiv 0(\text{mod } 12)$. Thus 6 is a Smarandache zero divisor. It is interesting to note that for $3 \in Z_{12}$, $3.4 \equiv 0(\text{mod } 12)$ is a zero divisor, but 3,4 is not a Smarandache zero divisor for there does not exist a $x,y \in Z_{12} \setminus \{0\}$ $x \neq y$ such that $3.x \equiv 0(\text{mod } 12)$ and $4y \equiv 0(\text{mod } 12)$ with $xy \not\equiv 0(\text{mod } 12)$.

This example leads us to the following theorem.

Theorem 2 Let S be a semigroup under multiplication with zero. Every Smarandache zero divisor is a zero divisor, but not reciprocally in general.

Proof: Given S is a multiplicative semigroup with zero. By the very definition of a Smarandache zero divisor in S we see it is a zero divisor in S. But if x is a zero divisor in S, it need not in general be a Smarandache zero divisor of S. We prove this by an example. Consider the semigroup Z_{12} given in example 1. Clearly 3 is a zero divisor in Z_{12} as $3.4 \equiv 0(12)$ but 3 is not a Smarandache zero divisor of 12.

Example 2 Let $S_{2\times2} = \left\{\begin{pmatrix} a & b \\ c & b \end{pmatrix} \Big/ a, b, c, d \in Z_2 = \{0,1\}\right\}$ be the set of all 2×2 matrices with entries from the ring of integers modulo 2. $S_{2\times2}$ is a semigroup under matrix multiplication modulo two. Now $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ in $S_{2\times2}$ is a zero divisor as $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in S_{2\times2}$ is such that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Now take $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ in $S_{2\times2}$. We have $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Finally, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a Smarandache zero divisor of the semigroup $S_{2\times2}$.

Example 3 Let $R_{3\times3} = \{(a_{ij}) \text{such that } a_{ij} \in Z_4 = \{0,1,2,3\}\}$ be the collection of all 3×3 matrices with entries from Z_4. Now $R_{3\times3}$ is a ring under matrix addition and multiplication modulo four. We have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in R_{3\times3} \text{ is a Smarandache zero divisor in } R_{3\times3}.$$

For

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&2\end{pmatrix}\begin{pmatrix}0&0&0\\0&1&0\\0&2&2\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \text{ and } \begin{pmatrix}0&0&0\\0&3&2\\0&0&2\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&2&2\end{pmatrix}\in R_{3\times3} \text{ such that}$$

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&2\end{pmatrix}\begin{pmatrix}0&0&0\\0&3&2\\0&0&2\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&3&2\\0&0&2\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&2\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&1&0\\0&2&2\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&2&2\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&2&2\end{pmatrix}\begin{pmatrix}0&0&0\\0&1&0\\0&2&2\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&2&0\end{pmatrix}\neq\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&3&2\\0&0&2\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&2&2\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&2&2\end{pmatrix}\begin{pmatrix}0&0&0\\0&3&2\\0&0&2\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&2&0\end{pmatrix}\neq\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

So $\begin{pmatrix}1&0&0\\0&0&0\\0&0&2\end{pmatrix}$ is Smarandache zero-divisor in $R_{3\times3}$.

Example 4: Let $Z_{20} = \{0, 1, 2, ..., 19\}$ be the ring of integers modulo 20. Clearly 10 is a Smarandache zero divisor. For $10 . 16 \equiv 0 (\text{mod } 20)$ and there exists $5, 6 \in Z_{20} \setminus \{0\}$ with

$5 \times 16 \equiv 0 \ (\text{mod } 20)$
$6 \times 10 \equiv 0 \ (\text{mod } 20)$
$6 \times 5 \equiv 10 (\text{mod } 20)$.

Theorem 3 Let R be a ring; a Smarandache zero divisor is a zero divisor , but not reciprocally in general.

Proof: By the very definition, we have every Smarandache zero divisor is a zero divisor. We have the following example to show that every zero divisor is not a Smarandache zero divisor. Let $Z_{10} = \{0,1,2,...,9\}$ be the ring of integers modulo 10.

Clearly 2 in Z_{12} is a zero divisor as $2.5 \equiv 0 (\text{mod } 10)$ which can never be a Smarandache zero divisors in Z_{10}. Hence the claim.

Theorem 4 Let R be a non-commutative ring. Suppose $x \in R \setminus \{0\}$ be a Smarandache zero divisor; with $xy = yx = 0$ and $a,b \in R \setminus \{0,x,y\}$ satisfying the following conditions:

1. $ax = 0$ and $xa \neq 0$,
2. $yb = 0$ and $by \neq 0$ and
3. $ab = 0$ and $ba \neq 0$.

Then we have $(xa + by)^2 = 0$.

Proof: Given $x \in R \setminus \{0\}$ is a Smarandache zero divisor such that $xy = 0 = yx$. We have $a,b \in R \setminus \{0,x,y\}$ such that $ax = 0$ and $xa \neq 0$, $yb = 0$ and $by \neq 0$ with $ab = 0$ and $ba \neq 0$. Consider $(xa + by)^2 = xaby + byxa + xaxa + byby$ using $ab = 0$, $yx = 0$, $ax = 0$ and $yb = 0$ we get $(xa + by)^2 = 0$.

Theorem 5 Let R be a ring having Smarandache zero divisor satisfying conditions of Theorem 5, then R has a nilpotent element of order 2.

Proof: By Theorem 5 the result is true.

We propose the following problems.

Problem 1: *Characterize rings R in which every zero divisor is a Smarandache zero divisor.*

Problem 2: *Find conditions or properties about rings so that it has Smarandache zero divisors.*

Problem 3: *Does there exists rings in which no zero divisor is a Smarandache zero divisor ?*

Problem 4: *Find group rings RG which has Smarandache zero divisors ?*

Problem 5: *Let G be a group having elements of finite order and F any field. Does the elements of finite order in G give way to Smarandache zero divisors ?*

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