THE FULFILLED EUCLIDEAN PLANE

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ABSTRACT. The fulfilled euclidean plane is the real projective plane $\overline{\Pi}$ completed with the infinite point of its infinite line denoted $\overline{\Pi}^c$. This new incidence structure is a structure with neighbouring elements, in which the unicity of the line through two distinct points is not assured. This new Geometry is a Smarandacheian structure introduced in [10] and [11], which generalizes and unites in the same time: Euclid, Bolyai Lobacewski Gauss and Riemann Geometries.

Key words: Non-euclidean Geometries, Hjelmslev-Barbilian Geometry, Smarandache Geometries, the fulfilled Euclidean plane.

1. HJELMSLEV-BARBILIAN INCIDENCE STRUCTURES

When the first Non-euclidean Geometry was introduced by Bolyai and Lobacewski even the great Gauss said that people were not prepared to receive a new Geometry. Now we know and accept many kinds of new Geometries. In 1969 Florentin Smarandache had put the problem to study a new Geometry in which the parallel from a point to a line to be unique only for some points of points and lines and for the others: none or more. More general: An axiom is said Smarandachely denied if the axiom behaves in at least two different ways within the same space (i.e., validated and invalided, or only invalided but in multiple distinct ways). Thus, a

Smarandache Geometry is a geometry which has at least one Smarandachely denied axiom.

Are nowadays people surprise for such new ideas and new Geometries? Certaintly not. After the formalized theories were introduced in Mathematics, a lot of new Geometries could be accepted and semantically to be proved to be non-contradictory by the models created for them as in [1], [2], [3], [4], [5], [6], [8], [9], [12].

Definition 1.1. We consider $\mathcal{P}, \mathcal{D}, I$ the sets which verify:

$$
\tag{1} \mathcal{P} \times \mathcal{D} = \emptyset
$$

$$
\tag{2} I \subset \mathcal{P} \times \mathcal{D}
$$

The elements of $\mathcal{P}$ are called points, the elements of $\mathcal{D}$ are called lines and I defines an incidence relation on the set $\mathcal{P} \times \mathcal{D}$. $(\mathcal{P}, \mathcal{D}, I)$ is called an **incidence structure**. If $(P, d) \in I$ we say that the point $P \in \mathcal{P}$ and the line $d \in \mathcal{D}$ are incident.

In the incidence structures introduced by D. Hilbert were accepted the axiom:

Axiom 1.1. $P_i \in \mathcal{P}$, $d_j \in \mathcal{D}$, $(P_i, d_j) \in I$, $i, j = 1, 2$ imply $P_1 = P_2$ or $d_1 = d_2$.

In [3] J. Hjelmslev generalized these incidence structures considering $(\mathcal{P}, \mathcal{D}, I)$ in which this axiom is denied, and the uniqueness of a line incident with two different points is not assured.

Definition 1.2. Two distinct points $P_1, P_2 \in \mathcal{P}$ of a $(\mathcal{P}, \mathcal{D}, I)$ are said to be **neighbouring**, denoted $P_1 \circ P_2$, if there are at least $d_1, d_2 \in \mathcal{D}$, $d_1 \neq d_2$ such that:

$$
\tag{3} (P_i, d_j) \in I, \quad i, j = 1, 2.
$$

An incidence structure $(\mathcal{P}, \mathcal{D}, I)$ with a neighbouring relation is denoted $(\mathcal{P}, \mathcal{D}, I, \circ)$.

D. Barbilian proved that such incidence structures are consistent, considering in [1] a Projective Geometry over a ring. Later such structures were studied in [2], [4], [5], [6], [8], [9], [12].

2. THE FULFILLED EUCLIDEAN PLANE

The mathematical model for the real projective plane $\overline{\overline{\Pi}}$ is:

$$(4) \qquad \mathcal{P}' := \{(\rho X, \rho Y, \rho Z) | \ X, Y, Z, \rho \in \mathbb{R}, \ \rho \neq 0\} \setminus \{(0, 0, 0)\}$$

where (X, Y, Z) are homogeneous coordinates for a point

$$(5) \qquad \mathcal{D}' = \{[qa, qb, qc] | \ a, b, c, q \in \mathbb{R}, \ q \neq 0\} \setminus \{[0, 0, 0]\}$$

is the set of the lines of the $\overline{\overline{\Pi}}$ plane.

The incidence between a point $M(X, Y, Z)$ and a line $[a, b, c]$ is defined through the condition

$$(6) \qquad aX + bY + cZ = 0$$

The **infinite line** denoted through $[\infty]$ has the equations $[0, 0, 1]$ or:

$$(7) \qquad Z = 0.$$

The infinite points of the $\overline{\overline{\Pi}}$ plane have homogeneous coordinates of the type:

$$(8) \qquad (X, Y, 0), \quad X^2 + Y^2 \neq 0$$

Let we observe that in $\overline{\overline{\Pi}}$ any line has its infinite point - except the infinite line $[\infty]$. In this note we introduce it.

Definition 2.1. The infinite point of the infinite line $[\infty]$ is $U(0, 0, 0)$ (the unique point which were not considered in $\mathcal{P}'$ in (4)).

From (6) and (8) we can see that $U(0, 0, 0)$ an infinite point incident with any line from $\mathcal{D}'$.

Definition 2.2. The real projective plane $\overline{\Pi}$ completed with the point $U(0,0,0)$ is called the completed real projective plane or the **fulfilled euclidean plane**, denoted $\overline{\Pi}^c$.

Definition 2.3. We denote $\mathcal{P}'' := \mathcal{P}' \cup \{(0,0,0)\}$ or $\mathcal{P}'' := \mathcal{P}' \cup \{U\}$. The incidence relation $I \subset \mathcal{P}' \times \mathcal{D}'$ now we prolonge it at I', $I' \subset \mathcal{P}'' \times \mathcal{D}'$ such that:

$$\text{(9)} \qquad\qquad I'|_{\mathcal{P}' \times \mathcal{D}'} = I$$

and

$$\text{(10)} \qquad\qquad U I' a, \ \forall \ a \in \mathcal{D}'$$

3. The incidence structure with neighbouring of order k

Definition 3.1. In an incidence structure $(\mathcal{P}, \mathcal{D}, I, \circ)$ with neighbouring elements we define an **order of neighbouring** of two lines $d_i \in \mathcal{D}$, $i = 1, 2$. The lines d_1 and d_2 are called neighbouring of order k if there are exactly k distinct points incident with them, that is:

$$\text{(11)} \qquad\qquad (d_i, P_j) \in I, \quad i = 1,2, \ j = 1,k$$

Definition 3.2. An incidence structure $(\mathcal{P}, \mathcal{D}, I, \circ)$ in which any two lines are neighbouring elements of order k is called a Hjelmslev Barbilian plane of order k.

Theorem 3.1. *The fulfilled euclidean plane $\overline{\Pi}^c$ is an incidence structure with neighbouring lines Hjelmslev Barbiliab of order two.*

Proof. Any two lines from $\overline{\Pi}$ are incident with exactly one point, $\overline{\Pi}$ being a projective plane. In $\overline{\Pi}^c$ any two lines are incident also with the point $U(0,0,0)$ which was not considered in $\overline{\Pi}$.

If two lines a and b from $\overline{\Pi}$ are incident with the P point, that is:

$$\text{(12)} \qquad\qquad P I a, b$$

then in $\overline{\overline{\Pi}}^c$ the lines a and b are incident with the two points P and U. Such we have:

$$(13) \qquad\qquad\qquad\qquad P \circ U$$

that is – after definition 1.2 – P and U are neighbouring points.

The lines a and b of $\mathcal{D}'$ are neighbouring lines of order two:

$$(14) \qquad\qquad\qquad\qquad a \circ_2 b,$$

because we have:

$$(15) \qquad\qquad\qquad\qquad P, U I' a, b, \ a \neq b,$$

for any two distinct lines from $\overline{\overline{\Pi}}^c$.

If a or b is the infinite line $[\infty]$ then P from (12) is an infinite point. If a and b are different of the line $[\infty]$ then P is a propre point of $\mathcal{P}'$.

In any case a and b are always incident with exactly two points from $\overline{\overline{\Pi}}^c$. Such we proved that $\overline{\overline{\Pi}}^c$ is a Hjelmslev-Barbilian plane of order two.

If $\overline{\Pi}$ is the real projective plane of a Π-euclidean plane we can see that:

$$(16) \qquad\qquad\qquad\qquad \Pi \subset \overline{\Pi} \subset \overline{\overline{\Pi}}^c$$

Definition 3.3. In the real space we consider a sphere $\mathcal{S}$ tangent in P to a Π euclidean plane and let be N the diametral opposite point of P on the $\mathcal{S}$. We define the stereographyc projection of the pole N from $\mathcal{S}$ to $\overline{\overline{\Pi}}^c$:

$$(17) \qquad\qquad\qquad\qquad f : \mathcal{S} \to \overline{\overline{\Pi}}^c$$

$$f(M) := M' \text{ where } \{M'\} = NM \cap \overline{\Pi}$$

and

$$f(N) := U.$$

Such through f we obtain a bijection between the all points of $\mathcal{S}$ and the points of $\overline{\Pi}^c$.

Some others applications of $\overline{\Pi}^c$ we gave in [14] as a transdisciplinary study given after the notions given in [7].

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