A Multiple Theorem with Isogonal and Concyclic Points

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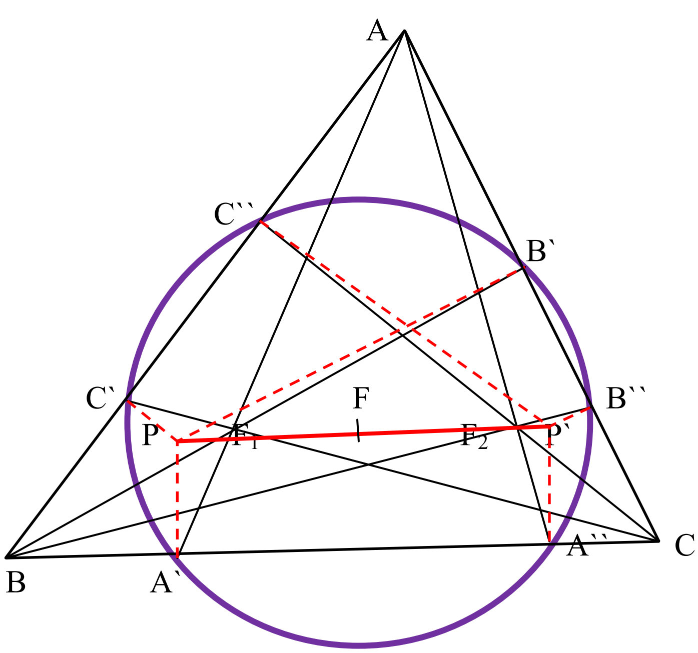
Let's consider A', B', C' three points on the sides (BC), (CA), (AB) of triangle ABC such that simultaneously are satisfied the following conditions:

i. $A'B^2 + B'C^2 + C'A^2 = A'C^2 + B'A^2 + C'B^2$
ii. The lines AA', BB', CC' are concurrent.

Prove that:

a) The perpendiculars drawn in A' on BC, in B' on AC, and in C' on AB are concurrent in a point P.
b) The perpendiculars drawn in A' on $B'C'$, in B' on $A'C'$, and in C' on $A'B'$ are concurrent in a point P'.
c) The points P and P' are isogonal.
d) If A'', B'', C'' are the projections of P' on BC, CA, respective AB, then the points A', A'', B', B'', C', C'', are concyclic points.
e) The lines AA'', BB'', CC'' are concurrent.

Proof:



a) Let P be the intersection of the perpendicular drawn in A' on BC with the perpendicular drawn in B' on AC. We have:

$$PB^2 - PC^2 = A'B^2 - A'C^2$$
$$PC^2 - PA^2 = B'C^2 - B'A^2.$$

By adding side by side these two relations, it results

$$PB^2 - PA^2 = A'B^2 - A'C^2 + B'C^2 - B'A^2. \quad (1)$$

If we note with C_1 the projection of P on AB, we have:

$$PB^2 - PA^2 = C_1B^2 - C_1A^2. \quad (2)$$

From the relations (1), (2), and (i) we obtain that $C_1 \equiv C'$, therefore P has as ponder triangle the triangle $A'B'C'$.

b) Let A_1, B_1, C_1 respective the orthogonal projections of the points A, B, C on $B'C', C'A'$ respectively $A'B'$.

We have

$$A_1C'^2 - A_1B'^2 = C'A^2 - B'A^2,$$
$$B_1C'^2 - B_1A'^2 = C'B^2 - A'B^2,$$
$$C_1A'^2 - C_1B'^2 = A'C^2 - B'C^2.$$

From these relations we deduct

$$A_1C'^2 + B_1A'^2 + C_1B'^2 = A_1B'^2 + B_1C'^2 + C_1B'^2$$

therefore, a relation of the same type as (i) for the triangle $A'B'C'$. By using a similar method it results that $A_1B_1C_1$ is the triangle ponder of a point P'.

c) The quadrilateral $AB'PC'$ is inscribable, therefore $\sphericalangle APB' \equiv \sphericalangle AC'B'$, and because these angles are the complements of the angles $\sphericalangle C'AP$ and $\sphericalangle B'AP'$, it results that these angles are congruent, therefore the Cevians AP and AP' are isogonal, similarly we can show that the Cevians BP and BP' are isogonal and also the Cevians CP and CP' are isogonal.

d) It is obvious that the medians of the segments $(A'A''), (B'B'')$ and $(C'C'')$ pass through F, which is the middle of the segment (PP'). We have to prove that F is the center of the circle that contains the given points of the problem.

We will use the median's theorem on the triangles $C'PP'$ and $B'PP'$ to compute $C'F$ and $B'F$.

We note $m\left(\widehat{P'AC}\right) = m(\sphericalangle PAB) = \alpha$, $AP = x$, $AP' = x'$;

then we have

$$4C'F^2 = 2\left(PC'^2 + P'C'^2\right) - PP'^2$$
$$4B'F^2 = 2\left(PB'^2 + P'B'^2\right) - PP'^2$$
$$PC' = x\sin\alpha,\ P'C'^2 = P'C''^2 + C''C'^2,\ P'C'' = x'\sin(A-\alpha)$$
$$AC'' = x'\cos(A-\alpha),\ AC' = x\cos\alpha,$$
$$P'C'^2 = x'^2 + \sin^2(A-\alpha) + \left(x'\cos(A-\alpha) - x\cos\alpha\right)^2 =$$

$$= x'^2 + x^2\cos^2\alpha - 2xx'\cos\alpha\cos(A-\alpha)$$

$$4C'F^2 = 2\left[x'^2 + x^2\cos^2\alpha - 2xx'\cos\alpha\cos(A-\alpha)\right] - PP'^2$$

$$4C'F^2 = 2\left[x'^2 + x^2 - 2xx'\cos\alpha\cos(A-\alpha)\right] - PP'^2$$

Similarly we determine the expression for $4B'F^2$, and then we obtain that $C'F = B'F$, therefore the points C', C'', B'', B' are concyclic.

We'll follow the same method to prove that $C'F = A'F$ which leads to the fact that the points C', C'', A', A'' are also concyclic, and from here to the requested statement.

e) From (ii) it results (from Ceva's theorem) that:

$$A'B \cdot B'C \cdot C'A = A'C \cdot B'A \cdot C'B. \quad (3)$$

Let's consider the points' A, B, C power respectively in rapport to the circle determined by the points $A', A'', B', B'', C', C''$,
we have

$$AB' \cdot AB'' = AC' \cdot AC''$$
$$BA' \cdot BA'' = BC' \cdot BC''$$
$$CA' \cdot CA'' = CB' \cdot CB''.$$

Multiplying these relations we obtain:

$$A'B \cdot BA'' \cdot B'C \cdot BC'' \cdot C'A \cdot AC'' = C'B \cdot BC'' \cdot B'A \cdot AB'' \cdot A'C \cdot CA''. \quad (4)$$

Taking into account the relation in (3), it results

$$BA'' \cdot CB'' \cdot AC'' = BC'' \cdot AB'' \cdot CA''.$$

This last relation along with Ceva's theorem will lead us to the conclusion that the lines AA'', BB'', CC'' are concurrent.

Reference:

F. Smarandache, *Problèmes avec et sans … problèmes!*, Somipress, Fès, Morocco, 1983.