ON RECURRENT STATIONARY SEQUENCES

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Abstract.
In this paper one studies in what conditions a recurrent sequence becomes stationary.

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Introduction.

Define a sequence $\{a_n\}$ by $a_1 = a$ and $a_{n+1} = f(a_n)$, where f is a real-valued function of real variable. For what values of a and for what functions f will this sequence be constant after a certain rank ?

In this note, the author answers for this question referring to F. Lazebnik and Y. Pilipenko's E 3036 proposed problem.

An interesting property of functions admitting fixed points is obtained.

Construction of a recurrent set.

Because $\{a_n\}$ is constant after a certain rank, it results that $\{a_n\}$ converges. Hence $(\exists)\ e \in R : e = f(e)$, that is the equation $f(x) - x = 0$ admits real solutions. Or f admits fixed points $((\exists)\ x \in R : f(x) = x)$.

Let $e_1, \ldots, e_m$ be all real solutions of this equation.

One constructs the recurrent set E, so:

(1) $e_1, \ldots, e_m \in E$;

(2) If $b \in E$, then all real solutions of the equation $f(x) = b$ belong to E;

(3) No other elements belong to E, except the elements obtained from rules (1) or (2) applied a finite number of times.

We prove that the set E, and the set A of values of a for which $\{a_n\}$ becomes constant after a certain rank, are indistinct.

"$E \subseteq A$"

(1) If $a = e_i$, $1 \leq i \leq m$, then $(\forall)\ n \in N^*\ a_n = e_i =$ $=$ constant.

(2) If for a = b, the sequence $a_1 = b$, $a_2 = f(b)$, ... becomes constant after a certain rank; let x_0 be a real solution of the equation $f(x) - b = 0$, the new formed sequence: $a'_1 = x_0$, $a'_2 = f(x_0) = b$, $a'_3 = f(b)$, ... is indistinct after a certain rank with the first one, hence it becomes constant too, having the same limit.

(3) Beginning from a certain rank, all these

sequences converge towards the same limit e (that is: they have the same value "e" from a certain rank), thus they are indistinct, equal to e.

"$A \supseteq E$"

Let "a" be a value such that: $\{a_n\}$ becomes constant (after a certain rank) equal to e. Of course $e \in \{e_1, ..., e_m\}$, because $e_1, ..., e_m$ are the only values towards with these sequences can tend.

If $a \in \{e_1, ..., e_m\}$, then $a \in E$.

Let $a \notin \{e_1, ..., e_m\}$. Then $(\exists)\ n_0 \in N$: $a_{n_0+1} = f(a_{n_0}) = e$, hence we obtain "a" applying the rules (1) or (2) a finite number of times. So, because $e \in \{e_1, ..., e_m\}$ and the equation $f(x) = e$ admits real solutions we find a_{n_0} among the real solutions of this equation; knowing a_{n_0} we find a_{n_0-1} because the equation $f(a_{n_0-1}) = a_{n_0}$ admits real solutions (because $a_{n_0} \in E$), and our method goes on until we find $a_1 = a$. Hence $a \in E$.

Remark.

For $f(x) = x^2 - 2$ we obtain the E 3036 Problem [1]. Here, the set E becomes equal to:

$$\{\pm 1, 0, \pm 2\} \cup \{\underbrace{\pm\sqrt{(2\pm\sqrt{(2\pm\sqrt{(...\sqrt{2})...))}}}}_{n \text{ times}}, n\in N^*\} \cup$$

$$\{\underbrace{\pm\sqrt{(2\pm\sqrt{(2...\pm\sqrt{(2\pm\sqrt{3})...))}}}}_{n \text{ times}}, n\in N\}.$$

Hence, for all $a \in E$ the sequence $a_1 = a$, $a_{n+1} = a_n^2 - 2$ becomes constant after a certain rank, and it converges (of course) towards -1 or 2:

$$(\exists)n_0 \in N^* : (\forall)\ n \geq n_0 \quad a_n = -1$$

or

$$(\exists)n_0 \in N^* : (\forall)\ n \geq n_0 \quad a_n = 2.$$

References:

[1] F. Lazebnik, Y. Pilipenko, “Problem E 3036”, Am. Math. Monthly, Vol. 91, No. 2, 140, 1984.

[2] F. Smarandache, “Collected Papers”, Vol. I, Tempus, Bucharest, 27-29, 1996.