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Generalized separation axioms

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Abstract In this paper we discuss new separation axioms using g-open sets.
Keywords G-closed sets, g-open sets, g_i spaces.

§1. Introduction

Norman Levine introduced generalized closed sets, K. Balachandaran and P. Sundaram studied generalized continuous functions and generalized homeomorphism. V. K. Sharma studied generalized separation axioms. Following V. K. Sharma we are going to further study the properties of generalized separation axioms. Throughout the paper a space X means a topological space (X, τ). For any subset A of X its complement, interior, closure, g-interior, g-closure are denoted respectively by the symbols A^c, A^0, A^-, $g(A)^0$ and $g(A)^-$. $(Y_{/A})^-$ represents the closure in the subspace Y in X with relativized topology $\tau_{/Y}$.

§2. Preliminaries

Definition 2.1. $A \subset X$ is called

(i) generalized closed [5](briefly g-closed) if $(A)^- \subseteq U$ whenever $A \subseteq U$ and U is open.

(ii) generalized open if its compliment is g-closed.

(iii) regularly open if $A = ((A)^-)^0$.

Definition 2.2. Let $A \subset X$ and $x \in X$. Then x is called g-limit point of A if each g-open set containing x intersects A. There exists a g-open set U containing x such that $x \in U \subseteq A$.

Note 1. The class of g-open sets and regularly open sets are denoted by $GO(X)$ and $RO(X)$ respectively. Clearly $RO(X) \subset \tau(X) \subset GO(X)$.

Definition 2.3. X is said to be

(i) $GO - compact$ [6] if every g-open cover has a finite subcover.

(ii) $r - T_0$ if for each pair of distinct points x, y of X, there exists a regular-open set G containing either x or y.

(iii) rT_1[resp : rT_2] if for each pair of distinct points x, y of X there exists [resp: disjoint] regular-open sets G and H such that G containing x but not y and H containing y but not x.

Definition 2.4. A function f: X$\rightarrow Y$ is said to be

(i) nearly continuous [resp: nearly-irresolute] if inverse image of every open [resp: regular-open] set is regular-open.

(ii) g-continuous [g-irresolute] [6] if the inverse image of any closed [g-closed] set in Y is a g-closed set in X.

(iii) g-open [6] if the image of every g-open set is g-open.

(iv) g-homeomophism [resp: gc-homeomophism] [6] if f is bijective, g-continuous [resp: g-irresolute] and g-open.

Theorem 2.1.

(i) If x is a g-limit point of any $A \subset X$, then every g-neighbourhood of x contains infinitely many distinct points.

(ii) Let $A \subseteq$Y$\subseteq$X and Y is regularly open subspace of X then A is g-open in X iff A is g-open in $\tau_{/Y}$.

Theorem 2.2. If f is g-continuous [resp: g-irresolute{g-homeomorphism}] and G is open [resp: g-open[g-closed]] set in Y, then $f^{-1}(G)$ is g-open [resp: g-open [g-closed]] in X.

Theorem 2.3.[2] Let Y and $\{X_\alpha : \alpha \in I\}$ be Topological Spaces. Let $f : Y \to \Pi X_\alpha$ be a function. If f is g-continuous, then $\pi_\alpha \circ f : Y \to X_\alpha$ is g-continuous.

Theorem 2.4.[2] If Y is $T_{\frac{1}{2}}$ and $\{X_\alpha : \alpha \in I\}$ be Topological Spaces. Let $f : Y \to \Pi X_\alpha$ be a function, then f is g-continuous iff π_α circ$f$: Y $\to$ X$_\alpha$ is g-continuous.

Corollary 2.1.[2] Let $f_\alpha : X_\alpha \to Y_\alpha$ be a function and let $f : \Pi X_\alpha \to \Pi Y_\alpha$ be defined by $f\left((x_\alpha)_{\alpha\in I}\right) = (f_\alpha(x_\alpha))_{\alpha\in I}$. If f is g-continuous then each f_α is g-continuous.

Corollary 2.2.[2] For each α, X_α be $T_{\frac{1}{2}}$ and let $f_\alpha : X_\alpha \to Y_\alpha$ be a function and let $f : \Pi X_\alpha \to \Pi Y_\alpha$ be defined by $f\left((x_\alpha)_{\alpha\in I}\right) = (f_\alpha(x_\alpha))_{\alpha\in I}$, then f is g-continuous iff each f_α is g-continuous.

§3. g_i spaces, i = 0, 1, 2.

Definition 3.1. X is said to be

(i) g_0 space if for each pair of distinct points x, y of X, there exists a g-open set G containing either x or y.

(ii) g_1 space if for each pair of distinct points x, y of X, there exists a g-open set G containing x but not y and a g-open set H containing y but not x.

(iii) g_2 space if for each pair of distinct points x, y of X, there exists disjoint g-open sets G and H such that G containing x but not y and H containing y but not x.

Note 2. By note 1. $r - T_i \Rightarrow T_i \Rightarrow g_i$ for $i = 0, 1, 2$ but the converse is not true in general.

Example 3.1. Let $X = \{a, b, c\}$ and $\tau = \{\phi$, {a, c}, X } then X is g_0 but not r-T_0 and T_0.

Example 3.2. Let $X = \{a, b, c, d\}$.

(i) $\tau = \wp(X)$ then X is g_i for $i = 0, 1, 2$.

(ii)$\tau = \{\phi$, {a}, {a, b}, {c, d}, {a, c, d}, X } then X is not g_i for $i = 0, 1, 2$.

Remark 3.1. If X is $T_{\frac{1}{2}}$ then T_i and g_i are one and the same for $i = 0, 1, 2$.

Proof. Since X is $T_{\frac{1}{2}}$, every g-closed set is closed and hence proof follows from the definitions.

Theorem 3.1. Every [regular] open subspace of g_i space is g_i, for $i = 0, 1, 2$.

Theorem 3.2. X is g_0 iff $\forall x \in X, \exists U \in GO(X)$ containing $x \ni$ the subspace U is g_0.

Theorem 3.3. If f: X$\rightarrow Y$ is injective, g-irresolute and Y is g_i then X is g_i for $i = 0, 1, 2$.

Theorem 3.4.

(i) The product of g_i [7] spaces is again g_i for $i = 0, 1, 2$.

(ii) X is g_0 [7] iff distinct points of X have disjoint g-closures.

Corollary 3.1.

(i) If f: X$\rightarrow Y$ is injective, g-continuous and Y is T_i then X is g_i for $i = 0, 1, 2$.

(ii) If f: X$\rightarrow Y$ is injective, nearly-irresolute [nearly-continuous] and Y is $r - T_i$ then X is g_i for $i = 0, 1, 2$.

(iii) The property of being a space is g$_i$ is a g-topological property for $i = 0, 1, 2$.

(iv) Let f: $X \rightarrow Y$ is a g-homeomorphism, then X is g_i if Y is g$_i$ for $i = 0, 1, 2$.

Theorem 3.5. The following are equivalent:

(i) X is g_1.

(ii) Each one point set is g-closed.

(iii) Each subset of X is the intersection of all g-open sets containing it.

(iv) For any $x \in X$, intersection of all g-open sets containing the point is the set {x}.

Theorem 3.6. If a space X is g_1 then distinct points of X have disjoint g-closures.

Theorem 3.7. Suppose x is a g-limit point of a subset A of a g_1 space X. Then every g-neighborhood of x contains infinitely many distinct points of A.

Theorem 3.8. Let X be T_1 and f: X $\rightarrow Y$ be g-closed sujection. Then X is g_1.

Remark 3.2. X is $g_2 \Rightarrow X$ is $g_1 \Rightarrow X$ is g_0.

Theorem 3.9. A space X is g_2 iff the intersection of all g-closed, g-neighbourhoods of each point of the space is reduced to that point.

Proof. Let X be g_2 and $x \in X$, then for each $y \neq x$ in X, there exists g-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Since $x \in U - V$, hence $X - V$ is a g-closed, g-neighbourhood of x to which y does not belong. Consequently, the intersection of all g-closed, g-neighbourhoods of x is reduced to $\{x\}$.

Conversely let $x, y \in X$, such that $y \neq x$ in X, then by hypothesis there exists a g-closed, g-neighbourhood U of x such that $y \notin U$. Now there exists a g-open set G such that $x \in G \subset U$. Thus G and $X - U$ are disjoint g-open sets containing x and y respectively. Hence X is g_2.

Theorem 3.10. If to each $x \in X$, there exists a g-closed, g-open subset of X containing x and which is also a g_2 subspace of X, then X is g_2.

Proof. Let $x \in X$, U a g-closed, g-open subset of X containing x and which is also a g_2 subspace of X, then the intersection of all g-closed, g-neighbourhood of x in U is reduced to x. U being g-closed, g-open, these are g-closed, g-neighbourhood of x in X. Thus the intersection of all g-closed, g-neighbourhoods of x is reduced to $\{x\}$. Hence X is g_2.

Theorem 3.11. If X is g_2 then the diagonal Δ in $X \times X$ is g-closed.

Proof. Suppose $(x, y) \in X \times X - \Delta$. As $(x, y) \notin \Delta$ and $x \neq y$. Since X is $g_2, \exists U; V \in GO(X) \ni x \in U$, $y \in V$ and $U \cap V = \phi$. $U \cap V = \phi \Rightarrow (U \times V) \cap \Delta = \phi$ and therefore $(U \times V) \subset X \times X - \Delta$. Further $(x, y) \in (U \times V)$ and $(U \times V)$ is g-open in $X \times X$ gives $X \times X - \Delta$ is g-open. Hence Δ is g-closed.

Corollary 3.2.

(i) In an $T_1[rT_1]$ space, each singleton set is g-closed.

(ii) If X is $T_2[r-T_2]$ then the diagonal Δ in $X \times X$ is g-closed.

Theorem 3.12. In g_2-space, g-limits of sequences, if exists, are unique.

Proof. Let (x_n) be a sequence in g_2-space X and if $(x_n) \to x$; $(x_n) \to y$ as n $\to \infty$. If $x \neq y$ then, for X is $g_2, \exists$ disjoint U; $V \in GO(X) \ni x \in U, y \in V$ and $U \cap V = \phi$. Then $\exists N_1, N_2 \in N \ni x_n \in U; \forall n \geq N_1$, and $x_n \in V$; $\forall n \geq N_2$. Let $m \in Z^+ \ni m > \{N_1; N_2\}$. Then $x_m \in U \cap V$ contradicting the fact $U \cap V = \phi$. So $x = y$ and thus the g-limits are unique.

Theorem 3.13. In a g_2 space, a point and disjoint $GO-compact$ subspace can be separated by disjoint g-open sets.

Proof. Let X be a g_2 space, x a point in X and C the disjoint $GO-compact$ subspace of X not containing x. Let y be a point in C then for $x \neq y$ in X and X is g_2, there exist disjoint g-neighborhoods G_x and H_y. Allowing this for each y in C, we obtain a class $\{H_y\}$ whose union covers C; and since C is $GO-compact$, some finite subclass, which we denote by $\{H_i, i = 1$ to $n\}$ covers C. If G_i is the g-neighborhood of x corresponding to H_i, we put $G = \bigcup_{i=1}^{n} G_i$ and $H = \bigcap_{i=1}^{n} H_i$, satisfying the required properties.

Theorem 3.14. Every $GO-compact$ subspace of a g_2 space X is g-closed.

Proof. Let C be $GO-compact$ subspace of a g_2 space. If x be any point in C^c, by above theorem x has a g-neighborhood G such that $x \in G \subset C^c$. This shows that C^c is the union of g-open sets and therefore C^c is g-open. Thus C is g-closed.

Corollary 3.3.

(i) Show that in a $T_2[r-T_2]$ space, a point and disjoint compact [r-compact] subspace can be separated by disjoint g-open sets.

(ii)Every compact [r-compact] subspace of a $T_2[r-T_2]$ space is g-closed.

Theorem 3.15. Every g-irresolute map from a $GO-compact$ space into a g_2 space is g-closed.

Proof. Suppose $f\colon X \to Y$ is g-irresolute where X is $GO-compact$ and Y is g_2. Let C be any g-closed subset of X. Then C is $GO-compact$ and so $f(C)$ is $GO-compact$. But then $f(C)$ is g-closed in Y. Hence the image of any g-closed set in X is g-closed set in Y. Thus f is g-closed.

Theorem 3.16.

(i) Any g-irresolute bijection from a $GO-compact$ space onto a g_2 space is a gc-homeomop -hism.

(ii) Any g-continuous bijection from a $GO-compact$ space onto a g_2 space is a gc-homeomop -hism.

Proof. (i) Let $f\colon X \to Y$ be a g-irresolute bijection from a $GO-compact$ space onto a g_2 space. Let G be an g-open subset of X. Then $X-G$ is g-closed and hence $f(X-G)$is g-closed. Since f is bijective $f(X-G) = Y - f(G)$. Therefore $f(G)$ is g-open in Y. This means that f is g-open. Hence f is bijective g-irresolute and g-open. Thus f is gc-homeomophism.

(ii) Similar to the previous and so omitted.

Theorem 3.17. The following are equivalent:

(i) X is g_2.

(ii) For each pair $x \neq y \in X, \exists$ a g-open, g-closed set $V \ni x \in V; y \notin V$, and

(iii) For each pair $x \neq y \in X, \exists$ a g-continuous function $f : X \to [0,1] \ni f(x) = 0$ and $f(C) = 1$.

Corollary 3.4. Let Y be a g_2 space. If f: X$\to Y$ be one-one and g-irresolute [nearly-irresolute]. Then X is g_2.

Theorem 3.18. If f: X$\to Y$ is g-irresolute and Y is g_2 then

(i) the set $A = \{(x_1, x_2) : f(x_1) = f(x_2)\}$ is g-closed in $X \times X$.

(ii) G(f), the graph of f is g-closed in $X \times Y$.

Proof. (i) Let $A = \{(x_1, x_2) : f(x_1) = f(x_2)\}$. If $(x_1, x_2) \in X \times X - A$, then $f(x_1) \neq f(x_2) \Rightarrow \exists$ disjoint $V_1; V_2 \in GO(Y) \ni f(x_1) \in V_1$ and $f(x_2) \in V_2$, then by g-irresoluteness of f, $f^{-1}(V_j) \in GO(X, x_j)$ for each j. Thus $(x_1, x_2) \in f^{-1}(V_1) \times f^{-1}(V_2) \in GO(X \times X)$. Therefore $f^{-1}(V_1) \times f^{-1}(V_2) \subset X \times X - A \Rightarrow X \times X - A$ is g-open. Hence A is g-closed.

(ii) Let $(x, y) \notin G(f) \Rightarrow y \neq f(x) \Rightarrow \exists$ disjoint $V; W \in GO(X) \ni f(x) \in V$ and y$\in W$. Since f is g-irresolute, $\exists U \in GO(X) \ni x \in U$ and $f(U) \subset W$. Therefore we obtain $(x, y) \in U \times V \subset X \times Y - G(f)$. Hence $X \times Y - G(f)$ is g-open. Hence G(f) is g-closed in $X \times Y$.

Theorem 3.19. If f: X$\to Y$ is g-open and the set $A = \{(x_1, x_2) : f(x_1) = f(x_2)\}$ is closed in $X \times X$. Then Y is g_2.

Theorem 3.20. Let X be an arbitrary space, R an equivalence relation in X and $p : X \to X/R$ the identification map. If $R \subset X \times X$ is $g-$closed in $X \times X$ and p is $g-$open map, then X/R is g_2.

Proof. Let p(x), p(y) be distinct members of X/R. Since x and y are not related, $R \subset X \times X$ is $g-$closed in $X \times X$. There are $U; V \in GO(X) \ni x \in U, y \in V$ and $U \times V \subset R^c$. Thus *p(U), p(V)* are disjoint and also $g-$open in X/R since p is $g-$open.

Theorem 3.21. The following four properties are equivalent:

(i) X is g_2.

(ii) Let $x \in X$. For each $y \neq x, \exists U \in GO(X) \ni x \in U$ and $y \notin g(U)^-$.

(iii) For each $x \in X, \cap\{g(U)^-/U \in GO(X)$ and $x \in U\} = \{x\}$.

(iv) The diagonal $\Delta = \{(x, x)/x \in X\}$ is $g-$closed in $X \times X$.

Proof. $(i) \Rightarrow (ii)$ Let $x \in X$ and $y \neq x$. Then $\exists$ disjoint $U; V \in GO(X) \ni x \in U$ and $y \in V$. Clearly V^c is $g-$closed, $g(U)^- \subset V^c, y \notin V^c$ and therefore $y \notin g(U)^-$.

$(ii) \Rightarrow (iii)$ If $y \neq x$, then $\exists U \in GO(X) \ni x \in U$ and $y \notin g(U)^-$. So $y \notin \cap\{g(U)^-/U \in GO(X)$ and $x \in U\}$.

$(iii) \Rightarrow (iv)$ We prove Δ^c is $g-$open. Let $(x, y) \notin \Delta$. Then $y \neq x$ and $\cap\{g(U)^-/U \in GO(X, \tau)$ and $x \in U\} = \{x\}$ there is some $U \in GO(X)$ with $x \in U$ and $y \notin g(U)^-$. Since $U \cap (g(U)^-)^c = \phi$, $U \times (g(U)^-)^c \in GO(X) \ni (x, y) \in U \times (g(U)^-)^c \subset \Delta^c$.

$(iv) \Rightarrow (i)$ $y \neq x$, then $(x, y) \notin \Delta$ and thus $\exists U; V \in GO(X) \ni (x, y) \in U \times V$ and $(U \times V) \cap \Delta = \phi$. Clearly, for $U; V \in GO(X)$ we have: $x \in U, y \in V$ and $U \cap V = \phi$.

§4. $g - R_i$ spaces i = 0, 1.

Definition 4.1. Let $x \in X$. Then

(i) g-kernel of x is denoted and defined by $ker_g\{x\} = \cap\{U : U \in GO(X)$ and $x \in U\}$.

(ii) $Ker_gF = \cap\{U : U \in$GO(X) and $F \subset$ U$\}$.

Lemma 4.1. Let $A \subset X$, then $ker_g\{A\} = \{x \in X : g\{x\}^- \cap A \neq \phi.\}$

Lemma 4.2. Let $x \in X$. Then $y \in Ker_g\{x\}$ iff $x \in g\{y\}^-$.

Proof. Suppose that $y \notin Ker_g\{x\}$. Then $\exists V \in GO(X)$ containing $x \ni y \notin V$. Therefore we have $x \notin g\{y\}^-$. The proof of converse case can be done similarly.

Lemma 4.3. For any points $x \neq y \in X$, the following are equivalent:

(1) $Ker_g\{x\} \neq Ker_g\{y\}$;

(2) $g\{x\}^- \neq g\{y\}^-$.

Proof. (1)$\Rightarrow$(2) Let $Ker_g\{x\} \neq Ker_g\{y\}$, then $\exists z \in X \ni z \in Ker_g\{x\}$ and $z \notin Ker_g\{y\}$. From $z \in Ker_g\{x\}$ it follows that $x \cap g\{z\}^- \neq \phi \Rightarrow x \in g\{z\}^-$. By $z \notin Ker_g\{y\}$, we have $\{y\} \cap g\{z\}^- = \phi$. Since $x \in g\{z\}^-, g\{x\}^- \subset g\{z\}^-$ and $\{y\} \cap g\{x\}^- = \phi$. Therefore $g\{x\}^- \neq g\{y\}^-$. Now $Ker_g\{x\} \neq Ker_g\{y\} \Rightarrow g\{x\}^- \neq g\{y\}^-$.

(2)$\Rightarrow$(1) If $g\{x\}^- \neq g\{y\}^-$. Then $\exists z \in X \ni z \in g\{x\}^-$ and $z \notin g\{y\}^-$. Then $\exists$ a g-open set containing z and therefore containing x but not y, namely, $y \notin Ker_g\{x\}$. Hence $Ker_g\{x\} \neq Ker_g\{y\}$.

Definition 4.2. X is said to be

(i) $g - R_0$ if and only if $g\{x\}^- \subseteq G$ whenever $x \in G \in g - O(\tau)$.

(ii) weakly $g - R_0$ iff $\cap g\{x\}^- = \phi$.

(iii) $g - R_1$ iff for $x, y \in X \ni g\{x\}^- \neq g\{y\}^-, \exists$ disjoint $U; V \in GO(X) \ni g\{x\}^- \subseteq U$ and $g\{y\}^- \subseteq V$.

Example 4.1.

(i) Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$, then X is weakly $g - R_0$ and $g - R_i$, i = 0, 1.

(ii) Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$, then X is neither weakly $g - R_0$ nor $g - R_i$, $i = 0, 1$.

Remark 4.1. $rR_i \Rightarrow R_i \Rightarrow gR_i$, $i = 0, 1$.

Theorem 4.1.

(i) Every weakly-R_0 space is weakly $g - R_0$.

(ii) Every subspace of $g - R_1$ space is again $g - R_1$.

(ii) Product of any two $g - R_1$ spaces is again $g - R_1$.

Lemma 4.4. Every $g - R_0$ space is weakly $g - R_0$.

Converse of the above lemma is not true in general by the following examples.

Example 4.2.

(i) Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}$. Clearly, the space (X, τ) is weakly $g - R_0$, since $\cap g\{x\}^- = \phi$. But it is not $g - R_0$, for $\{b\} \subset X$ is g-open and $g\{b\}^- = \{b, c\} \not\subset \{b\}$.

(ii) Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{b, c\}, X\}$. Clearly, the space (X, τ) is $g - R_0$ and R_0.

Theorem 4.2. X is $g - R_0$ iff given $x \neq y \in X; g\{x\}^- \neq g\{y\}^-$.

Proof. Assume X is $g - R_0$ and let $x,\ y$ be two distinct points of X. suppose U is a g-open set containing x but not y, then $y \in g\{y\}^- \subset X - U$ and so $x \notin g\{y\}^-$. Hence $g\{x\}^- \neq g\{y\}^-$.

Conversely, let $x \neq y$ be any two points in $X \ni g\{x\}^- \neq g\{y\}^- \Rightarrow g\{x\}^- \subset X - g\{y\}^- = U$(say) a g-open set in X. This is true for every $g\{x\}^-$. Thus $\cap g\{x\}^- \subseteq U$ where

$x \in g\{x\}^- \subseteq U \in GO(\tau)$, which in turn implies $\cap g\{x\}^- \subseteq U$ where $x \in U \in g - O(\tau)$. Hence X is $g - R_0$.

Theorem 4.3. X is weakly $g - R_0$ iff $ker_g\{x\} \neq X$ for any $x \in X$.

Proof. Let $x_0 \in X \ni \ker_g\{x_0\} = X$. This means that x_0 is not contained in any proper g-open subset of X. Thus x_0 belongs to the g-closure of every singleton set. Hence $x_0 \in \cap g\{x\}^-$, a contradiction.

Conversely assume that $ker_g\{x\} \neq X$ for any x$\in X$. If there is a point $x_0 \in X \ni x_0 \in \cap\{g\{x\}^-\}$, then every g-open set containing x_0 must contain every point of X. Therefore, the unique g-open set containing x_0 is X. Hence $ker_g\{x_0\} = X$, which is a contradiction. Thus X is weakly $g - R_0$.

Theorem 4.4. The following statements are equivalent:

(i) X is $g - R_0$ space.

(ii) For each x$\in X, g\{x\}^- \subset ker_g\{x\}$.

(iii) For any g-closed set F and a point x$\notin F, \exists U \in GO(X) \ni x \notin U$ and $F \subset U$.

(iv) Each g-closed set F can be expressed as $F = \cap\{G$: G is g-open and $F \subset G\}$.

(v) Each g-open set G can be expressed as the union of g-closed sets A contained in G.

(vi) For each g-closed set F, $x \notin F$ implies $g\{x\}^- \cap F = \phi$.

Proof. (i)$\Rightarrow$(ii) For any $x \in X$, we have $ker_g\{x\} = \cap\{U : U \in GO(X)$ and $x \in U\}$. Since X is $g - R_0$, each g-open set containing x contains $g\{x\}^-$. Hence $g\{x\}^- \subset ker_g\{x\}$.

(ii)$\Rightarrow$(iii) Let $F \in GC(X)\&x \in X - F$. Then for any $y \in F$, $g\{y\}^- \subset F$ and so $x \notin g\{y\}^- \Rightarrow y \notin g\{x\}^-$ that is $\exists U_y \in GO(X) \ni y \in U_y$ and $x \notin U_y$ for all $y \in F$. Let $U = \cup\{U_y : U_y$ is $g -$ open,$y \in U_y$ and $x \notin U_y\}$. Then U is g-open such that $x \notin U$ and $F \subset U$.

(iii)$\Rightarrow$(iv) Let $F \in GC(X)$ and $N = \cap\{G : G$ is g-open and $F \subset G\}$. Then $F \subset N \longrightarrow (1)$. Let $x \notin F$, then by (iii), $\exists G \in GO(X) \ni x \notin G$ and $F \subset G$, hence $x \notin N$ which implies $x \in N \Rightarrow x \in F$. Hence $N \subset F \longrightarrow (2)$.

Therefore from (1)&(2), each g-closed set $F = \cap\{G : G$ is g-open and $F \subset G\}$.

(iv)$\Rightarrow$(v) obvious.

(v)$\Rightarrow$(vi) Let F be any g-closed set and $x \notin F$. Then $X - F = G$ is a g-open set containing x. Then by (v), G can be expressed as the union of g-closed sets A contained in G, and so there is a g-closed set M such that $x \in M \subset G$; and hence $g\{x\}^- \subset G$ which implies $g\{x\}^- \cap F = \phi$.

(vi)$\Rightarrow$(i) Let G be any g-open set and $x \in G$. Then $x \notin X - G$, which is a g-closed set.

Therefore by (vi) $g\{x\}^- \cap X - G = \phi$, which implies that $g\{x\}^- \subseteq G$. Thus (X, τ) is $g - R_0$ space.

Theorem 4.5. Let f: $X \to Y$ be a g-closed one-one function. If X is weakly $g - R_0$, then so is Y.

Theorem 4.6. If X is weakly $g - R_0$, then for every space $Y, X \times Y$ is weakly $g - R_0$.

Proof. $\cap g\{(x, y)\}^- \subseteq \cap\{g\{x\}^- \times g\{y\}^-\} = \cap[g\{x\}^-] \times [g\{y\}^-] \subseteq \phi \times Y = \phi$. Hence $X \times Y$ is $g - R_0$.

Corollary 4.1.

(i) If X and Y are weakly $g - R_0$, then $X \times Y$ is weakly $g - R_0$.

(ii) If X and Y are (weakly-)R_0, then $X \times Y$ is weakly $g - R_0$.

(iii) If X and Y are $g - R_0$, then $X \times Y$ is weakly $g - R_0$.

(iv) If X is $g-R_0$ and Y are weakly R_0, then $X \times Y$ is weakly $g-R_0$.

Theorem 4.7. X is $g-R_0$ iff for any $x;y \in X, g\{x\}^- \neq g\{y\}^- \Rightarrow g\{x\}^- \cap g\{y\}^- = \phi$.

Proof. Let X is $g-R_0$ and $x;y \in X \ni g\{x\}^- \neq g\{y\}^-$ .Then, $\exists z \in g\{x\}^- \ni z \notin g\{y\}^-(orz \in g\{y\}^-) \ni z \notin g\{x\}^-$. There exists $V \in GO(X) \ni y \notin V$ and $z \in V$; Hence $x \in V$. Therefore, we have $x \notin g\{y\}^-$. Thus $x \in [Gy^-]^c \in GO(X)$, which implies $g\{x\}^- \subset [g\{y\}^-]^c$ and $g\{x\}^- \cap g\{y\}^- = \phi$. The proof for otherwise is similar.

Sufficiency: Let $V \in GO(X)$ and let $x \in V$. We show that $g\{x\}^- \subset V$. Let $y \notin V$, i.e., $y \in [V]^c$. Then $x \neq y$ and $x \notin g\{y\}^-$. Hence $g\{x\}^- \neq g\{y\}^-$. By assumption, $g\{x\}^- \cap g\{y\}^- = \phi$. Hence $y \notin g\{x\}^-$. Therefore $g\{x\}^- \subset V$.

Theorem 4.8. A space X is $g-R_0$ iff for any points $x;y \in X, Ker_g\{x\} \neq Ker_g\{y\} \Rightarrow Ker_g\{x\} \cap Ker_g\{y\} = \phi$.

Proof. Suppose X is $g-R_0$. Thus by lemma 4.3. for any points $x;y \in X$ if $Ker_g\{x\} \neq Ker_g\{y\}$ then $g\{x\}^- \neq g\{y\}^-$. Assume that $z \in Ker_g\{x\} \cap Ker_g\{y\}$. By $z \in Ker_g\{x\}$ and lemma 4.2, it follows that $x \in g\{z\}^-$. Since $x \in g\{x\}^-$, $g\{x\}^- = g\{z\}^-$. Similarly, we have $g\{y\}^- = g\{z\}^- = g\{x\}^-$. This is a contradiction. Therefore, we have $Ker_g\{x\} \cap Ker_g\{y\} = \phi$.

Conversely, let x and y be any two points in X, $\ni g\{x\}^- \neq g\{y\}^-$, then by lemma 4.3, $Ker_g\{x\} \neq Ker_g\{y\}$. Hence by hypothesis $Ker_g\{x\} \cap Ker_g\{y\} = \phi$ which implies $g\{x\}^- \cap g\{y\}^- = \phi$. Because $z \in g\{x\}^-$ implies that $x \in Ker_g\{z\}$ and therefore $Ker_g\{x\} \cap Ker_g\{z\} \neq \phi$. Therefore X is a $g-R_0$ space.

Theorem 4.9. The following properties are equivalent:

(1) X is a $g-R_0$ space.

(2) For any $A \neq \phi$ and $G \in GO(X,\tau) \ni A \cap G \neq \phi, \exists F \in GC(X,\tau) \ni A \cap F \neq \phi$ and $F \subset G$.

Proof. (1)$\Rightarrow$(2) Let $A \neq \phi$ and $G \in GO(X) \ni A \cap G \neq \phi$. There exists $x \in A \cap G$. Since $x \in G \in GO(X), g\{x\}^- \subset G$. Set $F = g\{x\}^-$, then $F \in GC(X), F \subset G$ and $A \cap F \neq \phi$.

(2)$\Rightarrow$(1) Let $G \in GO(X)$ and $x \in G$. Suppose $y \in Ker_g\{x\}$, then $x \in g\{y\}^-$ and $y \in G$. This implies that $g\{x\}^- \subset Ker_g\{x\} \subset G$. Hence X is a $g-R_0$ space.

Corollary 4.2. The following properties are equivalent:

(1) X is $g-R_0$.

(2) $g\{x\}^- = Ker_g\{x\} \forall x \in X$.

Proof. (1)$\Rightarrow$(2) Suppose X is $g-R_0$. By theorem 4.9, $g\{x\}^- \subset Ker_g\{x\} \forall x \in X$. Let $y \in Ker_g\{x\}$, then $x \in g\{y\}^-$ and by theorem 4.7. $g\{x\}^- = g\{y\}^-$. Therefore, $y \in g\{x\}^-$ and hence $Ker_g\{x\} \subset g\{x\}^-$. Hence $g\{x\}^- = Ker_g\{x\}$.

(2)$\Rightarrow$(1) This is obvious by theorem 4.9.

Theorem 4.10. The following properties are equivalent:

(1) X is a $g-R_0$ space;

(2) $x \in g\{y\}^-$ if and only if $y \in g\{x\}^-$, for any points x and y in X.

Proof. (1)$\Rightarrow$(2) Assume that X is $g-R_0$. Let $x \in g\{y\}^-$ and D be any g-open set such that $y \in D$. Now by hypothesis, $x \in D$. Therefore, every g-open set which containy contains x. Hence $y \in g\{x\}^-$.

(2)$\Rightarrow$(1) Let U be a g-open set and $x \in U$. If $y \notin U$, then $x \notin g\{y\}^-$ and hence $y \notin g\{x\}^-$. This implies that $g\{x\}^- \subset U$. Hence X is $g-R_0$.

Theorem 4.11. The following properties are equivalent:

(1) X is a $g - R_0$ space.

(2) If F is g-closed, then $F = Ker_g(F)$.

(3) If F is g-closed and $x \in F$, then $Ker_g\{x\} \subseteq F$.

(4) If $x \in X$, then $Ker_g\{x\} \subset g\{x\}^-$.

Proof. (1)⇒(2) Let $x \notin F \in GC(X) \Rightarrow X - F \in GO(X)$ and contains x. For X is $g - R_0, G(\{x\})^- \subset X - F$. Thus $G(\{x\})^- \cap F = \phi$ and $x \notin Ker_g(F)$. Hence $Ker_g(F) = F$.

(2)⇒(3) $A \subset B \Rightarrow Ker_g(A) \subset Ker_g(B)$. Therefore, by (2) $Ker_g\{x\} \subset Ker_g(F) = F$.

(3)⇒(4) Since $x \in g\{x\}^-$ and $g\{x\}^-$ is g-closed, by (3) $Ker_g\{x\} \subset g\{x\}^-$.

(4)⇒(1) Let $x \in g\{y\}^-$. Then by lemma 4.2, $y \in Ker_g\{x\}$. Since $x \in g\{x\}^-$ and $g\{x\}^-$ is g-closed, by (4) we obtain $y \in Ker_g\{x\} \subseteq g\{x\}^-$. Therefore $x \in g\{y\}^-$ implies $y \in g\{x\}^-$. The converse is obvious and X is $g - R_0$.

Recall that a filterbase F is called g-convergent to a point x in X, if for any g-open set U of X containing x, there exists $B \in F$ such that $B \subset U$.

Lemma 4.5. Let x and y be any two points in X such that every net in X g-converging to y g-converges to x. Then $x \in g\{y\}^-$.

Proof. Suppose that $x_n = y$ for each $n \in N$. Then $\{x_n\}_{n\in N}$ is a net in $g(\{y\})^-$. Since $\{x_n\}_{n\in N}$ $g-$converges to y, then $\{x_n\}_{n\in N}$ $g-$converges to x and this implies that $x \in g\{y\}^-$.

Theorem 4.12. The following statements are equivalent:

(1) X is a $g - R_0$ space.

(2) If $x, y \in X$, then $y \in g\{x\}^-$ iff every net in X g-converging to y g-converges to x.

Proof.(1)⇒(2) Let $x, y \in X \ni y \in g\{x\}^-$. Suppose that $\{x_\alpha\}_{\alpha\in\Lambda}$ is a net in $X \ni \{x_\alpha\}_{\alpha\in\Lambda}$ $g-$converges to y. Since $y \in g\{x\}^-$, by theorem 4.7. we have $g\{x\}^- = g\{y\}^-$. Therefore $x \in g\{y\}^-$. This means that $\{x_\alpha\}_{\alpha\in\Lambda}$ $g-$converges to x.

Conversely, let $x, y \in X$ such that every net in X $g-$converging to y g-converges to x. Then $x \in g\{y\}^-$ by theorem 4.4. By theorem 4.7, we have $g\{x\}^- = g\{y\}^-$. Therefore $y \in g\{x\}^-$.

(2)⇒(1) Let $x; y$ be any two points of $X \ni g\{x\}^- \cap g\{y\}^- \neq \phi$. Let $z \in g\{x\}^- \cap g\{y\}^-$. So $\exists$ a net $\{x_\alpha\}_{\alpha\in\Lambda}$ in $g\{x\}^- \ni \{x_\alpha\}_{\alpha\in\Lambda}$ $g-$converges to z. Since $z \in g\{y\}^-$, then $\{x_\alpha\}_{\alpha\in\Lambda}$ $g-$converges to y. It follows that $y \in g\{x\}^-$. Similarly we obtain $x \in g\{y\}^-$. Therefore $g\{x\}^- = g\{y\}^-$. Hence, X is $g - R_0$.

Theorem 4.13. A space X is $g - R_1$ iff given $x \neq y \in X, g\{x\}^- \neq g\{y\}^-$.

Theorem 4.14. Every g_2 space is $g - R_1$.

The converse is not true. However, we have the following result.

Theorem 4.15. Every g_1, $g - R_1$ space is g_2.

Proof. Let X be g_1 and $g - R_1$ space. Let $x \neq y \in X$. Since X is g_1, $\{x\}$ is g-closed set and $\{y\}$ is g-closed set such that $g\{x\}^- \neq g\{y\}^-$. Since X is $g - R_1$, there exists disjoint g-open sets U and V such that $x \in U$, $y \in V$. Hence X is g_2.

Corollary. X is g_2 iff it is $g - R_1$ and g_1.

Theorem 4.16. The following are equivalent

(i) X is $g - R_1$.

(ii) $\cap g\{x\}^- = \{x\}$.

(iii) For any $x \in X$, intersection of all g-neighborhoods of x is $\{x\}$.

Proof. (i)⇒(ii) Let $y \neq x \in X \ni y \in g\{x\}^-$. since X is $g - R_1$, therefore there is a g-open set U such that $y \in U$, x∉U or $x \in$U, $y \notin U$. In either case $y \notin g\{x\}^-$. Hence $\cap g\{x\}^- = \{x\}$.

(ii)⇒(iii) If $x, y \in X$ where $y \neq x$, then x$\notin \cap g\{y\}^-$, so there is a g-open set containing x but not y. Therefore y does not belong to the intersection of all g-neighborhoods of x. Hence intersection of all g-neighborhoods of x is $\{x\}$.

(iii)⇒(i) Let $x, y \in X$ where $y \neq x$. By hypothesis, y does not belong to the intersection of all g-neighborhoods of x and x does not belong to the intersection of all g-neighborhoods of y, whcih implies $g\{x\}^- \neq g\{y\}^-$, therefore by theorem 4.13, X is $g - R_1$.

Theorem 4.17. The following are equivalent:

(i) X is $g - R_1$.

(ii) For each pair $x, y \in X$ such that $g\{x\}^- \neq g\{y\}^-$, there exists a g-open, g-closed set V such that $x \in V$ and $y \notin V$, and

(iii) For each pair $x, y \in X$ such that $g\{x\}^- \neq g\{y\}^-$, there exists a g-continuous function $f\colon (X, \tau) \to [0, 1]$ such that f(x) = 0 and f(C) = 1.

Proof. (i)⇒(ii) Let $x, y \in X$ such that $g\{x\}^- \neq g\{y\}^-$, then there exists disjoint g-open sets U and W such that $g\{x\}^- \subset U$ and $g\{y\}^- \subset W$ and $V = g\{U\}^-$ is g-open and g-closed such that $x \in V$ and $y \notin V$.

(ii)⇒(iii) Let $x, y \in X$ such that $g\{x\}^- \neq g\{y\}^-$, and let V be g-open and g-closed such that $x \in V$ and $y \notin V$. Then $f\colon (X, \tau) \to [0, 1]$ defined by $f(z) = 0$ if $z \in V$ and $f(z) = 1$ if $z \notin V$ satisfied the desired properties.

(iii)⇒(i) Let $x, y \in X$ such that $g\{x\}^- \neq g\{y\}^-$, let $f : (X, \tau) \to [0, 1]$ such that f is g-continuous, $f(x) = 0$ and $f(y) = 1$. Then $U = f^{-1}([0, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 1])$ are disjoint g-open and g-closed sets in X, such that $g\{x\}^- \subset U$ and $g\{y\}^- \subset V$.

Theorem 4.18. If X is $g - R_1$, then X is $g - R_0$.

Proof. Let $x \in U \in GO(X)$. If $y \notin U$, then $g\{x\}^- \neq g\{y\}^-$. Hence, $\exists$ a g-open $V_y \ni g\{y\}^- \subset V_y$ and $x \notin V_y, \Rightarrow y \notin g\{x\}^-$. Thus $g\{x\}^- \subset U$. Therefore X is $g - R_0$.

Theorem 4.19. X is $g - R_1$ iff for $x, y \in X$, $Ker_g\{x\} \neq Ker_g\{y\}, \exists$ disjoint $U; V \in GO(X) \ni g\{x\}^- \subset U$ and $g\{y\}^- \subset V$.

Following diagram indicates the interrelation among the separation axioms.

§5. $g - C_i$ and $g - D_i$ spaces, i = 0, 1, 2.

Definition 5.1. X is said to be a

(i) $g - C_0$ space if for each pair of distinct points x, y of X there exists a $g-$open set G whose closure contains either of the point x or y.

(ii) $g - C_1$ space if for each pair of distinct points x, y of X there exists a $g-$open set G whose closure containing x but not y and a $g-$open set H whose closure containing y but not x.

(iii) $g - C_2$ space if for each pair of distinct points x, y of X there exists disjoint $g-$open sets G and H such that G containing x but not y and H containing y but not x.

Note. $g - C_2 \Rightarrow g - C_1 \Rightarrow g - C_0$ but converse need not be true in general as shown by the following example.

Example 5.1.

(i) Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, then X is gC_i, $i = 0, 1, 2$.

(ii) Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ then X is not gC_i, $i = 0, 1, 2$.

Theorem 5.1.

(i) Every subspace of $g - C_i$ space is $g - C_i$.

(ii) Every g_i spaces is $g - C_i$.

(iii) Product of $g - C_i$ spaces are $g - C_i$.

Theorem 5.2. Let (X, τ) be any $g - C_i$ space and A be any non empty subset of X then A is $g - C_i$ iff $(A, \tau_{/A})$ is $g - C_i$.

Theorem 5.3.

(i) If X is $g - C_1$ then each singleton set is $g-$closed.

(ii) In an $g - C_1$ space disjoint points of X has disjoint $g-$ closures.

Definition 5.2. $A \subset X$ is called a gDifference (shortly gD-set) set if there are two U, $V \in GO(X)$ such that $U \neq X$ and $A = U - V$.

Clearly every $g-$open set U different from X is a g D-set if $A = U$ and $V = \phi$.

Definition 5.3. X is said to be a

(i) $g-D_0$ if for any pair of distinct points x and y of X there exist a gD-set in X containing x but not y or a gD-set in X containing y but not x.

(ii) $g-D_1$ if for any pair of distinct points x and y in X there exist a gD-set of X containing x but not y and a gD-set in X containing y but not x.

(iii) $g - D_2$ if for any pair of distinct points x and y of X there exists disjoint gD-sets G and H in X containing x and y, respectively.

Example 5.2. (iii) Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ then X is gD_i. $i = 0, 1, 2$.

Remark 5.2.

(i) If X is $r - T_i$, then it is $g_i, i = 0, 1, 2$ and converse is false.

(ii) If X is g_i, then it is $g_{i-1}, i = 1, 2$.

(iii) If X is g_i, then it is $g - D_i, i = 0, 1, 2$.

(iv) If X is $g - D_i$, then it is $g - D_{i-1}, i = 1, 2$.

Theorem 5.4. The following statements are true:

(i) X is $g - D_0$ if and only if it is g_0.

(ii) X is $g - D_1$ if and only if it is $g - D_2$.

Proof. (i) The sufficiency is stated in remark 5.1(iii).

To prove necessity, let X be $g - D_0$. Then for each distinct pair of points x, $y \in X$, at least one of x, y, say x, belongs to a gD-set G but $y \notin G$. Let $G = U_1 - U_2$ where $U_1 \neq X$ and $U_1, U_2 \in GO(X, \tau)$. Then $x \in U_1$ and for $y \notin G$ we have two cases: (a) $y \notin U_1$; (b) $y \in U_1$ and $y \notin U_2$.

In case (a), $x \in U_1$ but $y \notin U_1$;

In case (b), $y \in U_2$ but $x \notin U_2$. Hence X is g_0.

(ii) Sufficiency. remark 5.1(iv).

Necessity. Suppose X is $g - D_1$. Then for each $x \neq y \in X$, we have gD-sets $G_1, G_2 \ni x \in G_1; y \notin G_1; y \in G_2, x \notin G_2$. Let $G_1 = U_1 - U_2, G_2 = U_3 - U_4$. From $x \notin G_2$, it follows that

either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.

(1) $x \notin U_3$. By $y \notin G_1$ we have two subcases:

(a) $y \notin U_1$. From $x \in U_1 - U_2$, it follows that $x \in U_1 - (U_2 \cup U_3)$ and by $y \in U_3 - U_4$ we have $y \in U_3 - (U_1 \cup U_4)$. Therefore $(U_1 - (U_2 \cup U_3)) \cap (U_3 - (U_1 \cup U_4) = \phi$.

(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 - U_2, y \in U_2$. $(U_1 - U_2) \cap U_2 = \phi$.

(2) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 - U_4, x \in U_4$. $(U_3 - U_4) \cap U_4 = \phi$. Therefore X is $g - D_2$.

Corollary 5.1. If X is $g - D_1$, then it is g_0.

Proof. remark 5.1(iv) and theorem 5.2(i).

Definition 5.4. A point $x \in X$ which has X as the unique $g-$neighborhood is called gc.c point.

Theorem 5.5. For an g_0 space X the following are equivalent:

(1) X is $g - D_1$.

(2) X has no gc.c point.

Proof. (1)$\Rightarrow$(2) Since X is $g - D_1$, then each point x of X is contained in a gD-set $O = U - V$ and thus in U. By definition $U \neq X$. This implies that x is not a gc.c point.

(2)$\Rightarrow$(1) If X is g_0, then for each $x \neq y \in X$, at least one of them, x (say) has a $g-$neighborhood U containing x and not y. Thus U which is different from X is a gD-set. If X has no gc.c point, then y is not a gc.c point. This means that there exists a $g-$neighborhood V of y such that $V \neq X$. Thus $y \in (V - U)$ but not x and $V - U$ is a gD-set. Hence X is $g - D_1$.

Corollary 5.2. A g_0 space X is $g - D_1$ if and only if there is a unique gc.c point in X.

Proof. Only uniqueness is sufficient to prove. If x_0 and y_0 are two gc.c points in X then since X is g_0, at least one of x_0 and y_0 say x_0, has a $g-$neighbourhood U such that $x_0 \in U$ and $y_0 \notin U$, hence $U \neq X$, x_0 is not a gc.c point, a contradiction.

Remark 5.2. It is clear that an g_0 space X is not $g - D_1$ if and only if there is a unique $g-$c.c point in X. It is unique because if x and y are both gc.c point in X, then at least one of them say x has a $g-$neighborhood U containing x but not y. But this is a contradiction since $U \neq X$.

Definition 5.5. X is $g-$symmetric if for x and y in X, $x \in g\{y\}^-$ implies $y \in g\{x\}^-$.

Theorem 5.6. X is $g-$symmetric if and only if {x} is gg-closed for each $x \in X$.

Proof. Assume that $x \in g\{y\}^-$ but $y \notin g\{x\}^-$. This means that $[g\{x\}^-]^c$ containsy. This implies that $g\{y\}^-$ is a subset of $[g\{x\}^-]^c$. Now $[g\{x\}^-]^c$ contains x which is a contradiction.

Conversely, suppose that $\{x\} \subset E \in g(X, \tau)$ but $g\{x\}^-$ is not a subset of E. This means that $g\{x\}^-$ and E^c are not disjoint. Let y belongs to their intersection. Now we have $x \in g\{y\}^-$ which is a subset of E^c and $x \notin E$. But this is a contradiction.

Corollary 5.3. If X is a g_1, then it is $g-$symmetric.

Proof. In a g_1 space, singleton sets are $g-$closed (theorem 2.2(ii)) and therefore $gg-$closed (remark 5.3). By theorem 5.6, the space is $g-$symmetric.

Corollary 5.4. The following are equivalent:

(1) X is $g-$symmetric and g_0.

(2) X is g_1.

Proof. By corollary 5.3 and remark 5.1 it suffices to prove only (1)→(2). Let $x \neq y$ and by g_0, we may assume that $x \in G_1 \subset \{y\}^c$ for some $G_1 \in GO(X,\tau)$. Then $x \notin g\{y\}^-$ and hence $y \notin g\{x\}^-$. There exists a $G_2 \in GO(X,\tau)$ such that $y \in G_2 \subset \{x\}^c$ and X is a g_1 space.

Theorem 5.7. For an $g-$symmetric space X the following are equivalent:

(1) X is g_0. (2) X is $g - D_1$. (3) X is g_1.

Proof. (1)⇒(3) corollary 5.4 and (3)⇒ (2) ⇒ 1) remark 5.1.

Theorem 5.8. If $f\colon X \to Y$ is a $g-$irresolute surjective function and E is a gD-set in Y, then $f^{-1}(E)$ is a gD-set in X.

Proof. Let E be a gD-set in Y. Then there $\exists U_1; U_2 \in GO(X) \ni E = U_1 - U_2$ and $U_1 \neq Y$. By the $g-$irresoluteness of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are $g-$open in X. Since $U_1 \neq Y$, we have $f^{-1}(U_1) \neq X$. Hence $f^{-1}(E) = f^{-1}(U_1) - f^{-1}(U_2)$ is a $g-$D-set.

Theorem 5.9. If (Y,σ) is $g - D_1$ and $f\colon (X,\tau) \to (Y,\sigma)$ is $g-$irresolute and bijective, then (X,τ) is $\mathrm{g} - \mathrm{D}_1$.

Proof. Suppose that Y is a $g - D_1$ space. Let $x \neq y \in X$ be any pair of points. Since f is injective and Y is $g - D_1$, there exist $g-$D-sets G_z and G_y of Y containing f(x)and f (y) respectively, such that $f(y) \notin G_z$ and $f(x) \notin G_y$. By theorem 5.8, $f^{-1}(G_z)$ and $f^{-1}(G_y)$ are $g-$D-sets in X containing x and y, respectively. This implies that X is a $g - D_1$ space.

Theorem 5.10. X is $g - D_1$ if and only if for each pair of distinct points x, y in X there exists a $g-$iresolute surjective function $f\colon (X,\tau) \to (Y,\sigma)$, where Y is a $g - D_1$ space such that f(x) and f(y) are distinct.

Proof. Necessity. For every $x \neq y \in X$, it suffices to take the identity function on X.

Sufficiency. Let x and y be any pair of distinct points in X. By hypothesis, there exists a $g-$irresolute, surjective function f of a space X onto a $g - D_1$ space Y such that $f(x) \neq f(y)$. Therefore, there exist disjoint $g-$D-sets G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is $g-$irresolute and surjective, by theorem 5.8, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint $g-$D-sets in X containing x and y respectively. Therefore X is $g - D_1$ space.

Corollary 5.5. Let $\{X_\alpha/\alpha \in I\}$ be any family of topological spaces. If X_α is $g - D_1$ for each $\alpha \in I$, then the product ΠX_α is $g - D_1$.

Proof. Let (x_α) and (y_α) be any pair of distinct points in ΠX_α. Then there exists an index $\beta \in I$ such that $x_\beta \neq y_\beta$. The natural projection $P_\beta : \Pi X_\alpha \to X_\beta$ almost continuous and almost open and $P_\beta((x_\alpha)) = P_\beta((y_\alpha))$. Since X_β is $g - D_1$, ΠX_α is $g - D_1$.

§4. Conclusion

In this paper we defined new separation axioms using g-open sets and studied their interrelations with other separation axioms.

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On the bounds of the largest eigen value and the Laplacian energy of certain class of graphs

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Abstract The energy of a graph G is defined as the sum of the eigen values of the adjacency matrix of the graph G. We report the upper Bounds for the Laplacian energy of the $L(S(K_n))$, $L(S(W_n))$, $L(S(T_{n,k}))$ and $L(S(L_n))$, where L and S stands for line graph and subdivision graph of G. The bounds for the largest eigen value of the line graph of the subdivision graph of same class of graphs are obtained.

Keywords Energy, subdivision graph, zagrab index, Laplacian matrix.

§1. Introduction

Let G be a simple graph and $A(G)$ be the *adjacency matrix* of G. The eigenvalues of G are just the eigenvalues of the matrix $A(G)$ [2]. The *energy* $E(G)$ of G is defined as the sum of the absolute values of the eigenvalues of G. The maximum eigen value of the adjacency matrix is denoted by λ_{max}. This notion was proposed by Gutman [3] and found applications in the molecular orbital theory of conjugated π-electron systems [4,6].

Let $D(G)$ be the degree diagonal matrix of the graph G. Then $L(G) = D(G) - A(G)$ is the *Laplacian matrix* of G. Denote by $\mu_1(G)$, $\mu_2(G)$,..., $\mu_n(G)$ the Laplacian eigenvalues of G, arranged in a nonincreasing order, where n is the number of vertices of G (see [8]). The *Laplacian energy* $LE(G)$ [7,15] of G is defined as the sum of the distance between *Laplacian eigenvalues* of G and the average degree $d(G)$ of G, for which more results may be found in [15,16].

The *first Zagreb index* [5,12] of a graph G is defined as

$$M(G) = \sum d^2(u),$$

where $d(u)$ is the degree of the vertex u in G. The *subdivision graph* $S(G)$ is the graph obtained from G by replacing each of its edge by a path of length 2, or equivalently by inserting an additional vertex into each edge of G [11]. The *line graph* $L(G)$ is the graph whose vertices corresponds to the edges of G with two vertices being adjacent if and only if the corresponding edges in G have a vertex in common.

The *tadpole graph* $T_{n,k}$ [13] is the graph obtained by joining a cycle graph C_n to a path of length k. The *wheel graph* W_{n+1} is defined as the graph $K_1 + C_n$ where K_1 is the singleton

graph and C_n is the cycle graph [9]. The *helm graph* H_{n+1} is the graph obtained from the wheel Graph W_{n+1}, by adjoining a pendent edge at each node of the cycle. The ladder graph $L_n = K_2 \Box P_n$, where P_n is a path graph. In [10], the authors obtained some results using the notion ladder graph.

§2. Preliminaries

The following results will be useful in our further investigation.

Theorem 2.1.[14] Let G be a graph with n vertices and $m \geq 1$ edges. Then the line graph $L(G)$ have m vertices and $\frac{M(G)}{2} - m$ edges.

Theorem 2.2.[14] Let G be a graph with n vertices and $m \geq 1$ edges, then

1. $E(L(G)) \leq 4m - 2$.
2. $LE(L(G)) \leq (2M(G) - 4m)(1 - \dfrac{1}{m})$.

Theorem 2.3.[1] The maximal eigen value $\lambda_{max}(G)$ of the bipartite graph G with e number of edges is

$$\lambda_{max}(G) \leq \sqrt{e}.$$

§3. The maximal eigen value and energy

This section deals with the bounds for the *laplacian energy* and the *largest eigen value* of the *line graph* of the *subdivision graph* of the tadpole graph, wheel graph, complete graph and ladder graph.

Theorem 3.1. For the line graph of the subdivision graph of the tadpole graph $T_{n,k}$, the bounds for the laplacian energy and the maximal eigen value is given by theorem 2.2. Then

$$LE(L(S(T_{n,k}))) \leq \frac{2(2n+2k+1)(2n+2k-1)}{n+k}) \quad \text{and} \quad \lambda_{\max}(\mathrm{L(S(T_{n,k}))}) \leq \sqrt{2\mathrm{n}+2\mathrm{k}+1}.$$

Proof. The cardinality of the subdivision graph of the tadpole graph $T_{n,k}$ is $2(n+k)$ of which one vertex of degree 3, one pendent vertex and $2n+2k-2$ vertices of degree 2. Hence the zagreb index of the subdivision graph of the tadpole graph $T_{n,k}$ is

$$M(S(T_{n,k})) = 8n + 8k + 2. \tag{1}$$

Hence from equation 1 and theorem 2.2 the Laplacian energy of the line graph of the subdivision graph of the tadpole graph $T_{n,k}$ written as

$$LE(L(S(T_{n,k}))) \leq \frac{2(2n+2k+1)(2n+2k-1)}{n+k}.$$

The graph $S(T_{n,k})$ is a bipartite graph having the cardinality of the edge set $2n+2k$. Hence

$$\lambda_{max}(S(T_{n,k})) \leq \sqrt{2(n+k)}.$$

From equation (1) and theorem 2.1, the number of edges in the line graph of the subdivision graph of the tadpole graph

$$2n + 2k + 1. \tag{2}$$

So, the maximum eigen value of $L(S(T_{n,k})$ written from equation (2) as

$$\lambda_{max}(L(S(T_{n,k}))) \leq \sqrt{2n+2k+1}.$$

Theorem 3.2. For the line graph of the subdivision graph of the wheel graph W_{n+1}, the bounds for the Laplacian energy and the maximal eigen value is

$$LE(L(S(W_{n+1}))) \leq \frac{1}{2}(n+9)(4n-1) \quad \text{and} \quad \lambda_{\max}(\mathrm{L(S(W_{n+1}))}) \leq \sqrt{\frac{\mathrm{n(n+9)}}{2}}.$$

Proof. The cardinality of the vertex set of the subdivision graph of the wheel graph W_{n+1} is $3n+1$ among which the hub of the wheel of degree n, the $2n$ vertices of degree 2 and n vertices of degree 3. So the zagreb index of $S(W_{n+1})$ is

$$M(S(W_{n+1})) = n(n+17). \tag{3}$$

Hence from the equation (3) the laplacian energy of the line graph of the subdivision graph of the wheel graph is

$$LE(L(S(W_{n+1}))) \leq \frac{1}{2}(n+9)(4n-1).$$

The graph $S(W_{n+1})$ contains $4n$ edges. So the cardinality of the edge set of $L(S(W_{n+1}))$ written using equation (3) and theorem 2.1 as

$$\frac{n(n+9)}{2}. \tag{4}$$

Using equation (4), then

$$\lambda_{max}(L(S(W_{n+1}))) \leq \sqrt{\frac{n(n+9)}{2}}.$$

Theorem 3.3. For the line graph of the subdivision graph of the complete graph S_n, the bounds for the Laplacian energy and the maximal eigen value is

$$LE(L(S(S_n))) \leq 2(n-1)(n^2-n-1) \quad \text{and} \quad \lambda_{\max}(\mathrm{L(S(S_n))}) \leq \sqrt{\frac{\mathrm{n(n-1)^2}}{2}}.$$

Proof. The subdivision graph of the complete graph S_n, the vertex set is of cardinality $\frac{n(n+1)}{2}$. Out of which the n vertices of degree $n-1$, and $\frac{n(n-1)}{2}$ vertices of degree 2. Hence the zagreb index of the subdivision graph of the complete graph.

$$M(S(S_n)) = n(n^2-1). \tag{5}$$

For the line graph of the subdivision graph of the complete graph S_n, the bounds for the Laplacian energy is given from equation (5) and theorem 2.2

$$LE(L(S(S_n))) \leq 2(n-1)(n^2-n-1).$$

The cardinality of the edge set of the subdivision graph of the complete graph S_n is $4n$ so that

$$\lambda_{max}(S(S_n)) \leq 2\sqrt{n}.$$

The number of edges in the line graph of the subdivision graph of the complete graph S_n can be written using equation (5) and theorem 2.1 as

$$\frac{n(n-1)^2}{2}. \tag{6}$$

Using equation (6) the maximal eigen value of $L(S(S_n))$ is

$$\lambda_{max}(L(S(S_n))) \leq \sqrt{\frac{n(n-1)^2}{2}}.$$

Theorem 3.4. For the line graph of the subdivision graph of the ladder graph L_n, the bounds for the Laplacian energy and the maximal eigen value is

$$LE(L(S(L_n))) \leq 2(n-1)(n^2-n-1) \quad \text{and} \quad \lambda_{\max}(\mathrm{L(S(L_n))}) \leq \sqrt{9\mathrm{n}-10}.$$

Proof. The cardinality of the vertex set of the subdivision graph of the ladder graph L_n is $5n-2$ among which $3n+2$ vertices of degree 2 and $2n-4$ vertices of degree 3. Hence the zagreb index of the subdivision graph of the ladder graph L_n is

$$M(S(L_n)) = 30n - 28. \tag{7}$$

For the line graph of the subdivision graph of the ladder graph L_n, the bounds for the Laplacian energy is given from equation (7) and theorem 2.2 as

$$LE(L(S(L_n))) \leq 2\frac{(9n-10)(6n-5)}{3n-2}.$$

The cardinality of the edge set of $S(L_n)$ is $6n-4$. Hence the bounds for the largest eigen value of $S(L_n)$ is

$$\lambda_{max}(S(L_n)) \leq \sqrt{6n-4}.$$

The number of edges in the line graph of the subdivision graph of the ladder graph L_n can be written using equation (7) and theorem 2.1 as

$$9n - 10. \tag{8}$$

From equation (8) the largest eigen value of the line graph of the subdivision graph of the ladder graph is

$$\lambda_{max}(L(S(L_n))) \leq \sqrt{9n-10}.$$

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On the hybrid mean value of the Smarandache kn-digital sequence and Smarandache function [1]

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Abstract The main purpose of this paper is using the elementary method to study the hybrid mean value properties of the Smarandache kn-digital sequence and Smarandache function, and give an interesting asymptotic formula for it.

Keywords Smarandache kn-digital sequence, Smarandache function, hybrid mean value, as -ymptotic formula, elementary method.

§1. Introduction

For any positive integer k, the famous Smarandache kn-digital sequence $a(k,n)$ is defined as all positive integers which can be partitioned into two groups such that the second part is k times bigger than the first. For example, Smarandache $2n$ and $3n$ digital sequences $a(2,n)$ and $a(3,n)$ are defined as $\{a(2,n)\} = \{12, 24, 36, 48, 510, 612, 714, 816, \cdots\}$ and $\{a(3,n)\} = \{13, 26, 39, 412, 515, 618, 721, 824, \cdots\}$.

Recently, Professor Gou Su told me that she studied the hybrid mean value properties of the Smarandache kn-digital sequence and the divisor sum function $\sigma(n)$, and proved that the asymptotic formula

$$\sum_{n\le x} \frac{\sigma(n)}{a(k,n)} = \frac{3\pi^2}{k\cdot 20\cdot \ln 10}\cdot \ln x + O(1)$$

holds for all integers $1 \le k \le 9$.

When I read professor Gou Su's work, I found that the method is very new, and the results are also interesting. This paper as a note of Gou Su's work, we consider the hybrid mean value properties of the Smarandache kn-digital sequence and Smarandache function $S(n)$, which is defined as the smallest positive integer m such that $n|m!$. That is, $S(n) = \min\{m : n|m!,\ m \in N\}$. In this paper, we will use the elementary and analytic methods to study a similar problem, and prove a new conclusion. That is, we shall prove the following:

Theorem. Let $1 \le k \le 9$, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} \frac{S(n)}{a(k,n)} = \frac{3\pi^2}{k\cdot 20}\cdot \ln\ln x + O(1).$$

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§2. Proof of the theorem

In this section, we shall use the elementary and combinational methods to complete the proof of our theorem. First we need following:

Lemma. For any real number $x > 1$, we have

$$\sum_{n\le x}\frac{S(n)}{n}=\frac{\pi^2}{6}\cdot\frac{x}{\ln x}+O\left(\frac{x}{\ln^2 x}\right).$$

Proof. For any real number $x > 2$, from [4] we have the asymptotic formula

$$\sum_{n\le x}S(n)=\frac{\pi^2}{12}\cdot\frac{x^2}{\ln x}+O\left(\frac{x^2}{\ln^2 x}\right). \tag{1}$$

Then from Euler summation formula (see theorem 3.1 of [3]) we can deduce that

$$\begin{aligned}\sum_{1<n\le x}\frac{S(n)}{n} &= \frac{1}{x}\left(\frac{\pi^2}{12}\cdot\frac{x^2}{\ln x}+O\left(\frac{x^2}{\ln^2 x}\right)\right)+\int_1^x\left(\frac{\pi^2}{12}\cdot\frac{t^2}{\ln t}+O\left(\frac{t^2}{\ln^2 t}\right)\frac{1}{t^2}\right)dt\\ &= \frac{\pi^2}{12}\cdot\frac{x}{\ln x}+O\left(\frac{x}{\ln^2 x}\right)+\frac{\pi^2}{12}\cdot\frac{x}{\ln x}+\frac{13\pi^2}{12}\int_1^x\frac{1}{\ln^2 t}\,dt\\ &= \frac{\pi^2}{6}\cdot\frac{x}{\ln x}+O\left(\frac{x}{\ln^2 x}\right).\end{aligned}$$

This proves our Lemma.

Now we take $k = 2$ (or $k = 4$), then for any real number $x > 1$, there exists a positive integer M such that

$$5\cdot 10^M\le x<5\cdot 10^{M+1},$$

then we can deduce that

$$M=\frac{1}{\ln 10}\cdot\ln x+O(1). \tag{2}$$

So from the definition of $a(2,n)$ we have

$$\begin{aligned}\sum_{1\le n\le x}\frac{S(n)}{a(2,n)} &= \sum_{n=1}^{4}\frac{S(n)}{a(2,n)}+\sum_{n=5}^{49}\frac{S(n)}{a(2,n)}+\sum_{n=50}^{499}\frac{S(n)}{a(2,n)}+\cdots+\sum_{n=5\cdot 10^{M-1}}^{5\cdot 10^M-1}\frac{S(n)}{a(2,n)}\\ &\quad+\sum_{5\cdot 10^M\le n\le x}\frac{S(n)}{a(2,n)}\\ &= \sum_{n=1}^{4}\frac{S(n)}{n\cdot(10+2)}+\sum_{n=5}^{49}\frac{S(n)}{n\cdot(10^2+2)}+\sum_{n=50}^{499}\frac{S(n)}{n\cdot(10^3+2)}+\cdots\\ &\quad+\sum_{n=5\cdot 10^{M-1}}^{5\cdot 10^M-1}\frac{S(n)}{n\cdot(10^{M+1}+2)}+\sum_{5\cdot 10^M\le n\le x}\frac{S(n)}{n\cdot(10^{M+2}+2)}\end{aligned} \tag{3}$$

and

$$\begin{aligned}\sum_{1\le n\le x}\frac{S(n)}{a(4,n)} &= \sum_{n=1}^{2}\frac{S(n)}{a(4,n)}+\sum_{n=3}^{24}\frac{S(n)}{a(4,n)}+\sum_{n=25}^{249}\frac{S(n)}{a(4,n)}+\cdots+\sum_{n=\frac{1}{4}\cdot 10^{M-1}}^{\frac{1}{4}\cdot 10^{M}-1}\frac{S(n)}{a(4,n)}\\ &\quad+\sum_{\frac{1}{4}\cdot 10^{M}\le n\le x}\frac{S(n)}{a(4,n)}\\ &= \sum_{n=1}^{2}\frac{S(n)}{n\cdot(10+4)}+\sum_{n=3}^{24}\frac{S(n)}{n\cdot(10^2+4)}+\sum_{n=25}^{249}\frac{S(n)}{n\cdot(10^3+4)}+\cdots\\ &\quad+\sum_{n=\frac{1}{4}\cdot 10^{M-1}}^{\frac{1}{4}\cdot 10^{M}-1}\frac{S(n)}{n\cdot(10^M+4)}+\sum_{\frac{1}{4}\cdot 10^{M}\le n\le x}\frac{S(n)}{n\cdot(10^{M+1}+4)}. \qquad (4)\end{aligned}$$

Then from (2), (3) and Lemma we may immediately deduce

$$\begin{aligned}\sum_{n=5\cdot 10^{k-1}}^{5\cdot 10^{k}-1}\frac{S(n)}{n\cdot(10^{k+1}+2)} &= \sum_{n\le 5\cdot 10^{k}-1}\frac{S(n)}{n\cdot(10^{k+1}+2)}-\sum_{n\le 5\cdot 10^{k-1}}\frac{S(n)}{n\cdot(10^{k+1}+2)}\\ &= \frac{\pi^2}{6}\cdot\frac{5\cdot 10^{k}-5\cdot 10^{k-1}}{10^{k+1}+2}\cdot\frac{1}{\ln(5\cdot 10^{k})}+O\left(\frac{1}{k^2}\right)\\ &= \frac{3\pi^2}{40}\cdot\frac{1}{k}+O\left(\frac{1}{k^2}\right) \qquad (5)\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{n=\frac{1}{4}\cdot 10^{k-1}}^{\frac{1}{4}\cdot 10^{k}-1}\frac{S(n)}{n\cdot(10^{k}+4)} &= \sum_{n\le \frac{1}{4}\cdot 10^{k}-1}\frac{S(n)}{n\cdot(10^{k}+4)}-\sum_{n\le \frac{1}{4}\cdot 10^{k-1}}\frac{S(n)}{n\cdot(10^{k}+4)}\\ &= \frac{\pi^2}{6}\cdot\frac{\frac{1}{4}\cdot 10^{k}-\frac{1}{4}\cdot 10^{k-1}}{10^{k}+4}\cdot\frac{1}{\ln(\frac{1}{4}\cdot 10^{k})}+O\left(\frac{1}{k^2}\right)\\ &= \frac{3\pi^2}{80}\cdot\frac{1}{k}+O\left(\frac{1}{k^2}\right). \qquad (6)\end{aligned}$$

Noting that the identity $\sum_{n=1}^{\infty}\frac{1}{n^2}=\pi^2/6$ and the asymptotic formula

$$\sum_{1\le k\le M}\frac{1}{k}=\ln M+\gamma+O\left(\frac{1}{M}\right),$$

where γ is the Euler constant.

From (2), (3) and (5) we have

$$\begin{aligned}\sum_{1\le n\le x}\frac{S(n)}{a(2,n)} &= \sum_{n=1}^{4}\frac{S(n)}{a(2,n)}+\sum_{n=5}^{49}\frac{S(n)}{a(2,n)}+\sum_{n=50}^{499}\frac{S(n)}{a(2,n)}+\cdots+\sum_{n=5\cdot 10^{M-1}}^{5\cdot 10^{M}-1}\frac{S(n)}{a(2,n)}\\ &\quad+\sum_{5\cdot 10^{M}\le n\le x}\frac{S(n)}{a(2,n)}\\ &= \sum_{k=1}^{M}\frac{3\pi^2}{40}\cdot\frac{1}{k}+O\left(\sum_{k=1}^{M}\frac{1}{k^2}\right)\\ &= \frac{3\pi^2}{40}\ln\ln x+O(1).\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{1\le n\le x}\frac{S(n)}{a(4,n)} &= \sum_{n=1}^{2}\frac{S(n)}{a(4,n)}+\sum_{n=3}^{24}\frac{S(n)}{a(4,n)}+\sum_{n=25}^{249}\frac{S(n)}{a(4,n)}+\cdots+\sum_{n=\frac{1}{4}\cdot 10^{M-1}}^{\frac{1}{4}\cdot 10^{M}-1}\frac{S(n)}{a(4,n)}\\ &\quad+\sum_{\frac{1}{4}\cdot 10^{M}\le n\le x}\frac{S(n)}{a(4,n)}\\ &= \sum_{k=1}^{M}\frac{3\pi^2}{80}\cdot\frac{1}{k}+O\left(\sum_{k=1}^{M}\frac{1}{k^2}\right)\\ &= \frac{3\pi^2}{80}\ln\ln x+O(1).\end{aligned}$$

For using the same methods, we can also prove that the theorem holds for all integers $k=1,3,5,6,7,8,9$. This completes the proof of our theorem.

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M^*-paranormal composition operators on Weighted Hardy spaces

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Abstract If T is an analytic function mapping the unit disk D into itself, we define the composition operator C_T on the space $H^2(\beta)$ by $C_T f = f \circ T$. In this paper, we investigate the relationship between properties of the symbol T and the M^*-paranormality of the operators C_T and C_T^*.

Keywords Weighted Hardy space, M^*-paranormal, hyponormal, partial isometry, composi -tion operators.
Mathematics Subject Classification: Primary 47B38, Secondary 47B37, 47B35.

§1. Preliminaries

Let f be an analytic map on the open unit disk D given by the Taylor's series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\beta_0 = 1$ and $\frac{\beta_{n+1}}{\beta_n} \to 1$ as n$\to \infty$. The set $H^2(\beta)$ of formal complex power series $f(z) = \sum_{n=0}^{\infty} \mathrm{a}_n \mathrm{z}^n$ such that

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty$$

is a Hilbert Space of function analytic in the unit disk with the inner product $< f, g >_\beta = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2$ for f as above and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

Let D be the Open Unit disk in the complex plane and let $T : D \to D$ be an analytic self-map of the unit disk and consider the corresponding composition operator C_T acting on $H^2(\beta)$ i.e., $C_T(f) = f \circ T$, $f \in H^2(\beta)$.

The operators C_T are not necessarily defined on all of $H^2(\beta)$. They are everywhere defined in some special cases on the classical Hardy Space H^2 (the case when $\beta_n = 1$ for all n). See for example of [10] and on a general space $H^2(\beta)$ if the function T is analytic on some open set containing the closed unit disk having supremum norm strictly smaller than one (see [13]).

There are a lot of other known properties of composition operators, on the classical Hardy Space H^2 (see for example [3], [7] and [10]) and on more general space $H^2(\beta)$. (see [5], [6], [11], [12] and [13]).

In [4], Cowen's and Kriete obtained a nice correlation between hyponormality of composition operators on H^2 and the Denjoy-Wolff point of the induced map.

In [14], Nina Zorboska obtained some results on the hyponormality of Composition Operators and their adjoints.

Let ω be a point on the open disk.

Define $k_w^\beta(z) = \sum_{n=0}^{\infty} \frac{Z^n \overline{w}^n}{\beta_n^2}$.

Then the function k_w^β is a point evaluation for $H^2(\beta)$.

Then k_w^β is in $H^2(\beta)$ and $\left\|k_w^\beta\right\|^2 = \sum_{n=0}^{\infty} \frac{|w|^{2n}}{\beta_n^2}$. Thus, $\|k_w\|$ is an increasing function of $|w|$.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then $< f, k_w^\beta >_\beta = \sum_{n=0}^{\infty} \frac{a_n w^n \beta_n^2}{\beta_n^2} = f(w)$.

Therefore $< f, k_w^\beta >_\beta = f(w)$ for all f and k_w^β is known as the point evaluation kernal at w. It can be easily shown that $C_T^* k_w^\beta = k_{T(w)}^\beta$ and $k_0^\beta = 1$ (the function identically equal to 1).

In this article, we are interested in the M^*-paranormal composition operators and their adjoints on $H^2(\beta)$.

§2. M*-paranormal composition operators

An operator T defined on a Hilbert Space H is said to be M^* paranormal [1] if $\|T^*x\|^2 \leq M \left\|T^2x\right\|$ for each vector $x \in H$ or equivalently $\|T^*x\|^2 \leq M \left\|T^2x\right\| \|x\|$ for all in $x \in H$.

An operator T defined on a Hilbert space H is said to be [2]:

(i) Co-isometry if $TT^* = I$ or equivalently $< T^*(\mathrm{x}), T^*(y) > = < TT^*(x), y > = < x, y >$ for all $x, y \in H$.

(ii) Partial isometry if $T^*TT^* = T^*$ or equivalently $TT^*T = T$.

An operator T on a complex Hilbert Space H is of class (M, k), $k \geq 2$ if $T^{*k}T^k \geq (T^*T)^k$ [9].

Arora and Thukral [1] proved that T is M^*-paranormal if and only if $M^2T^{*2}T^2 - 2\lambda TT^* + \lambda^2 \geq 0$, for each $\lambda > 0$.

Theorem 2.1. If composition operator C_T on $H^2(\beta)$ is M^*-paranormal then $\left\|k_{T(0)}\right\|_\beta^2 \leq M$.

Proof. If C_T is M^*-paranormal

$\Rightarrow \mathrm{M}^2\mathrm{C}_T^{*2}\mathrm{C}_T^2 - 2\,\lambda\mathrm{C}_T\mathrm{C}_T^* + \lambda^2 \geq 0$ for all $\lambda > 0$.

$\Rightarrow < (\mathrm{M}^2\mathrm{C}_T^{*2}\mathrm{C}_T^2 - 2\lambda\mathrm{C}_T\mathrm{C}_T^* + \lambda^2)\ \mathrm{f}, \mathrm{f} > \geq 0$ for all $\mathrm{f} \in \mathrm{H}^2(\beta)$.

$\Rightarrow \mathrm{M}^2 < \mathrm{C}_T^{*2}\mathrm{C}_T^2\mathrm{f}, \mathrm{f} > - 2\,\lambda < \mathrm{C}_T\mathrm{C}_T^*\ \mathrm{f}, \mathrm{f} > + \lambda^2 < \mathrm{f}, \mathrm{f} > \geq 0$.

$\Rightarrow \mathrm{M}^2 < \mathrm{C}_T^2\ \mathrm{f}, \mathrm{C}_T^2\ \mathrm{f} > - 2\,\lambda < \mathrm{C}_T^*\mathrm{f}, \mathrm{C}_T^*\mathrm{f} > + \lambda^2 < \mathrm{f}, \mathrm{f} > \geq 0$.

$\Rightarrow \mathrm{M}^2 \left\|C_T^2 f\right\|^2 - 2\,\lambda \left\|C_T^* f\right\|^2 + \lambda^2 \|f\|^2 \geq 0$.

Let $\mathrm{f} = \mathrm{k}_0^\beta \in H^2(\beta)$,

$\mathrm{M}^2 \left\|C_T C_T k_0^\beta\right\|_\beta^2 - 2\lambda \left\|C_T^* k_0^\beta\right\|_\beta^2 + \lambda^2 \left\|k_0^\beta\right\|_\beta^2 \geq 0$.

$\mathrm{M}^2 \left\|C_T k_0^\beta\right\|_\beta^2 - 2\lambda \left\|k_{T(0)}^\beta\right\|_\beta^2 + \lambda^2 \left\|k_0^\beta\right\|_\beta^2 \geq 0.$

M^2 - 2 $\lambda \left\|k_{T(0)}^\beta\right\|_\beta^2 + \lambda^2 \geq 0$ [14].

By elementary property of real quadratic form we get $\left\|k_{T(0)}^\beta\right\|_\beta^2 \leq M$.

Corollary 2.2. If hyponormal operator C_T on $H^2(\beta)$ is M^*-paranormal then $M \geq 1$.

Theorem 2.3. If partial isometry C_T on $H^2(\beta)$ is M^*-paranormal and if co-isometry then $M^2 \geq 1$.

Proof. C_T is M^*-paranormal

$\mathrm{M}^2 C_T^{*2} C_T^2$ - $2\lambda C_T C_T^* + \lambda^2 \geq 0$, for all $\lambda > 0$.

$(\mathrm{M}^2 C_T^{*2} C_T^2$ - $2\lambda C_T C_T^* + \lambda^2$) $C_T^* C_T \geq 0$.

$\mathrm{M}^2 C_T^{*2} C_T C_T C_T^* C_T$ - 2 $\lambda C_T C_T^* C_T^* C_T + \lambda^2 C_T^* C_T \geq 0$.

Since C_T is partial isometry.

$\mathrm{M}^2 C_T^{*2} C_T C_T$ - $2\lambda C_T C_T^* C_T^* C_T + \lambda^2 C_T^* C_T \geq 0$.

Since C_T is co-isometry.

$\mathrm{M}^2 C_T^{*2} C_T^2$ - $2\lambda C_T^* C_T + \lambda^2 C_T^* C_T \geq 0$.

$\mathrm{M}^2 \left\|C_T^2 f\right\|^2$ - $2\lambda \left\|C_T f\right\|^2 + \lambda^2 \left\|C_T f\right\|^2 \geq 0$ for all f $\in H^2(\beta)$.

Let f = $\mathrm{k}_0^\beta \in H^2(\beta)$,

$\mathrm{M}^2 \left\|C_T C_T k_0^\beta\right\|_\beta^2$ - 2 $\lambda \left\|C_T k_0^\beta\right\|_\beta^2 + \lambda^2 \left\|C_T k_0^\beta\right\|_\beta^2 \geq 0$.

$\mathrm{M}^2 \left\|C_T k_0^\beta\right\|_\beta^2$ - $2\lambda \left\|k_0^\beta\right\|_\beta^2 + \lambda^2 \left\|k_0^\beta\right\|_\beta^2 \geq 0$ [14].

$\mathrm{M}^2 \left\|k_0^\beta\right\|_\beta^2$ - 2 $\lambda \left\|k_0^\beta\right\|_\beta^2 + \lambda^2 \left\|k_0^\beta\right\|_\beta^2 \geq 0$.

M^2 - $2\lambda + \lambda^2 \geq 0$ [14].

By elementary properties of real quadratic form, we get $\mathrm{M}^2 \geq 1$.

Theorem 2.4. If composition operator C_T is on $H^2(\beta)$ and C_T^* is M^*-paranormal then $M^2 \left\|k_{T^2(0)}^\beta\right\|^2 \geq 1$.

Proof. C_T^* is M^*-paranormal,

$M^2 C_T^2 C_T^{*2}$ - 2 $\lambda C_T^* C_T + \lambda^2 \geq 0$ for all $\lambda > 0$.

$\Rightarrow <$ $(\mathrm{M}^2 C_T^2 C_T^{*2}$ - $2\lambda C_T^* C_T + \lambda^2)$f, f $> \geq 0$, for all f $\in H^2(\beta)$.

$\Rightarrow \mathrm{M}^2 < C_T^2 C_T^{*2} f, f>$ - 2 $\lambda < C_T^* C_T f$, f$>$ + $\lambda^2 <$f, f$> \geq 0$.

$\Rightarrow \mathrm{M}^2 < \mathrm{C}_T^{*2} f, C_T^{*2} f>$ - $2\lambda < C_T f, C_T f >$ + $\lambda^2 <$f, f$> \geq 0$.

$\mathrm{M}^2 \left\|C_T^{*2} f\right\|^2$ - $2\lambda \left\|C_T f\right\|^2 + \lambda^2 \left\|f\right\|^2 \geq 0$.

Let f = $k_0^\beta \in H^2(\beta)$.

$\mathrm{M}^2 \left\|C_T^{*2} k_0^\beta\right\|_\beta^2$ - 2 $\lambda \left\|C_T k_0^\beta\right\|_\beta^2 + \lambda^2 \left\|k_0^\beta\right\|_\beta^2 \geq 0$.

$\mathrm{M}^2 \left\|C_T^* C_T^* k_0^\beta\right\|_\beta^2$ - 2 $\lambda \left\|C_T k_0^\beta\right\|_\beta^2 + \lambda^2 \left\|k_0^\beta\right|_\beta^2 \geq 0$.

$\mathrm{M}^2 \left\|C_T^* k_{T(0)}^\beta\right\|$ - 2 $\lambda \left\|k_0^\beta\right\|_\beta^2 + \lambda^2 \left\|k_0^\beta\right\| \geq 0$ [14].

$\mathrm{M}^2 \left\|k_{T^2(0)}^\beta\right\|$ - 2 $\lambda + \lambda^2 \geq 0$ [14].

$\Rightarrow \mathrm{M}^2 \left\| k^{\beta}_{T^2(0)} \right\| \geq 1.$

Theorem 2.5. If composition operator C_T on $H^2(\beta)$ is of class $(M,3)$, then $\left\| C_T k^{\beta}_{T(0)} \right\|^2_{\beta} \leq 1.$

Proof. C_T is of class $(M,3)$ if $C_T^{*3} C_T^3 \geq (C_T^* C_T)^3$
$\Rightarrow < (C_T^{*3} C_T^3 - (C_T^* C_T)^3) f, f > \geq 0$ for all $f \in H^2(\beta)$.
$< (C_T^{*3} C_T^3) f, f >$ - $< (C_T^* C_T)^3 f, f > \geq 0$.
$< (C_T^* C_T)^3 f, f > \leq < (C_T^{*3} C_T^3) f, f >$
$< (C_T^* C_T C_T^* C_T C_T^* C_T) f, f > \leq < C_T^3 f, C_T^3 f >$
$< C_T C_T^* C_T f, C_T C_T^* C_T f > \leq < C_T^3 f, C_T^3 f >$
$\| C_T C_T^* C_T f \|^2 \leq \| C_T^3 f \|^2.$

Let f $= k_0^{\beta} \in H^2(\beta)$, we have
$\left\| C_T C_T^* C_T k_0^{\beta} \right\|^2_{\beta} \leq \left\| C_T^3 k_0^{\beta} \right\|^2_{\beta}$
$\left\| C_T C_T^* k_0^{\beta} \right\|^2_{\beta} \leq \left\| C_T^2 C_T k_0^{\beta} \right\|^2_{\beta}$
$\left\| C_T k^{\beta}_{T(0)} \right\|^2_{\beta} \leq \left\| C_T C_T k_0^{\beta} \right\|^2_{\beta}$ [14]
$\left\| C_T k^{\beta}_{T(0)} \right\|^2_{\beta} \leq \left\| C_T k_0^{\beta} \right\|^2_{\beta}$
$\left\| C_T k^{\beta}_{T(0)} \right\|^2_{\beta} \leq \left\| k_0^{\beta} \right\|^2_{\beta}$
$\left\| C_T k^{\beta}_{T(0)} \right\|^2_{\beta} \leq 1.$

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On the mean value of some new sequences

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Abstract The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of the Smarandache repetitional sequence, and give two asymptotic formulas for it.

Keywords Euler's summation formula, Abel's identity, Smarandache repetitional sequence.

§1. Introduction

Let k be a fixed positive integer. The famous Smarandache repetitional generalized sequence $S(n,k)$ is defined as following: $n1n, n22n, n333n, n4444n, n55555n, n666666n, n777777 7n, n88888888n, n999999999n, n10101010101010101010n, \cdots$. In problem 3 of reference [1], Professor Mihaly Benze asked us to study the arithmetical properties about this sequence. It is interesting for us to study this problem. But it's a pity none had studied it before. At least we haven't seen such a paper yet. In this paper, we shall use the elementary and analytic methods to study the arithmetical properties of the special Smarandache repetitional generalized sequence,and give a sharper asymptotic formula for it. That is, we shall prove the following:

Theorem. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{k\leq x} S(n,k) = \sum_{i=0}^{m} di(x-i)^3 + fi(2i+1)(x-i)^2 + h\cdot 10^x + O(\frac{100\cdot n}{9}\cdot 10^x) + A.$$

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem. First we give three simple lemmas which are necessary in the proof of our theorem. The proofs of these lemmas can be found in reference [7].

Lemma 1. For any real number $x \geq 1$ and $\alpha > 0$, we have the asymptotic formula

$$\sum_{n\leq x} n^\alpha = \frac{x^{1+\alpha}}{1+\alpha} + O\left(x^\alpha\right).$$

Lemma 2. If f has a continuous derivative f' on the interval $[x,y]$, where $0 < y < x$,

$$\sum_{y<k\leq x} f(n) = \int_y^x f(t)dt + \int_y^x (t-[t])f'(t)dt + f(x)([x]-x) - f(y)([y]-y).$$

Lemma 3. For any arithmetical function $a(n)$, let $A(x) = \sum_{n\le x} a(n)$, If f has a continuous derivative f' on the interval $[y, x]$, where $0 < y < x$,

$$\sum_{y<k\le x} a(n)f(n) = A(x)f(x) - A(y)f(y) + \int_y^x A(t)f'(t)dt.$$

Now we use these lemmas to prove our conclusion. First we use the elementary method to obtain an asymptotic formula. To accomplish our theorem easily, we can get the following equations by observing the classification of the Smarandache repetitional sequence.

$S(n,1) = n\cdot 10^2 + 1\cdot 10^1 + n\cdot 10^0$

$S(n,2) = n\cdot 10^3 + 2\cdot 10^2 + 2\cdot 10^1 + n\cdot 10^0$

$S(n,3) = n\cdot 10^4 + 3\cdot 10^3 + 3\cdot 10^2 + 3\cdot 10^1 + n\cdot 10^0$

$S(n,4) = n\cdot 10^5 + 4\cdot 10^4 + 4\cdot 10^3 + 4\cdot 10^2 + 4\cdot 10^1 + n\cdot 10^0$

$\cdots\cdots$

$S(n,a-2) = n\cdot 10^{a-1} + (a-2)\cdot 10^{a-2} + \cdots + (a-2)\cdot 10^2 + (a-2)\cdot 10^1 + n\cdot 10^0$

$S(n,a-1) = n\cdot 10^{a} + (a-1)\cdot 10^{a-1} + \cdots + (a-1)\cdot 10^2 + (a-1)\cdot 10^1 + n\cdot 10^0$

$S(n,a) = n\cdot 10^{a+1} + a\cdot 10^{a} + \cdots + a\cdot 10^2 + a\cdot 10^1 + n\cdot 10^0.$

Now we estimate the right hand side of the above equations, by lemma 2 we have

$$\begin{aligned}\sum_{k\le x} S(n,k) &= \sum_{k\le x} n\cdot[\frac{100\cdot(10^k-1)}{9}] + \sum_{k\le x}\frac{k(k+1)}{2}\cdot 10^1 + \sum_{k\le x}\frac{(k-1)(k+2)}{2}\cdot 10^2 \\ &\quad + \sum_{k\le x}\frac{(k-2)(k+3)}{2}\cdot 10^3 + \cdots + \sum_{k\le x}\frac{[k+(k-2)][k-(k-3)]}{2}\cdot 10^{k-2} \\ &\quad + \sum_{k\le x}\frac{[k+(k-1)][k-(k-2)]}{2}\cdot 10^{k-1} + k\cdot 10^k. \qquad (1)\end{aligned}$$

Now we estimate the one part of the right hand side of the above equations, by lemma 2 and lemma 3 we have

$$\begin{aligned}\sum_{k\le x} k(k+1) &= [\frac{x^2}{2} + O(x)](x+1) - 1 - O(2) - \int_1^x [\frac{t^2}{2} + O(t)]dt \\ &= \frac{x^3}{2} + \frac{x^2}{2} - 1 - \int_1^x [\frac{t^2}{2} + O(t)]dt + O(x^2) \\ &= \frac{x^3}{2} + \frac{x^2}{2} - 1 - \frac{t^2}{2}dt + O(\int_1^x tdt) + O(x^2) \\ &= \frac{x^3}{3} + \frac{x^2}{2} + O(x^2) + C1, \qquad (2)\end{aligned}$$

where we have used the identity $C1$.

Similarly, we also have the asymptotic formulae

$$\begin{aligned}\sum_{k\le x}(k-1)(k+2) &= [\frac{(x-1)^2}{2}+O(x-1)(x+2)-\int_1^x[\frac{(t-1)^2}{2}+O(t-1)]dt\\ &= \frac{(x-1)^2(x+2)}{2}]-\int_1^x\frac{(t-1)^2}{2}dt+\int_1^x O(t-1)dt+O(x-1)^2)\\ &= \frac{(x-1)^2(x-1+3)}{2}-\frac{(x-1)^3}{6}+O(\int_1^x(t-1)dt)+O(x^2)\\ &= \frac{(x-1)^3}{3}+\frac{3(x-1)^2}{2}+O((x-1)^2)+C2, \qquad (3)\end{aligned}$$

where we have used the identity $C2$. Similarly, we also have the asymptotic formulae

$$\begin{aligned}\sum_{k\le x}(k-a+1)(k+a) &= [\frac{(x-a+1)^2}{2}+O(x-a+1)](x+a)-\int_1^x[\frac{(t-a+1)^2}{2}\\ &\quad +O(t-a+1)]dt\\ &= \frac{(x-a+1)^2(x+a)}{2}-\int_1^x\frac{(t-a+1)^2}{2}dt+\int_1^x O(t-a+1)dt\\ &\quad +O(x-a+1)^2\\ &= \frac{(x-a+1)^2(x-a+1+2a-1)}{2}-\frac{(x-a+1)^3}{6}\\ &\quad +O(\int_1^x t-a+1dt)\\ &= \frac{(x-a+1)^3}{3}+\frac{(2a-1)(x-a+1)^2}{2}+O((x-a+1)^2)\\ &\quad +Ca-1. \qquad (4)\end{aligned}$$

$$\begin{aligned}\sum_{k\le x}(k-a)(k+a+1) &= [\frac{(x-a)^2}{2}+O(x-a)](x+a+1)-\int_1^x[\frac{(t-a)^2}{2}+O(t-a)]dt\\ &= \frac{(x-a)^2(x+a+1)}{2}-\int_1^x\frac{(t-a)^2}{2}dt+\int_1^x O(t-a)dt+O(x-a)^2\\ &= \frac{(x-a)^2(x-a+2a+1)}{2}-\frac{(x-a)^3}{6}+O(\int_1^x t-adt)\\ &= \frac{(x-a)^3}{3}+\frac{(2a+1)(x-a)^2}{2}+O((x-a)^2)+Ca. \qquad (5)\end{aligned}$$

Finally ,we can use the lemma 2 to get the following formulae

$$\begin{aligned}\sum_{k\le x}k\cdot 10^k &= [\frac{x^2}{2}+O(x)]\cdot 10^x-\int_1^x\ln 10[\frac{t^2}{2}+O(t)]10^t dt\\ &= \frac{x^2}{2}\cdot 10^x+O(x\cdot 10^x)-\frac{1}{2}\int_1^x t^2 d10^t-\int_1^x O(t\cdot 10^t)dt\\ &= \frac{x^2}{2}\cdot 10^x+O(x\cdot 10^x)-\frac{x^2}{2}\cdot 10^x-O(\int_1^x t\cdot 10^t dt)\\ &= \frac{x\cdot 10^x}{\ln 10}-\frac{10^x}{(\ln 10)^2}+O(x\cdot 10^x)+A1 \qquad (6)\end{aligned}$$

From (1) (2), (3), (4), (5) and (6) we deduce the asymptotic formula

$$\sum_{k\leq x} S(n,k) = \sum_{i=0}^{m} di(x-i)^3 + fi(2i+1)(x-i)^2 + h\cdot 10^x + O(\frac{100\cdot n}{9}\cdot 10^x) + A.$$

Now combining two methods we may immediately deduce our theorem.

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Eigenvalues and eigenfunctions of Sturm-Liouville problem with two-point discontinuities containing eigenparameter-dependent boundary conditions

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Abstract In this work, we extend some spectral properties of regular Sturm-Liouville problems to the problems consisting of a Sturm-Liouville equation with piecewise continuous potential together with eigenparameter-dependent boundary conditions and four supplementary transmission conditions. We give an operator-theoretical formulation, construct fundamental solutions and investigate some properties of the eigenvalues and corresponding eigenfunctions of the discontinuous Sturm-Liouville problem.

Keywords Discontinuous Sturm-Liouville problems, the asymptotic of eigenvalues and eigenfunc -tions.
2000 Mathematics subject classification: 34B24, 34L10, 34L20.

§1. Introduction

It is well known that many topics in mathematical physics lead to the Sturm-Liouville type boundary-value problems. The Sturmian theory is one of the most actual and extensively developing fields in theoretical and applied mathematics. Particularly, in recent years, there have been increasing interests in spectral analysis of boundary value problems with eigenvalue-dependent boundary conditions. There are quite substantial literatures on such problems. Here we mention the results of [1-7, 11 and 13, 15-17] and corresponding references cited therein. Basically, boundary-value problems with continuous coefficients at the highest derivative of the equation have been investigated. Note that, discontinuous Sturm-Liouville problems with eigen-dependent boundary conditions and with two supplementary transmission conditions at the point of discontinuity were investigated in [2, 8-11]. In this paper, we shall investigate following discontinuous eigenvalue problem

$$\tau u := -a(x)u'' + q(x)u = \lambda u \tag{1}$$

on $x \in [a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b]$ with boundary conditions at $x = a$

$$L_1 u := \alpha_1 u(a) + \alpha_2 u'(a) = 0 \tag{2}$$

four transmission conditions at the points of discontinuities $x = \xi_1$ and $x = \xi_2$,

$$L_2 u \quad := \quad \gamma_1 u(\xi_1 - 0) - \delta_1 u(\xi_1 + 0) = 0, \tag{3}$$

$$L_3 u := \gamma_1^{'} u'(\xi_1 - 0) - \delta_1^{'} u'(\xi_1 + 0) = 0, \tag{4}$$

$$L_4 u \quad := \quad \gamma_2 u(\xi_2 - 0) - \delta_2 u(\xi_2 + 0) = 0, \tag{5}$$

$$L_5 u := \gamma_2^{'} u'(\xi_2 - 0) - \delta_2^{'} u'(\xi_2 + 0) = 0, \tag{6}$$

and eigendependent boundary conditions at $x = b$

$$L_6(\lambda) u := \lambda[\beta_1^{'} u(b) - \beta_2^{'} u'(b)] + [\beta_1 u(b) - \beta_2 u'(b)] = 0. \tag{7}$$

Where $a(x) = a_1^2$ for $x \in [a, \xi_1)$, $a(x) = a_2^2$ for $x \in (\xi_1, \xi_2)$, $a(x) = a_3^2$ for $x \in (\xi_2, b]$; $a_1 > 0, a_2 > 0$ and $a_3 > 0$ are given real numbers; $q(x)$ is a given real-valued function continuous in $[a, \xi_1]$, $[\xi_1, \xi_2]$ and $[\xi_2, b]$ (that is, continuous in $[a, \xi_1), (\xi_1, \xi_2)$ and $(\xi_2, b]$ and has finite limits $q(\xi_1 \pm) := \lim_{x \to \xi_1 \pm} q(x), q(\xi_2 \pm) := \lim_{x \to \xi_2 \pm} q(x)$); λ is a complex eigenvalue parameter; the coefficients of the boundary and transmission conditions are real numbers. We assume that $|\alpha_1| + |\alpha_2| \neq 0$, $|\gamma_i| + |\delta_i| \neq 0, |\gamma_i^{'}| + |\delta_i^{'}| \neq 0$ $(i = 1, 2)$ and $\rho := \beta_1^{'} \beta_2 - \beta_1 \beta_2^{'} > 0$.

§2. Preliminaries

For convenience let us introduce the next notations:

$$\Omega_1 = [a, \xi_1], \qquad \Omega_2 = [\xi_1, \xi_2], \qquad \Omega_3 = [\xi_2, b].$$

$$u_{(1)}(x) := \begin{cases} u(x) & x \in [a, \xi_1) \\ \lim_{x \to \xi_1 -} & x = \xi_1 \end{cases} \qquad u_{(2)}(x) := \begin{cases} u(x) & x \in (\xi_1, \xi_2) \\ \lim_{x \to \xi_1 +} & x = \xi_1 \end{cases}$$

$$u_{(3)}(x) := \begin{cases} u(x) & x \in (\xi_1, \xi_2) \\ \lim_{x \to \xi_2 -} & x = \xi_2 \end{cases} \qquad u_{(4)}(x) := \begin{cases} u(x) & x \in (\xi_2, b] \\ \lim_{x \to \xi_2 +} & x = \xi_2 \end{cases}$$

$$(u)_\beta := \lim_{x \to b} (\beta_1 u(x) - \beta_2 u'(x)), \ (u)_\beta^{'} := \lim_{x \to b} \left(\beta_1^{'} u(x) - \beta_2^{'} u'(x)\right), \ \widetilde{u}(x) = \begin{cases} u(x), & x \in [a, b) \\ (u)_\beta^{'}, & x = b. \end{cases}$$

Note that, everywhere in below we shall assume that $\gamma_i \gamma_i^{'} \delta_i \delta_i^{'} > 0$ $(i = 1, 2)$ and for the Lebesque measurable subsets $M \subset [a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b]$ with Lebesque measure $\mu_L(M)$ we shall define a new positive measure $\mu_\rho(M)$ by

$$\mu_\rho(M) := \frac{1}{a_1^2} \frac{\gamma_1 \gamma_1^{'}}{\delta_1 \delta_1^{'}} \mu_L(M \cap [a, \xi_1)) + \frac{1}{a_2^2} \mu_L(M \cap (\xi_1, \xi_2)) + \frac{1}{a_3^2} \frac{\delta_2 \delta_2^{'}}{\gamma_2 \gamma_2^{'}} \mu_L(M \cap (\xi_2, b]) + \frac{\delta_2 \delta_2^{'}}{\gamma_2 \gamma_2^{'}} \frac{b(M)}{\rho}.$$

Where $b(M) := \begin{cases} 0 & if \quad b \notin M \\ 1 & if \quad b \in M. \end{cases}$ Let $\langle .,. \rangle_{H_\rho}$ denote the scalar product in the Hilbert space $H_\rho := L^2([a,b];\mu_\rho)$. In this space we define a linear operator A by the domain of definition

$$D(A) := \left\{ \begin{array}{l} u \in H_\rho | u_{(i)}, u'_{(i)} are \quad absolutely \quad continuous \quad in \quad \Omega_i (i=1,2,3), \tau\widetilde{u} \in L^2[a,b] \\ \alpha_1 \widetilde{u}(a) + \alpha_2 \widetilde{u}'(a) = 0, \gamma_1 \widetilde{u}(\xi_1 - 0) = \delta_1 \widetilde{u}(\xi_1 + 0), \gamma'_1 \widetilde{u}'(\xi_1 - 0) = \delta'_1 \widetilde{u}'(\xi_1 + 0) \\ \gamma_2 \widetilde{u}(\xi_2 - 0) = \delta_2 \widetilde{u}(\xi_2 + 0), \gamma'_2 \widetilde{u}'(\xi_2 - 0) = \delta'_2 \widetilde{u}'(\xi_2 + 0), \widetilde{u}(b) = (\widetilde{u}_\beta)' \end{array} \right\}$$

and

$$(A\widetilde{u})(x) = \begin{cases} (\tau\widetilde{u})(x) & for \quad x \in [a,\xi_1) \cup (\xi_1,\xi_2) \cup (\xi_2,b] \\ -(\widetilde{u})_\beta & for \quad x = b. \end{cases}$$

So we can pose the problem (1-7) in the operator-equation form $A\widetilde{u} = \lambda\widetilde{u}$ i.e., the problem (1-7) can be considered as the eigenvalue problem for the operator A.

Theorem 2.1. The operator A is symmetric.

Proof. Let $f, g \in D(A)$. By two partial integrations we get

$$\langle Af, g \rangle_{H_\rho} - \langle f, Ag \rangle_{H_\rho} = \frac{\gamma_1 \gamma'_1}{\delta_1 \delta'_1} [W(f,\overline{g};\xi_1 - 0) - W(f,\overline{g};a)] + W(f,\overline{g};\xi_2 - 0) -$$

$$W(f,\overline{g};\xi_1 + 0) + \frac{\delta_2 \delta'_2}{\gamma_2 \gamma'_2} [W(f,\overline{g};b) - W(f,\overline{g};\xi_2 + 0)] - \frac{\delta_2 \delta'_2}{\gamma_2 \gamma'_2} \frac{1}{\rho} [(f)'_\beta (\overline{g})_\beta - (f)_\beta (\overline{g})'_\beta] \tag{8}$$

where as usual,

$$W(f,g;x) = f(x)g'(x) - f'(x)g(x) \tag{9}$$

denotes the Wronskian's of the functions f and g. Since f and $\overline{g}$ satisfy the boundary condition (2) it follows that

$$W(f,\overline{g};a) = 0. \tag{10}$$

From the transmission conditions (3)-(6) we get

$$\gamma_i \gamma'_i W(f,g;\xi_i - 0) = \delta_i \delta'_i W(f,g;\xi_i + 0) \quad (i = 1,2). \tag{11}$$

Further, it is easy to verify that

$$(f)_\beta (\overline{g})'_\beta - (f)'_\beta (\overline{g})_\beta = \rho W(f,\overline{g};b). \tag{12}$$

Finally, substituting (9)-(12) in (8) yield the required equality

$$\langle Af, g \rangle_{H_\rho} = \langle f, Ag \rangle_{H_\rho} \quad (f, g \in H_\rho). \tag{13}$$

Corollary 2.1. All eigenvalues of the considered problem (1-7) are real.

We can now assume that all eigenfunctions are real-valued.

Corollary 2.2. If λ_1 and λ_2 are two different eigenvalues of the problem (1-7) then corresponding eigenfunctions u_1 and u_2 of this problem satisfy the following equality:

$$\frac{1}{a_1^2}\frac{\gamma_1\gamma_1^{'}}{\delta_1\delta_1^{'}}\int_a^{\xi_1} u_1(x)u_2(x)dx \; + \; \frac{1}{a_2^2}\int_{\xi_1}^{\xi_2} u_1(x)u_2(x)dx + \frac{1}{a_3^2}\frac{\delta_2\delta_2^{'}}{\gamma_2\gamma_2^{'}}\int_{\xi_2}^{b} u_1(x)u_2(x)dx$$
$$+ \; \frac{1}{\rho}\frac{\delta_2\delta_2^{'}}{\gamma_2\gamma_2^{'}}(u_1)_{\beta}^{'}(u_2)_{\beta}^{'} = 0. \tag{14}$$

In fact, this formula means the orthogonality of eigenfunctions u_1 and u_2 in the Hilbert space H_ρ. We need the following Lemma, which can be proved similarly to theorem 2 in [2].

Lemma 2.1. Let the real-valued function $q(x)$ be continuous in $[a,b]$ and $f(\lambda)$, $g(\lambda)$ be given entire functions. Then for any $\chi(x,\lambda_0)$ the equation $-u''+q(x)u=\lambda u$, $x\in[a,b]$ has a unique solution $u=u(x,\lambda)$ satisfying the initial conditions

$$u(a)=f(\lambda), u'(a)=g(\lambda) \; or \; (u(b)=f(\lambda), u'(b)=g(\lambda)).$$

For each $x\in[a,b]$, $u(x,\lambda)$ is an entire function of λ.

We shall define two solutions

$$\phi_\lambda(x)=\begin{cases} \phi_{1\lambda}(x), & x\in[a,\xi_1) \\ \phi_{2\lambda}(x), & x\in(\xi_1,\xi_2) \\ \phi_{3\lambda}(x), & x\in[\xi_2,b) \end{cases}$$

and

$$\chi_\lambda(x)=\begin{cases} \chi_{1\lambda}(x), & x\in[a,\xi_1) \\ \chi_{2\lambda}(x), & x\in(\xi_1,\xi_2) \\ \chi_{3\lambda}(x), & x\in[\xi_2,b) \end{cases}$$

of the equation (1) as follows: Let $\phi_{1\lambda}(x)=\phi_1(x,\lambda)$ be the solution of equation (1) on $[a,\xi_1)$, which satisfies the initial conditions

$$u(a)=\alpha_2, \; u'(a)=-\alpha_1. \tag{15}$$

Using lemma 2.1, after defining this solution we may define the solution $\phi_2(x,\lambda)$ of equation (1) on $[\xi_1,\xi_2]$ by means of the solution $\phi_1(x,\lambda)$ by the nonstandard initial conditions

$$u(\xi_1+0)=\frac{\gamma_1}{\delta_1}\phi_{1\lambda}(\xi_1-0,\lambda), \; u'(\xi_1+0)=\frac{\gamma_1^{'}}{\delta_1^{'}}\phi_{1\lambda}^{'}(\xi_1-0,\lambda). \tag{16}$$

After defining this solution, we may define the solution $\phi_3(x,\lambda)$ of equation (1) on $[\xi_2,b]$ by means of the solution $\phi_2(x,\lambda)$ by the nonstandard initial conditions

$$u(\xi_2+0)=\frac{\gamma_2}{\delta_2}\phi_{2\lambda}(\xi_2-0,\lambda), \; u'(\xi_2+0)=\frac{\gamma_2^{'}}{\delta_2^{'}}\phi_{2\lambda}^{'}(\xi_2-0,\lambda). \tag{17}$$

Hence, $\phi(x,\lambda)$ satisfies the equation (1) on $[a,\xi_1)\cup(\xi_1,\xi_2)\cup(\xi_2,b]$, the boundary condition (2) and the transmission conditions (3-6). Analogically first we define the solution $\chi_{3\lambda}=\chi_3(x,\lambda)$ on $[\xi_2,b]$ by the initial conditions

$$u(b)=\beta_2^{'}\lambda+\beta_2, \; u'(b)=\beta_1^{'}\lambda+\beta_1. \tag{18}$$

Again, after defining this solution we define the solution $\chi_{2\lambda} = \chi_2(x,\lambda)$ of the equation (1) on $[\xi_1, \xi_2]$ by the initial conditions

$$u(\xi_2 - 0) = \frac{\delta_1}{\gamma_1}\chi_{3\lambda}(\xi_2 + 0, \lambda),\ u'(\xi_2 - 0) = \frac{\delta_1^{'}}{\gamma_1^{'}}\chi_{3\lambda}^{'}(\xi_2 + 0, \lambda). \tag{19}$$

Using this solution, we define the solution $\chi_{1\lambda} = \chi_1(x,\lambda)$ of (1) on $[\xi_2, b]$ by conditions

$$u(\xi_1 - 0) = \frac{\delta_2}{\gamma_2}\chi_{3\lambda}(\xi_1 + 0, \lambda),\ u'(\xi_1 - 0) = \frac{\delta_2^{'}}{\gamma_2^{'}}\chi_{3\lambda}^{'}(\xi_1 + 0, \lambda). \tag{20}$$

Hence, $\chi(x,\lambda)$ satisfies the equality (1) on $[a,\xi_1) \cup (\xi_1,\xi_2) \cup (\xi_2, b]$, the boundary condition (7) and the transmission conditions (3-6). Further it follows from (1) that the Wronskians

$$\omega_i(\lambda) := W_\lambda(\phi_i, \chi_i; x) := \phi_i(x,\lambda)\chi_i^{'}(x,\lambda) - \phi_i^{'}(x,\lambda)\chi_i(x,\lambda), x \in \Omega_i\ (i = 1,2,3)$$

are independent of $x \in \Omega_i$. Moreover, these functions are entire in λ.

Lemma 2.2. For each $\lambda \in \mathbb{C}, \gamma_1\gamma_1^{'}\gamma_2\gamma_2^{'}\omega_1(\lambda) = \delta_1\delta_1^{'}\gamma_2\gamma_2^{'}\omega_2(\lambda) = \delta_1\delta_2\delta_1^{'}\delta_2^{'}\omega_3(\lambda)$.

Proof. In view of (16-17) and (19-20), short calculation gives

$$\gamma_1\gamma_1^{'}\gamma_2\gamma_2^{'}W(\phi_1, \chi_1; \xi_1 - 0) = \delta_1\delta_1^{'}\gamma_2\gamma_2^{'}W(\phi_2, \chi_2; \xi_1 + 0) = \delta_1\delta_2\delta_1^{'}\delta_2^{'}W(\phi_3, \chi_3; \xi_2 + 0),$$

so $\gamma_1\gamma_1^{'}\gamma_2\gamma_2^{'}\omega_1(\lambda) = \delta_1\delta_1^{'}\gamma_2\gamma_2^{'}\omega_2(\lambda) = \delta_1\delta_2\delta_1^{'}\delta_2^{'}\omega_3(\lambda)$ for each $\lambda \in \mathbb{C}$.

Now we may introduce the characteristic function

$$\omega(\lambda) := \gamma_1\gamma_1^{'}\gamma_2\gamma_2^{'}\omega_1(\lambda) = \delta_1\delta_1^{'}\gamma_2\gamma_2^{'}\omega_2(\lambda) = \delta_1\delta_2\delta_1^{'}\delta_2^{'}\omega_3(\lambda).$$

Theorem 2.2. The eigenvalues of the problem (1-7) are the zeros of the function $\omega(\lambda)$.

Proof. Let $\omega(\lambda_0) = 0$. Then $W_{\lambda_0}(\phi_1, \chi_1; x) = 0$ and therefore the functions $\phi_{1\lambda_0}(x)$ and $\chi_{1\lambda_0}(x)$ are linearly dependent i.e. $\chi_{1\lambda_0}(x) = k_1\phi_{1\lambda_0}(x), x \in [a,\xi_1]$ for some $k_1 \neq 0$. From this it follows that $\chi(x,\lambda_0)$ satisfies also the first boundary condition (2), so $\chi(x,\lambda_0)$ is an eigenfunctions for the eigenvalue λ_0. Now let $u_0(x)$ be any eigenfunction corresponding to eigenvalue λ_0, but $\omega(\lambda_0) \neq 0$. Then the functions $\phi_1, \chi_1, \phi_2, \chi_2$, and ϕ_3, χ_3 would be linearly independent on $[a,\xi_1]$, $[\xi_1,\xi_2]$ and $[\xi_2, b]$ respectively. Therefore $u_0(x)$ may be represented as the form

$$u_0(x) = \begin{cases} c_1\phi_1(x,\lambda_0) + c_2\chi_1(x,\lambda_0), & x \in [a,\xi_1) \\ c_3\phi_2(x,\lambda_0) + c_4\chi_2(x,\lambda_0), & x \in (\xi_1,\xi_2) \\ c_5\phi_3(x,\lambda_0) + c_6\chi_3(x,\lambda_0), & x \in [\xi_2, b) \end{cases}$$

where at least one of the constants $c_1, c_2, c_3, c_4, c_5, c_6$ is not zero. Considering the equations

$$L_\nu(u_0(x)) = 0, \quad \nu = \overline{1,6}. \tag{21}$$

as a system of linear equations of the variables $c_i, i = \overline{1,6}$, and taking (16-17) and (19-20) into account, it can be shown that the determinant of this system is different from zero. Therefore, the system (21) has only the trivial solution $c_i = 0, i = \overline{1,6}$. Thus we get a contradiction, which completes the proof.

Lemma 2.3. If $\lambda = \lambda_0$ is an eigenvalue, then $\phi(x,\lambda_0)$ and $\chi(x,\lambda_0)$ are linearly dependent.

Proof. Let $\lambda = \lambda_0$ be eigenvalue. From theorem 2, $W(\phi_{i\lambda_0}, \chi_{i\lambda_0}; x) = \omega_i(\lambda_0) = 0$ and therefore

$$\chi_{i\lambda_0}(x) = k_i \phi_{i\lambda_0}(x) \quad i = (1, 2, 3) \tag{22}$$

for some $k_1 \neq 0, k_2 \neq 0$ and $k_3 \neq 0$. We must show that $k_1 = k_2 = k_3$. Suppose, if possible that $k_1 \neq k_2$. Using the definitions of $\phi_i(x, \lambda_0)$ and $\chi_i(x, \lambda_0)$ and the equalities (22) we have

$$\begin{aligned} \delta_1(k_1 - k_2)\phi_{2\lambda}(\xi_1 + 0) &= \delta_1 k_1 \phi_{2\lambda}(\xi_1 + 0) - \delta_1 k_2 \phi_{2\lambda}(\xi_1 + 0) \\ &= k_1 \gamma_1 \phi_{1\lambda}(\xi_1 - 0) - k_2 \delta_1 \phi_{2\lambda}(\xi_1 + 0) \\ &= \gamma_1 \chi_{1\lambda}(\xi_1 - 0) - \delta_1 \chi_{2\lambda}(\xi_1 + 0) = 0. \end{aligned}$$

Hence

$$\phi_{2\lambda_0}(\xi_1 + 0) = 0. \tag{23}$$

Analogically, starting from $\delta_1^{'}(k_1 - k_2)\phi_{2\lambda}^{'}(\xi_1 + 0)$ and following the same procedure we can derive that

$$\phi_{2\lambda_0}^{'}(\xi_1 + 0) = 0. \tag{24}$$

From the fact that $\phi_{2\lambda_0}(x)$ is a solution of the differential equation (1) on $[\xi_1, \xi_2]$ and satisfies the initial conditions (23-24) it follows that $\phi_{2\lambda_0}(x) = 0$ identically on $[\xi_1, \xi_2]$. Making use of (16-17) and (23-24) we may also derive that

$$\phi_{1\lambda_0}(\xi_1 - 0) = \phi_{1\lambda_0}^{'}(\xi_1 - 0) = 0 \quad \textit{and} \quad \phi_{3\lambda_0}(\xi_2 + 0) = \phi_{3\lambda_0}^{'}(\xi_2 + 0) = 0$$

respectively. From this by the same argument as for $\phi_{2\lambda_0}(x)$ it follows that $\phi_{1\lambda_0}(x) = 0$ identically on $[a, \xi_1]$ and $\phi_{3\lambda_0}(x) = 0$ identically on $[\xi_2, b]$. Hence $\phi(x, \lambda_0) = 0$ identically on $[a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b]$. However, this contradicts (15), since $|\alpha_1| + |\alpha_2| \neq 0$.

Corollary 2.3. If $\lambda = \lambda_0$ is an eigenvalue, then both $\phi(x, \lambda_0)$ and $\chi(x, \lambda_0)$ are eigenfunctions corresponding to this eigenvalue.

Lemma 2.4. All eigenvalues λ_n are simple zeros of $\omega(\lambda)$.

Proof. Using the well-known Lagrange's formula [12] it can be shown that

$$\frac{1}{a_1^2}\int_a^{\xi_1} \phi_\lambda(x)\phi_{\lambda_n}(x)dx + \frac{1}{a_2^2}\int_{\xi_1}^{\xi_2} \phi_\lambda(x)\phi_{\lambda_n}(x)dx \frac{1}{a_3^2}\int_{\xi_2}^{b} \phi_\lambda(x)\phi_{\lambda_n}(x)dx = \frac{W(\phi_\lambda, \phi_{\lambda_n}; b)}{\lambda - \lambda_n} \tag{25}$$

for any λ. Since $\chi_{\lambda_n}(x) = k_n \phi_{\lambda_n}(x),\ x \in [a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b]$ for some $k_n \neq 0$ $(n = 1, 2, ...)$.

Using this equality for the right side of (25) we have

$$\begin{aligned} W(\phi_\lambda, \phi_{\lambda_n}; b) &= \frac{1}{k_n} W(\phi_\lambda, \chi_{\lambda_n}; b) \\ &= \frac{1}{k_n}\left[\lambda_n (\phi_\lambda)_\beta^{'} + (\phi_\lambda)_\beta\right] \\ &= \frac{1}{k_n}\left[\omega(\lambda) + (\lambda - \lambda_n)(\phi_\lambda)_\beta^{'}\right] \\ &= (\lambda - \lambda_n)\frac{1}{k_n}\left[\frac{\omega(\lambda)}{\lambda - \lambda_n} - (\phi_\lambda)_\beta^{'}\right]. \end{aligned} \tag{26}$$

Substituting this formula in (25) and letting $\lambda \to \lambda_n$ we get

$$\frac{1}{a_1^2}\int_a^{\xi_1}[\phi_{\lambda_n}(x)]^2\,dx+\frac{1}{a_2^2}\int_{\xi_1}^{\xi_2}[\phi_{\lambda_n}(x)]^2\,dx+\frac{1}{a_3^2}\int_{\xi_2}^{b}[\phi_{\lambda_n}(x)]^2\,dx=\frac{1}{k_n}\left[\omega'(\lambda_n)-(\phi_{\lambda_n})'_\beta\right]. \quad (27)$$

Now putting $(\phi_{\lambda_n})'_\beta=\frac{1}{k_n}(\chi_{\lambda_n})'_\beta=\frac{\rho}{k_n}$ in (26) seems that $\omega'(\lambda_n)\neq 0$.

§3. Asymptotic approximate formulas of the characteristic function

Lemma 3.1. Let $\phi(x,\lambda)$ be the solutions of equation (1) defined in section 2 and let $\lambda=s^2$. Then the next integral equations are hold:

$$\begin{aligned}\phi_{1\lambda}^{(k)}(x) &= \alpha_2\left[\cos\frac{s(x-a)}{a_1}\right]^{(k)}-\alpha_1\frac{a_1}{s}\left[\sin\frac{s(x-a)}{a_1}\right]^{(k)}\\ &\quad+\frac{1}{a_1 s}\int_a^x\left[\sin\frac{s(x-y)}{a_1}\right]^{(k)}q(y)\phi_{1\lambda}(y)dy, \end{aligned}\quad (28)$$

$$\begin{aligned}\phi_{2\lambda}^{(k)}(x) &= \frac{\gamma_1}{\delta_1}\phi_{1\lambda}(\xi_1-0)\left[\cos\frac{s(x-\xi_1)}{a_2}\right]^{(k)}\frac{a_2}{s}\frac{\gamma_1'}{\delta_1'}\phi_{1\lambda}'(\xi_1-0)\left[\sin\frac{s(x-\xi_1)}{a_2}\right]^{(k)}\\ &\quad+\frac{1}{a_2 s}\int_{\xi_1}^x\left[\sin\frac{s(x-y)}{a_2}\right]^{(k)}q(y)\phi_{2\lambda}(y)dy, \end{aligned}\quad (29)$$

$$\begin{aligned}\phi_{3\lambda}^{(k)}(x) &= \frac{\gamma_2}{\delta_2}\phi_{2\lambda}(\xi_2-0)\left[\cos\frac{s(x-\xi_2)}{a_3}\right]^{(k)}\frac{a_3}{s}\frac{\gamma_2'}{\delta_2'}\phi_{2\lambda}'(\xi_2-0)\left[\sin\frac{s(x-\xi_2)}{a_3}\right]^{(k)}\\ &\quad+\frac{1}{a_3 s}\int_{\xi_2}^x\left[\sin\frac{s(x-y)}{a_3}\right]^{(k)}q(y)\phi_{3\lambda}(y)dy, \end{aligned}\quad (30)$$

where $(\bullet)^{(k)}=\frac{d^k}{dx^k}(\bullet)$.

Proof. It is enough to substitute $s^2\phi_{1\lambda}(y)+a_1^2\phi_{1\lambda}''(y)$, $s^2\phi_{2\lambda}(y)+a_2^2\phi_{2\lambda}''(y)$ and $s^2\phi_{3\lambda}(y)+a_3^2\phi_{3\lambda}''(y)$ instead of $q(y)\phi_{1\lambda}(y)$, $q(y)\phi_{2\lambda}(y)$ and $q(y)\phi_{3\lambda}(y)$ in the integral terms of the equations (27), (28) and (29) for $k=0,1$, respectively and integrate by parts twice.

Lemma 3.2. Let $\lambda=s^2$, $Ims=t$. Then the functions $\phi_{i\lambda}(x)$ have the following asymptotic representations for $|\lambda|\to\infty$, which hold uniformly for $x\in\Omega_i$ $(i=1,2,3)$ and $k=0,1$:

If $\alpha_2\neq 0$,

$$\phi_{1\lambda}^{(k)}(x)=\alpha_2\left[\cos\frac{s(x-a)}{a_1}\right]^{(k)}+O\left(|s|^{k-1}exp\frac{|t|(x-a)}{a_1}\right), \quad (31)$$

$$\begin{aligned}\phi_{2\lambda}^{(k)}(x) &= \alpha_2\frac{\gamma_1}{\delta_1}\left[\cos\frac{s(x-\xi_1)}{a_2}\right]^{(k)}\cos\frac{s(\xi_1-a)}{a_1}\\ &- \alpha_2\frac{a_2}{a_1}\frac{\gamma_1'}{\delta_1'}\left[\sin\frac{s(x-\xi_1)}{a_2}\right]^{(k)}\sin\frac{s(\xi_1-a)}{a_1}\\ &+ O\left(|s|^{k-1}exp\frac{|t|\left[a_1(x-\xi_1)+a_2(\xi_1-a)\right]}{a_1a_2}\right), \end{aligned}\quad (32)$$

$$\begin{aligned}
\phi_{3\lambda}^{(k)}(x) &= \alpha_2 \frac{\gamma_1\gamma_2}{\delta_1\delta_2}\left[\cos\frac{s(x-\xi_2)}{a_3}\right]^{(k)}\cos\frac{s(\xi_2-\xi_1)}{a_2}\cos\frac{s(\xi_1-a)}{a_1} \\
&- \alpha_2\frac{a_2}{a_1}\frac{\gamma_1^{'}\gamma_2}{\delta_1^{'}\delta_2}\left[\cos\frac{s(x-\xi_2)}{a_3}\right]^{(k)}\sin\frac{s(\xi_2-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1} \\
&- \alpha_2\frac{a_3}{a_2}\frac{\gamma_1\gamma_2^{'}}{\delta_1\delta_2^{'}}\left[\sin\frac{s(x-\xi_2)}{a_3}\right]^{(k)}\sin\frac{s(\xi_2-\xi_1)}{a_2}\cos\frac{s(\xi_1-a)}{a_1} \\
&- \alpha_2\frac{a_3}{a_1}\frac{\gamma_1^{'}\gamma_2^{'}}{\delta_1^{'}\delta_2^{'}}\left[\sin\frac{s(x-\xi_2)}{a_3}\right]^{(k)}\cos\frac{s(\xi_2-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1} \\
&+ O\left(|s|^{k-1}exp\frac{|t|\left[a_1a_2(x-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)\right]}{a_1a_2a_3}\right). \qquad (33)
\end{aligned}$$

If $\alpha_2 = 0$,

$$\phi_{1\lambda}^{(k)}(x) = -\frac{a_1}{s}\alpha_1\left[\sin\frac{s(x-a)}{a_1}\right]^{(k)} + O\left(|s|^{k-2}exp\frac{|t|(x-a)}{a_1}\right), \qquad (34)$$

$$\begin{aligned}
\phi_{2\lambda}^{(k)}(x) &= -\frac{a_1\alpha_1}{s}\frac{\gamma_1}{\delta_1}\left[\cos\frac{s(x-\xi_1)}{a_2}\right]^{(k)}\sin\frac{s(\xi_1-a)}{a_1} \\
&- \frac{a_2\alpha_1}{s}\frac{\gamma_1^{'}}{\delta_1^{'}}\left[\sin\frac{s(x-\xi_1)}{a_2}\right]^{(k)}\cos\frac{s(\xi_1-a)}{a_1} \\
&+ O\left(|s|^{k-2}exp\frac{|t|\left[a_1(x-\xi_1)+a_2(\xi_1-a)\right]}{a_1a_2}\right), \qquad (35)
\end{aligned}$$

$$\begin{aligned}
\phi_{3\lambda}^{(k)}(x) &= -\frac{a_1\alpha_1}{s}\frac{\gamma_1\gamma_2}{\delta_1\delta_2}\left[\cos\frac{s(x-\xi_2)}{a_3}\right]^{(k)}\cos\frac{s(\xi_2-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1} \\
&- \frac{a_2\alpha_1}{s}\frac{\gamma_1^{'}\gamma_2}{\delta_1^{'}\delta_2}\left[\cos\frac{s(x-\xi_2)}{a_3}\right]^{(k)}\sin\frac{s(\xi_2-\xi_1)}{a_2}\cos\frac{s(\xi_1-a)}{a_1} \\
&+ \frac{\alpha_1}{s}\frac{a_1a_3}{a_2}\frac{\gamma_1\gamma_2^{'}}{\delta_1\delta_2^{'}}\left[\sin\frac{s(x-\xi_2)}{a_3}\right]^{(k)}\sin\frac{s(\xi_2-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1} \\
&- \frac{a_3\alpha_1}{s}\frac{\gamma_1^{'}\gamma_2^{'}}{\delta_1^{'}\delta_2^{'}}\left[\sin\frac{s(x-\xi_2)}{a_3}\right]^{(k)}\cos\frac{s(\xi_2-\xi_1)}{a_2}\cos\frac{s(\xi_1-a)}{a_1} \\
&+ O\left(|s|^{k-2}exp\frac{|t|\left[a_1a_2(x-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)\right]}{a_1a_2a_3}\right). \qquad (36)
\end{aligned}$$

Proof. Since the proof of formulas for $\phi_{1\lambda}(x)$ are identical to the Titchmarsh's proof of similar results for $\phi_\lambda(x)$ [14], we may formulate them without proving. But the similar formulas for $\phi_{2\lambda}(x)$ and $\phi_{3\lambda}(x)$ need individual consideration, since the last solutions are defined by initial conditions of special nonstandard forms. We shall only prove the formula (31) for $k = 0$. Let $\alpha_2 \neq 0$. Then according to (30),

$$\phi_{1\lambda}(\xi_1 - 0) = \alpha_2\cos\frac{s(\xi_1-a)}{a_1} + O\left(|s|^{-1}exp\frac{|t|\left[(\xi_1-a)\right]}{a_1}\right)$$

and

$$\phi_{1\lambda}^{'}(\xi_1 - 0) = -s\frac{\alpha_2}{a_1}\sin\frac{s(\xi_1-a)}{a_1} + O\left(exp\frac{|t|\left[(\xi_1-a)\right]}{a_1}\right).$$

Substituting this asymptotic expressions into (28) for $k=0$ we get

$$\begin{aligned}\phi_{2\lambda}(x) &= a_2\alpha_2\left[\frac{1}{a_2}\frac{\gamma_1}{\delta_1}\cos\frac{s(x-\xi_1)}{a_2}\cos\frac{s(\xi_1-a)}{a_1}-\frac{1}{a_1}\frac{\gamma_1^{'}}{\delta_1^{'}}\sin\frac{s(x-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1}\right]\\ &+ \frac{1}{a_2 s}\int_{\xi_1}^{x}\sin\frac{s(x-y)}{a_2}q(y)\phi_{2\lambda}(y)dy\\ &+ O\left(|s|^{-1}exp\frac{|t|\left[a_1(x-\xi_1)+a_2(\xi_1-a)\right]}{a_1a_2}\right). \qquad (37)\end{aligned}$$

Multiplying through by $exp\left(-\frac{|t|[a_1(x-\xi_1)+a_2(\xi_1-a)]}{a_1a_2}\right)$ and denoting

$$F_{2\lambda}(x) := exp\left(-\frac{|t|\left[a_1(x-\xi_1)+a_2(\xi_1-a)\right]}{a_1a_2}\right)\phi_{2\lambda}(x),$$

we have

$$\begin{aligned}F_{2\lambda}(x) &:= a_2\alpha_2 exp\left(-\frac{|t|\left[a_1(x-\xi_1)+a_2(\xi_1-a)\right]}{a_1a_2}\right)\left[\frac{1}{a_2}\frac{\gamma_1}{\delta_1}\cos\frac{s(x-\xi_1)}{a_2}\cos\frac{s(\xi_1-a)}{a_1}\right.\\ &- \left.\frac{1}{a_1}\frac{\gamma_1^{'}}{\delta_1^{'}}\sin\frac{s(x-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1}\right]\\ &+ \frac{1}{a_2 s}\int_{\xi_1}^{x}\sin\frac{s(x-y)}{a_2}q(y)exp\left(-\frac{|t|\left[a_1(x-\xi_1)+a_2(\xi_1-a)\right]}{a_1a_2}\right)F_{2\lambda}(y)dy+O\left(|s|^{-1}\right).\end{aligned}$$

Denoting $M(\lambda) := \max_{x\in[\xi_1,\xi_2]}|F_{2\lambda}(x)|$ from the last formula it follows that

$$M(\lambda)\le\left|\frac{\alpha_2\gamma_1}{\delta_1}\right|+\frac{a_2}{a_1}\left|\frac{\alpha_2\gamma_1^{'}}{\delta_1^{'}}\right|+\frac{M(\lambda)}{|s|a_2}\int_{\xi_1}^{\xi_2}|q(y)|dy+\frac{M_0}{|s|}$$

for some $M_0>0$. From this it follows that $M(\lambda)=O(1)$ as $\lambda\to\infty$, so

$$\phi_{2\lambda}(x)=O\left(exp\frac{|t|\left[a_1(x-\xi_1)+a_2(\xi_1-a)\right]}{a_1a_2}\right).$$

Substituting back into the integral on the right of (36) yields (31) for $k=0$. The other assertions can be proved similarly.

Theorem 3.1. Let $\lambda=s^2$, $t=Ims$. Then the characteristic function $\omega(\lambda)$ has the following asymptotic representations:

Case 1: If $\beta_2^{'}\neq 0, \alpha_2\neq 0$, then

$$\begin{aligned}\omega_3(\lambda) &= \alpha_2\beta_2^{'}s^3\frac{1}{a_3}\frac{\gamma_1\gamma_2}{\delta_1\delta_2}\sin\frac{s(b-\xi_2)}{a_3}\cos\frac{s(\xi_2-\xi_1)}{a_2}\cos\frac{s(\xi_1-a)}{a_1}\\ &- \alpha_2\beta_2^{'}s^3\frac{a_2}{a_1a_3}\frac{\gamma_1^{'}\gamma_2}{\delta_1^{'}\delta_2}\sin\frac{s(b-\xi_2)}{a_3}\sin\frac{s(\xi_2-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1}\\ &+ \alpha_2\beta_2^{'}s^3\frac{1}{a_2}\frac{\gamma_1\gamma_2^{'}}{\delta_1\delta_2^{'}}\cos\frac{s(b-\xi_2)}{a_3}\sin\frac{s(\xi_2-\xi_1)}{a_2}\cos\frac{s(\xi_1-a)}{a_1}\\ &+ \alpha_2\beta_2^{'}s^3\frac{1}{a_1}\frac{\gamma_1^{'}\gamma_2^{'}}{\delta_1^{'}\delta_2^{'}}\cos\frac{s(b-\xi_2)}{a_3}\cos\frac{s(\xi_2-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1}\\ &+ O\left(|s|^2exp\frac{|t|\left[a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)\right]}{a_1a_2a_3}\right). \qquad (38)\end{aligned}$$

Case 2: If $\beta_2^{'} \neq 0, \alpha_2 = 0$, then

$$\begin{aligned}
\omega_3(\lambda) &= -\alpha_1\beta_2^{'}s^2\frac{a_1}{a_3}\frac{\gamma_1\gamma_2}{\delta_1\delta_2}\sin\frac{s(b-\xi_2)}{a_3}\cos\frac{s(\xi_2-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1} \\
&- \alpha_1\beta_2^{'}s^2\frac{a_2}{a_3}\frac{\gamma_1^{'}\gamma_2}{\delta_1^{'}\delta_2}\sin\frac{s(b-\xi_2)}{a_3}\sin\frac{s(\xi_2-\xi_1)}{a_2}\cos\frac{s(\xi_1-a)}{a_1} \\
&- \alpha_1\beta_2^{'}s^2\frac{a_1}{a_2}\frac{\gamma_1\gamma_2^{'}}{\delta_1\delta_2^{'}}\cos\frac{s(b-\xi_2)}{a_3}\sin\frac{s(\xi_2-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1} \\
&+ \alpha_1\beta_2^{'}s^2\frac{\gamma_1^{'}\gamma_2^{'}}{\delta_1^{'}\delta_2^{'}}\cos\frac{s(b-\xi_2)}{a_3}\cos\frac{s(\xi_2-\xi_1)}{a_2}\cos\frac{s(\xi_1-a)}{a_1} \\
&+ O\left(|s|exp\frac{|t|\left[a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)\right]}{a_1a_2a_3}\right). \qquad (39)
\end{aligned}$$

Case 3: If $\beta_2^{'} = 0, \alpha_2 \neq 0$, then

$$\begin{aligned}
\omega_3(\lambda) &= \alpha_2\beta_1^{'}s^2\frac{\gamma_1\gamma_2}{\delta_1\delta_2}\cos\frac{s(b-\xi_2)}{a_3}\cos\frac{s(\xi_2-\xi_1)}{a_2}\cos\frac{s(\xi_1-a)}{a_1} \\
&- \alpha_2\beta_1^{'}s^2\frac{a_2}{a_1}\frac{\gamma_1^{'}\gamma_2}{\delta_1^{'}\delta_2}\cos\frac{s(b-\xi_2)}{a_3}\sin\frac{s(\xi_2-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1} \\
&- \alpha_2\beta_1^{'}s^2\frac{a_3}{a_2}\frac{\gamma_1\gamma_2^{'}}{\delta_1\delta_2^{'}}\sin\frac{s(b-\xi_2)}{a_3}\sin\frac{s(\xi_2-\xi_1)}{a_2}\cos\frac{s(\xi_1-a)}{a_1} \\
&+ \alpha_2\beta_1^{'}s^2\frac{a_3}{a_1}\frac{\gamma_1^{'}\gamma_2^{'}}{\delta_1^{'}\delta_2^{'}}\sin\frac{s(b-\xi_2)}{a_3}\cos\frac{s(\xi_2-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1} \\
&+ O\left(|s|exp\frac{|t|\left[a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)\right]}{a_1a_2a_3}\right). \qquad (40)
\end{aligned}$$

Case 4: If $\beta_2^{'} = 0, \alpha_2 = 0$, then

$$\begin{aligned}
\omega_3(\lambda) &= -\alpha_1\beta_1^{'}sa_1\frac{\gamma_1\gamma_2}{\delta_1\delta_2}\cos\frac{s(b-\xi_2)}{a_3}\cos\frac{s(\xi_2-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1} \\
&- \alpha_1\beta_1^{'}sa_2\frac{\gamma_1^{'}\gamma_2}{\delta_1^{'}\delta_2}\cos\frac{s(b-\xi_2)}{a_3}\sin\frac{s(\xi_2-\xi_1)}{a_2}\cos\frac{s(\xi_1-a)}{a_1} \\
&- \alpha_1\beta_1^{'}s\frac{a_1a_3}{a_2}\frac{\gamma_1\gamma_2^{'}}{\delta_1\delta_2^{'}}\sin\frac{s(b-\xi_2)}{a_3}\sin\frac{s(\xi_2-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1} \\
&+ \alpha_1\beta_1^{'}sa_3\frac{\gamma_1^{'}\gamma_2^{'}}{\delta_1^{'}\delta_2^{'}}\sin\frac{s(b-\xi_2)}{a_3}\cos\frac{s(\xi_2-\xi_1)}{a_2}\sin\frac{s(\xi_1-a)}{a_1} \\
&+ O\left(exp\frac{|t|\left[a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)\right]}{a_1a_2a_3}\right). \qquad (41)
\end{aligned}$$

Where $k = 0, 1$.

Proof. The proof is immediate by substituting (32) and (35) for $k = 0$ in the representation

$$\begin{aligned}
\omega_3(\lambda) &= \lambda\left[\beta_1^{'}\phi_{3\lambda}(b) - \beta_2^{'}\phi_{3\lambda}^{'}(b)\right] + \left[\beta_1\phi_{3\lambda}(b) - \beta_2\phi_{3\lambda}^{'}(b)\right] \\
&= -\lambda\beta_2^{'}\phi_{3\lambda}^{'}(b) + \lambda\beta_1^{'}\phi_{3\lambda}(b) + \beta_1\phi_{3\lambda}(b) - \beta_2^{'}\phi_{3\lambda}^{'}(b). \qquad (42)
\end{aligned}$$

Corollary 3.1. The eigenvalues of the problem (1-7) are bounded below.

Proof. Putting $s = it \quad (t > 0)$ in the above formulae it follows that $\omega_3(-t^2) \to \infty$ as $t \to \infty$. Hence $\omega_3(\lambda) \neq 0$ for λ negative and sufficiently large.

§4. Asymptotic formula for eigenvalues and eigenfunctions

Here we can obtain the asymptotic approximation formula for the eigenvalues of the problem (1-7). Since the eigenvalues coincide with the zeros of the entire function $\omega_3(\lambda)$, it follows that they have no finite limit. Moreover, we know from corollary 2.1 and 2.2 that all eigenvalues are real and bounded below. Therefore, we may renumber them as $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ...$, listed according to their multiplicity. In this section for the sake of simplicity, we shall assume that $a_i\gamma_i\delta_i^{'} = a_{i+1}\gamma_i^{'}\delta_i$ $(i = 1, 2)$.

Theorem 4.1. The eigenvalues $\lambda_n = s_n^2$ $(n = 0, 1, 2, ...)$ of the problem (1-7) have the following asymptotic representation for $n \to \infty$:

Case 1: If $\beta_2^{'} \neq 0,\ \alpha_2 \neq 0$, then

$$s_n = \frac{a_1a_2a_3}{a_1a_2(b-\xi_2) + a_1a_3(\xi_2-\xi_1) + a_2a_3(\xi_1-a)}\pi\,(n-1) + O\left(\frac{1}{n}\right). \tag{43}$$

Case 2: If $\beta_2^{'} \neq 0,\ \alpha_2 = 0$, then

$$s_n = \frac{a_1a_2a_3}{a_1a_2(b-\xi_2) + a_1a_3(\xi_2-\xi_1) + a_2a_3(\xi_1-a)}\pi\left(n-\frac{1}{2}\right) + O\left(\frac{1}{n}\right). \tag{44}$$

Case 3: If $\beta_2^{'} = 0,\ \alpha_2 \neq 0$, then

$$s_n = \frac{a_1a_2a_3}{a_1a_2(b-\xi_2) + a_1a_3(\xi_2-\xi_1) + a_2a_3(\xi_1-a)}\pi\left(n-\frac{1}{2}\right) + O\left(\frac{1}{n}\right). \tag{45}$$

Case 4: If $\beta_2^{'} = 0,\ \alpha_2 = 0$, then

$$s_n = \frac{a_1a_2a_3}{a_1a_2(b-\xi_2) + a_1a_3(\xi_2-\xi_1) + a_2a_3(\xi_1-a)}\pi n + O\left(\frac{1}{n}\right). \tag{46}$$

Proof. We shall only consider the first case (the other cases may be considered analogically). Denoting $\omega_0(s) := \omega_3(s^2) = \omega_3(\lambda)$,

$$\omega_1(s) = \frac{\alpha_2\beta_2^{'}\gamma_1\gamma_2}{a_3\delta_1\delta_2}s^3\sin\left(\frac{a_1a_2(b-\xi_2) + a_1a_3(\xi_2-\xi_1) + a_2a_3(\xi_1-a)}{a_1a_2a_3}s\right)$$

and

$$\omega_2(s) := \omega_0(s) - \omega_1(\lambda).$$

We write $\omega(\lambda)$ as $\omega_0(s) = \omega_1(s) + \omega_2(\lambda)$. In view (37) from elementary considerations we have

$$\omega_2(s) = O\left(|s|^2 exp\frac{|t|\left[a_1a_2(b-\xi_2) + a_1a_3(\xi_2-\xi_1) + a_2a_3(\xi_1-a)\right]}{a_1a_2a_3}\right).$$

We shall apply the well-known Rouche theorem which asserts that if $f(s)$ and $g(s)$ analytic inside and on a closed contour C, and $|g(s)| < |f(s)|$ on C, then $f(s)$ and $f(s) + g(s)$ have the

same number zeros inside C, provided that each zero is counted according to their multiplicity. For sufficiently large, It is readily shown that $|\omega_1(s)| > |\omega_2(s)|$ on the contours

$$C_n := \left\{ s \in \mathbb{C} \,|\quad |s| = \frac{a_1a_2a_3}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)} \left(n+\frac{1}{2}\right)\pi \right\}$$

for sufficiently large n. Let $\lambda_0 \leq \lambda_1 \leq ...$ are zeros of $\omega(\lambda)$ and $\lambda_n = s_n^2$. Since inside the contour C_n, $\omega_1(s)$ has zeros at points $s = 0$ (with multiplicity 4) and

$$s = \frac{a_1a_2a_3}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)} \frac{k\pi}{2}, \quad k = \pm 1, \pm 2, ..., \pm n \tag{47}$$

(with multiplicity 1), and so the number of zeros is $2n+4$, it follows that

$$s_n = \frac{a_1a_2a_3}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}(n-1)\pi + \delta_n, \tag{48}$$

where $\delta_n = O(1)$, more precisely

$$|\delta_n| < \frac{a_1a_2a_3}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)} \frac{\pi}{4}$$

for sufficiently large n. By substituting in (37) we derive that $\delta_n = O\left(\frac{1}{n}\right)$, which completes the proof. The next approximation for the eigenvalues may be obtained by following procedure. For this, we shall suppose that $q(y)$ is of bounded variation in $[a,b]$. We only consider the case $\beta_2^{'} \neq 0, \alpha_2 \neq 0$ (since the other cases may be considered similarly). Putting $x = \xi_1$ in (27), $x = \xi_2$ in (28) and then substituting in (29), we derive that

$$\begin{aligned}
\phi_{3\lambda}^{'}(b) &= -\frac{s\alpha_2\gamma_1\gamma_2}{a_2\delta_1\delta_2} \sin\left(\frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2a_3}s\right) \\
&- \frac{\alpha_1\gamma_1^{'}\gamma_2^{'}}{\delta_1^{'}\delta_2^{'}} \cos\left(\frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2a_3}s\right) \\
&+ \frac{\gamma_1\gamma_2^{'}}{a_1a_2\delta_1\delta_2^{'}} \int_a^{\xi_1} \cos\left(\frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-y)}{a_1a_2a_3}s\right) q(y)\phi_{1\lambda}(y)dy \\
&+ \frac{\gamma_2}{a_2a_3\delta_2} \int_{\xi_1}^{\xi_2} \cos\left(\frac{a_2(b-\xi_2)+a_3(\xi_2-y)}{a_2a_3}s\right) q(y)\phi_{2\lambda}(y)dy \\
&+ \frac{1}{a_3^2} \int_{\xi_2}^{b} \cos\left(\frac{(b-y)}{a_3}s\right) q(y)\phi_{3\lambda}(y)dy.
\end{aligned}$$

Substituting (30), (31) and (32) in the right side of the last integral equality then gives

$$\begin{aligned}
\phi_{3\lambda}^{'}(b) &= -\frac{s\alpha_2\gamma_1\gamma_2}{a_2\delta_1\delta_2}\sin\left(\frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2a_3}s\right)\\
&- \frac{\alpha_1\gamma_1^{'}\gamma_2^{'}}{\delta_1^{'}\delta_2^{'}}\cos\left(\frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2a_3}s\right)\\
&+ \frac{\alpha_2\gamma_1\gamma_2^{'}}{a_1a_2\delta_1\delta_2^{'}}\int_a^{\xi_1}\cos\left(\frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-y)}{a_1a_2a_3}s\right)q(y)\\
&\quad \cos\left(\frac{s(y-a)}{a_1}\right)dy\\
&+ \frac{\alpha_2\gamma_1\gamma_2}{a_2a_3\delta_1\delta_2}\int_{\xi_1}^{\xi_2}\cos\left(\frac{a_2(b-\xi_2)+a_3(\xi_2-y)}{a_2a_3}s\right)q(y)\\
&\quad \cos\left(\frac{a_1(y-\xi_1)+a_2(\xi_1-a)}{a_1a_2}s\right)dy\\
&+ \frac{\alpha_2\gamma_1\gamma_2}{a_3^2\delta_1\delta_2}\int_{\xi_2}^{b}\cos\left(\frac{(b-y)}{a_3}s\right)q(y)\\
&\quad \cos\left(\frac{a_1a_2(y-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2a_3}s\right)dy\\
&+ O\left(|s|^{-1}exp\frac{|t|\left[a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)\right]}{a_1a_2a_3}\right).
\end{aligned}$$

On the other hand, from (32) it follows that

$$\begin{aligned}
\phi_{3\lambda}(b) &= \frac{\alpha_2\gamma_1\gamma_2}{\delta_1\delta_2}\cos\left(\frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2a_3}s\right)\\
&\quad +O\left(|s|^{-1}exp\frac{|t|\left[a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)\right]}{a_1a_2a_3}\right).
\end{aligned}$$

Putting these formulas in (41) we have

$$\begin{aligned}
\omega_3(\lambda) &= s^3\frac{\beta_2^{'}\alpha_2\gamma_1\gamma_2}{a_3\delta_1\delta_2}\sin\left(\frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2a_3}s\right)\\
&+ s^2\left(\alpha_2\beta_1^{'}\frac{\gamma_1\gamma_2}{\delta_1\delta_2}+\alpha_1\beta_2^{'}\frac{\gamma_1^{'}\gamma_2^{'}}{\delta_1^{'}\delta_2^{'}}\right)\cos\left(\frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2a_3}s\right)\\
&- s^2\frac{\beta_2^{'}}{a_1a_2}\frac{\gamma_1\gamma_2^{'}}{\delta_1\delta_2^{'}}\int_a^{\xi_2}\cos\left(\frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-y)}{a_1a_2a_3}s\right)q(y)dy\\
&- s^2\frac{\beta_2^{'}}{a_2a_3}\frac{\gamma_2}{\delta_2}\int_{\xi_1}^{\xi_2}\cos\left(\frac{a_2(b-\xi_2)+a_3(\xi_2-y)}{a_2a_3}s\right)q(y)\phi_{2\lambda}(y)dy\\
&- s^2\frac{\beta_2^{'}}{a_3^2}\int_{\xi_2}^{b}\cos\left(\frac{(b-y)}{a_3}s\right)q(y)\phi_{3\lambda}(y)dy\\
&+ O\left(|s|exp\frac{|t|\left[a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)\right]}{a_1a_2a_3}\right).
\end{aligned}$$

Putting (42) in the last equality we find that

$$\sin\delta_n = -\frac{\cos\delta_n}{s_n}\left[a_3\frac{\beta_1^{'}}{\beta_2^{'}}+a_1\frac{\alpha_1}{\alpha_2}-\frac{Q}{2}+O\left(\frac{1}{n}\right)\right]+O\left(|s_n|^{-2}\right), \tag{49}$$

where

$$Q=\frac{1}{a_1}\int_a^{\xi_1}q(y)dy+\frac{1}{a_2}\int_{\xi_1}^{\xi_2}q(y)dy+\frac{1}{a_3}\int_{\xi_2}^{b}q(y)dy.$$

Recalling that $q(y)$ is of bounded variation in $[a,b]$ and applying the well-known Riemann-Lebesque Lemma [18] to the third integral on the right in (47), this term is $O\left(\frac{1}{n}\right)$. Consequently, from (47) it follows that

$$\delta_n=-\frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2a_3}\frac{1}{\pi(n-1)}\left[a_3\frac{\beta_1'}{\beta_2'}+a_1\frac{\alpha_1}{\alpha_2}-\frac{Q}{2}\right]+O\left(\frac{1}{n^2}\right).$$

Substituting in (46), we have

$$\begin{aligned}s_n &= \frac{a_1a_2a_3}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi(n-1)\\ &\quad-\frac{1}{\pi(n-1)}\left[a_3\frac{\beta_1'}{\beta_2'}+a_1\frac{\alpha_1}{\alpha_2}-\frac{Q}{2}\right]+O\left(\frac{1}{n^2}\right).\end{aligned}$$

Similar formulae in the other cases are as follows:
Case 2:

$$\begin{aligned}s_n &= \frac{a_1a_2a_3}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi\left(n-\frac{1}{2}\right)\\ &\quad+\frac{1}{\pi\left(n-\frac{1}{2}\right)}\left[a_3\frac{\beta_1'}{\beta_2'}-\frac{Q}{2}\right]+O\left(\frac{1}{n^2}\right).\end{aligned}$$

Case 3:

$$\begin{aligned}s_n &= \frac{a_1a_2a_3}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi\left(n-\frac{1}{2}\right)\\ &\quad+\frac{1}{\pi\left(n-\frac{1}{2}\right)}\left[a_1\frac{\alpha_1}{\alpha_2}-\frac{1}{a_3}\frac{\beta_2}{\beta_1'}-\frac{Q}{2}\right]+O\left(\frac{1}{n^2}\right).\end{aligned}$$

Case 4:

$$\begin{aligned}s_n &= \frac{a_1a_2a_3}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi n\\ &\quad+\frac{1}{\pi n}\left[\frac{1}{a_3}\frac{\beta_2}{\beta_1'}+\frac{Q}{2}\right]+O\left(\frac{1}{n^2}\right).\end{aligned}$$

Recalling that $\phi(x,\lambda_n)$ is an eigenfunction according to eigenvalue λ_n, by putting (42) in the (30), (31) and (32) for $k=0$ we derive that

$$\phi_{1\lambda_n}=\alpha_2\cos\left(\frac{a_2a_3(x-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\right)+O\left(\frac{1}{n}\right),$$

$$\phi_{2\lambda_n}=\alpha_2\frac{\gamma_1}{\delta_1}\cos\left(\frac{a_1a_3(x-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\right)+O\left(\frac{1}{n}\right)$$

and

$$\phi_{3\lambda_n} = \alpha_2 \frac{\gamma_1\gamma_2}{\delta_1\delta_2} \cos\left(\frac{a_1a_2(x-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\right) + O\left(\frac{1}{n}\right)$$

in the first case. Hence if $\beta_2^{'} \neq 0\, and\, \alpha_2 \neq 0$ then the eigenfunction $\phi(x,\lambda_n)$ has the asymptotic representation

$$\phi(x,\lambda_n) = \begin{cases} \alpha_2 \cos\left[\frac{a_2a_3(x-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi(n-1)\right] + O\left(\frac{1}{n}\right), & x \in [a,\xi_1), \\ \alpha_2 \frac{\gamma_1}{\delta_1} \cos\left[\frac{a_1a_3(x-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi(n-1)\right] + O\left(\frac{1}{n}\right), & x \in (\xi_1,\xi_2), \\ \alpha_2 \frac{\gamma_1\gamma_2}{\delta_1\delta_2} \cos\left[\frac{a_1a_2(x-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi(n-1)\right] + O\left(\frac{1}{n}\right), & x \in [\xi_2,b). \end{cases}$$

Which hold uniformly for $x \in [a,\xi_1) \cup [\xi_1,\xi_2) \cup [\xi_2,b)$. Similar formulae in the other cases are as follows:

In case 2:

$$\phi(x,\lambda_n) = \begin{cases} -\alpha_1 \frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_2a_3} \frac{1}{\pi\left(n-\frac{1}{2}\right)} \\ \times \sin\left[\frac{a_2a_3(x-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi\left(n-\frac{1}{2}\right)\right] + O\left(\frac{1}{n^2}\right), x \in [a,\xi_1), \\ -\alpha_1 \frac{\gamma_1}{\delta_1} \frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_2a_3} \frac{1}{\pi\left(n-\frac{1}{2}\right)} \\ \times \sin\left[\frac{a_1a_3(x-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi\left(n-\frac{1}{2}\right)\right] + O\left(\frac{1}{n^2}\right), x \in (\xi_1,\xi_2), \\ -\alpha_1 \frac{\gamma_1\gamma_2}{\delta_1\delta_2} \frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_2a_3} \frac{1}{\pi\left(n-\frac{1}{2}\right)} \\ \times \sin\left[\frac{a_1a_2(x-\xi_2)+a_1a_3(\xi_-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi\left(n-\frac{1}{2}\right)\right] + O\left(\frac{1}{n^2}\right), x \in (\xi_2,b]. \end{cases}$$

In case 3:

$$\phi(x,\lambda_n) = \begin{cases} \alpha_2 \cos\left[\frac{a_2a_3(x-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi\left(n-\frac{1}{2}\right)\right] + O\left(\frac{1}{n}\right), & x \in [a,\xi_1) \\ \alpha_2 \frac{\gamma_1}{\delta_1} \cos\left[\frac{a_1a_3(x-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi\left(n-\frac{1}{2}\right)\right] + O\left(\frac{1}{n}\right), & x \in (\xi_1,\xi_2) \\ \alpha_2 \frac{\gamma_1\gamma_2}{\delta_1\delta_2} \cos\left[\frac{a_1a_2(x-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi\left(n-\frac{1}{2}\right)\right] + O\left(\frac{1}{n}\right), & x \in [\xi_2,b) \end{cases}$$

In case 4:

$$\phi(x,\lambda_n) = \begin{cases} -\alpha_1 \frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_2a_3} \frac{1}{\pi n} \\ \times \sin\left[\frac{a_2a_3(x-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi n\right] + O\left(\frac{1}{n^2}\right), \ x \in [a,\xi_1), \\ -\alpha_1 \frac{\gamma_1}{\delta_1} \frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_2a_3} \frac{1}{\pi n} \\ \times \sin\left[\frac{a_1a_3(x-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi n\right] + O\left(\frac{1}{n^2}\right), \ x \in (\xi_1,\xi_2). \\ -\alpha_1 \frac{\gamma_1\gamma_2}{\delta_1\delta_2} \frac{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}{a_2a_3} \frac{1}{\pi n} \\ \times \sin\left[\frac{a_1a_2(x-\xi_2)+a_1a_3(\xi_-\xi_1)+a_2a_3(\xi_1-a)}{a_1a_2(b-\xi_2)+a_1a_3(\xi_2-\xi_1)+a_2a_3(\xi_1-a)}\pi n\right] + O\left(\frac{1}{n^2}\right), \ x \in (\xi_2,b]. \end{cases}$$

All these asymptotic approximations hold uniformly for $x \in [a,\xi_1) \cup [\xi_1,\xi_2) \cup [\xi_2,b)$.

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On the reduced residue system modulo m

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Abstract Let m, k, b be integers, $k \neq 0$ and $m > 1$. It is well known that if $a_1, a_2, \ldots, a_m$ is a complete system of incongruent residues modulo m and $(k, m) = 1$, then $ka_1 + b$, $ka_2 + b$,...,$ka_m + b$ is also a complete system of incongruent residues modulo m for any integer b. In this paper, we give a necessary and sufficient condition for both $a_1, a_2, \ldots, a_{\phi(m)}$ and $ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b$ to be reduced residue systems modulo m.

Keywords Complete system, reduced residue system, necessary and sufficient condition.

§1. Introduction

We all know the following theorem [1,3]: if $a_1, a_2, \ldots, a_m$ is a complete system of incongruent residues modulo m and $(k, m) = 1$, then $ka_1 + b, ka_2 + b, \cdots, ka_m + b$ is also a complete system of incongruent residues modulo m for any integer b. If we replace the complete system of incongruent residues modulo m in this theorem by a reduced residue system modulo m, the result is not true. For example, 1, 5 modulo 6 is a reduced residue system modulo 6, but $1 \times 1 + 2$, $1 \times 5 + 2$ modulo 6 is not a reduced residue system modulo 6. We can easily find that $(k, m) = 1$ and $m|b$ are sufficient for $ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b$ to be a reduced residue system modulo m, but are they necessary? We answer it by the following example. Both 1, 3, 7, 9 modulo 10 and $2 \times 1 + 5, 2 \times 3 + 5, 2 \times 7 + 5, 2 \times 9 + 5$ modulo 10 are reduced residue systems modulo 10, but neither $(k, m) = 1$ nor $m|b$. In this paper, we give a necessary and sufficient condition for both $a_1, a_2, \ldots, a_{\phi(m)}$ and $ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b$ to be reduced residue systems modulo m.

§2. Main results

Theorem 1. Let m, k, b be integers, $k \neq 0$ and $m > 1$. Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$ $(p_1 < p_2 < \ldots < p_s)$ be the standard factorization of m. Suppose that $a_1, a_2, \ldots, a_{\phi(m)}$ is a reduced residue system modulo m, then $ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b$ is also a reduced residue system modulo m if and only if one of the following two conditions hold:

i) $(k, m) = 1$, $p_1 \ldots p_s | b$;

ii) $(k, m) = 2$, $m = 4t + 2$ $(t = 0, 1, 2, \ldots)$, $2 \nmid b$ and $p_2 \ldots p_s | b$.

Before the proof, we give the following lemma.

Lemma 2. Let n, d be positive integers, $n > 1$ and $d|n$. Then every reduced residue system modulo n can be divided into $\phi(n)/\phi(d)$ reduced residue systems modulo d.

Proof. It's sufficient for us to prove that if $(r, d) = 1$, then the set

$$A = \{r + d\ell : \ell = 1, 2, \ldots, n/d\}$$

has $\phi(n)/\phi(d)$ numbers prime to n.

Let

$$n = n_1 n_2,$$

where n_1 has the same prime factors with d and $(n_2, d) = 1$. Since

$$(r + d\ell, d) = (r, d) = 1,$$

we have

$$(r + d\ell, n_1) = 1.$$

Hence $(r + d\ell, n) = 1$ is equivalent to $(r + d\ell, n_2) = 1$. There are $\phi(n_2)$ numbers prime to n_2 in the set $\{1, 2, \ldots, n_2\}$. Take a number from it arbitrarily and denote it by s. Since $(n_2, d) = 1$, the congruence

$$r + d\ell \equiv s (\text{mod } n_2)$$

has only one solution. Hence there are $(\frac{n}{d}/n_2) \cdot \phi(n_2)$, namely $\phi(n)/\phi(d)$ numbers prime to n in the set A. This proves lemma 2.

Next we divide theorem 1 into the following two theorems and give the proof respectively.

Theorem 3. Let m, b be integers and $m > 1$. Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$ $(p_1 < p_2 < \ldots < p_s)$ be the standard factorization of m. If $a_1, a_2, \ldots, a_{\phi(m)}$ is a reduced residue system modulo m, then $a_1 + b, a_2 + b, \ldots, a_{\phi(m)} + b$ is also a reduced residue system modulo m if and only if $p_1 \ldots p_s | b$.

Proof. Sufficiency. It is obvious that $a_1 + b, a_2 + b, \cdots, a_{\phi(m)} + b$ is a set of incongruent residues modulo m, so it suffices to prove that $(a_i + b, m) = 1$ for $i = 1, 2, \ldots, \phi(m)$. Suppose that there exists an integer j such that $1 \le j \le \phi(m)$ and $(a_j + b, m) = d > 1$, we choose a prime factor p of d and then $p|m$. Since $p_1 \ldots p_s | b$, we have $p|b$. But $p|a_j + b$, we obtain $p|a_j$ and then $p|(a_j, m)$ which contradicts with $(a_j, m) = 1$. Therefore $a_1 + b, a_2 + b, \ldots, a_{\phi(m)} + b$ is also a reduced residue system modulo m.

Necessity. If $p_1 \ldots p_s \nmid b$, then there exists an integer i such that $1 \le i \le s$ and $p_i \nmid b$. Let

$$b \equiv j (\text{mod } p_i) \quad \text{and} \quad 1 \le j \le p_i - 1, \tag{1}$$

then

$$(p_i - j, p_i) = 1.$$

By lemma 2, there exists an integer ℓ such that $1 \le \ell \le \phi(m)$ and

$$a_\ell \equiv p_i - j \ (\text{mod } p_i). \tag{2}$$

From (1) and (2), we have

$$a_\ell + b \equiv p_i - j + j \equiv 0 \pmod{p_i},$$

i.e.,

$$(a_\ell + b, m) > 1.$$

Therefore

$$a_1 + b, a_2 + b, \ldots, a_{\phi(m)} + b$$

is not a reduced residue system modulo m which is a contradiction. Hence $p_1 \ldots p_s | b$. This completes the proof.

Corollary 4. Let m, k, b be integers, $k \neq 0$ and $m > 1$. Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ ($p_1 < p_2 < \ldots < p_s$) be the standard factorization of m. Suppose that $a_1, a_2, \ldots, a_{\phi(m)}$ is a reduced residue system modulo m and $(k, m) = 1$, then $ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b$ is also a reduced residue system modulo m if and only if $p_1 \ldots p_s \mid b$.

Proof. Since $(k, m) = 1$, $ka_1, ka_2, \ldots, ka_{\phi(m)}$ is a reduced residue system modulo m. By theorem 3, we can get this corollary immediately.

Theorem 5. Let m, k, b be integers, $k \neq 0$ and $m > 1$. Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$ ($p_1 < p_2 < \ldots < p_s$) be the standard factorization of m. Suppose that $a_1, a_2, \ldots, a_{\phi(m)}$ is a reduced residue system modulo m and $(k, m) > 1$, then $ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b$ is also a reduced residue system modulo m if and only if all the following conditions hold: $(k, m) = 2$, $m = 4t + 2$ ($t = 0, 1, 2, \cdots$), $2 \nmid b$ and $p_2 \cdots p_s | b$.

Proof. Necessity. Since $ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b$ is a reduced residue system modulo m, we have $ka_i + b \not\equiv ka_j + b \pmod{m}$ for $1 \leq i < j \leq \phi(m)$. It follows that $m \nmid k(a_i - a_j)$, i.e.,

$$\frac{m}{(m, k)} \nmid a_i - a_j, \ 1 \leq i < j \leq \phi(m).$$

If $\phi(m) > \phi(\frac{m}{(m,k)})$, by lemma 2, then there exist integers i_0, j_0 such that $1 \leq i_0 < j_0 \leq \phi(m)$ and

$$a_{i_0} \equiv a_{j_0} \pmod{\frac{m}{(m, k)}}.$$

It follows that

$$\frac{m}{(m, k)} \mid a_{j_0} - a_{i_0},$$

which is a contradiction. Hence we have

$$\phi(m) = \phi(\frac{m}{(m, k)}).$$

It follows that

$$m = 4t + 2 \ (t = 0, 1, 2, \ldots) \ \text{ and } \ (m, k) = 2. \tag{3}$$

Since for any i with $1 \leq i \leq \phi(m)$, we have that

$$(ka_i + b, m) = 1.$$

From (3), $2|k$ and $2|m$, it follows that $2 \nmid b$. By lemma 2, we have that both $a_1, a_2, \ldots, a_{\phi(m)}$ and $ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b$ are reduced residue systems modulo $\frac{m}{2}$. Since $(k, \frac{m}{2}) = 1$, by corollary 4, we obtain $p_2 \ldots p_s \mid b$.

Sufficiency. If $m = 4t+2$ $(t = 0, 1, 2, \ldots)$, then $\phi(m) = \phi(\frac{m}{2})$. By lemma 2, $a_1, a_2, \ldots, a_{\phi(m)}$ is a reduced residue system modulo $\frac{m}{2}$. Since

$$(k, m) = 2 \text{ and } p_2 \ldots p_s \mid b,$$

by corollary 4,

$$ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b,$$

is also a reduced residue system modulo $\frac{m}{2}$. Then $ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b$ are all incongruent modulo $\frac{m}{2}$, and so they are all incongruent modulo m. Besides, we have

$$(ka_i + b, \frac{m}{2}) = 1$$

for $i = 1, 2, \ldots, \phi(m)$. Since $2 \nmid b$, we have $2 \nmid ka_i + b$, and then $(ka_i + b, m) = 1$ for $i = 1, 2, \ldots, \phi(m)$. Hence $ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b$ is also a reduced residue system modulo m.

By theorem 3 and theorem 5 above, we have proved theorem 1.

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Several identities involving the classical Catalan numbers

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Abstract This paper mainly study some summation problems involving the classical Catalan numbers. By using the elementary method and the properties of the Catalan numbers, we give some interesting identities for it.

Keywords Catalan numbers, elementary method, identities.

§1. Introduction

For any positive integer n, the classical Catalan numbers b_n are defined as follows:

$$b_n = \frac{\binom{2n}{n}}{n+1},\ n = 0,\ 1,\ 2,\ 3, \cdots$$

the first several values of b_n are: b_0=1, b_1=1, b_2=2, b_3=5, b_4=14, b_5=42, b_6=132,$\cdots$. This sequence has some wide applications in combinational mathematics and graph theory, so it had been studied by many people, some related results can be found in references [1] and [2]. In this paper, we shall study the calculating problem of the summation

$$\sum_{a_1+a_2+\cdots+a_k=n} b_{2a_1} b_{2a_2} \cdots b_{2a_k} \tag{1}$$

and

$$\sum_{a_1+a_2+\cdots+a_k=n} b_{2a_1+1} b_{2a_2+1} \cdots b_{2a_k+1}, \tag{2}$$

where $\sum\limits_{a_1+a_2+\cdots+a_k=n}$ denotes the summation over all k-tuples with no-negative integer coordinates $(a_1, a_2, \cdots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$.

We shall use the elementary method to give two exact calculating formulas for (1) and (2). That is, we shall prove the following conclusions:

Theorem 1. For any positive integers n and k with $2 \le k \le n$, if k is an even number,

then we have

$$\begin{aligned}&\sum_{a_1+a_2+\cdots+a_k=n} b_{2a_1}b_{2a_2}\cdots b_{2a_k}\\ =\ & 4^{2n}\sum_{l=0}^{q-1}\sum_{i=0}^{n+q}\sum_{j=0}^{l}(-1)^{i+j+1}\binom{2q}{2l+1}\binom{l}{j}\binom{q-l-1}{2n+k-2i-j}\frac{1}{2^i i!}\cdot c(1,i),\end{aligned}\tag{3}$$

where $q=k/2$, and

$$c(1,i)=\begin{cases}1, & \text{if } i=0;\\ \prod_{s=0}^{i-1}(1-2s), & \text{if } i\geq 1.\end{cases}$$

If k is an odd number, then

$$\begin{aligned}&\sum_{a_1+a_2+\cdots+a_k=n} b_{2a_1}b_{2a_2}\cdots b_{2a_k}\\ =\ & 4^{2n}\sum_{l=0}^{q}\sum_{i=0}^{2n+k}\sum_{j=0}^{l}\binom{l}{j}\binom{q-l}{2n+k-i-j}\\ &\times\left[(-1)^j\binom{2q+1}{2l}+(-1)^{1-i-j}\binom{2q+1}{2l+1}\right]\frac{1}{2^i i!}\cdot c(1,i),\end{aligned}\tag{4}$$

where $q=\dfrac{k-1}{2}$.

Theorem 2. For any positive integers n and k with $2\leq k\leq n$, we have the identity

$$\begin{aligned}&\sum_{a_1+a_2+\cdots+a_k=n} b_{2a_1+1}b_{2a_2+1}\cdots b_{2a_k+1}\\ =\ & \sum_{m=0}^{k}\sum_{l=0}^{m}\sum_{i=0}^{2n+2k}2^{2n-k-m+2}(-1)^{m+i}\binom{k}{m}\binom{m}{l}\binom{2n+2k}{i}\frac{c(l,i)\cdot c(m-l,2n+2k-i)}{(2n+2k)!},\end{aligned}$$

where

$$c(l,i)=\begin{cases}1, & \text{if } i=0;\\ \prod_{s=0}^{i-1}(l-2s), & \text{if } i\geq 1.\end{cases}$$

$$c(m-l,2n+2k-i)=\begin{cases}1, & \text{if } 2n+2k-i=0;\\ \prod_{s=0}^{2n+2k-i-1}(m-l-2s), & \text{if } 2n+2k-i\geq 1.\end{cases}$$

Corollary 1. For $k=2,\ 3$, we have the identities

$$\sum_{a_1+a_2=n} b_{2a_1}b_{2a_2}=\frac{2^{3n}(-1)^n\cdot c(1,n+1)}{(n+1)!}\tag{5}$$

and

$$\sum_{a_1+a_2+a_3=n} b_{2a_1}b_{2a_2}b_{2a_3}=2^{2n}\left[\frac{c(1,2n+3)}{(2n+3)!}-\frac{c(1,2n+2)}{(2n+2)!}\right].\tag{6}$$

Corollary 2. For any integer $n \geq 2$, we have

$$\sum_{a_1+a_2=n} b_{2a_1+1}b_{2a_2+1} = -\frac{2^{2n+1}c(1,2n+4)}{(2n+4)!} + \frac{2^{3n+1}(-1)^n \cdot c(1,n+2)}{(n+2)!}.$$

§2. Proof of the theorem

In this section, we shall use the elementary methods and the properties of the Catalan numbers to prove our theorems directly. First we prove theorem 1 and corollary 1. From the properties of the Catalan numbers we know that

$$2(1-\sqrt{1-x}) = x\sum_{n=0}^{\infty}\frac{b_n x^n}{4^n}.$$

Let

$$f(x) = 2(1-\sqrt{1-x}) = \sum_{n=0}^{\infty}\frac{b_n x^{n+1}}{4^n},$$

then

$$f(-x) = 2(1-\sqrt{1+x}) = \sum_{n=0}^{\infty}\frac{b_n (-x)^{n+1}}{4^n},$$

so

$$f(x)-f(-x) = 2(\sqrt{1+x}-\sqrt{1-x}) = 2x\sum_{n=0}^{\infty}\frac{b_{2n}x^{2n}}{4^{2n}}.$$

Then from the properties of the power series we have

$$(\sqrt{1+x}-\sqrt{1-x})^k = x^k\sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_k=n}\frac{b_{2a_1}b_{2a_2}\cdots b_{2a_k}}{4^{2n}}\right)x^{2n}. \tag{7}$$

On the other hand,

$$(\sqrt{1+x}-\sqrt{1-x})^k = \sum_{m=0}^{k}\binom{k}{m}(\sqrt{1+x})^m(-\sqrt{1-x})^{k-m}.$$

(a) If k is an even number, we discuss m in the following two cases:

(1) If m is an even number, let $m = 2l, k = 2q$, then

$$\sum_{m=0,m=2l}^{k}\binom{k}{m}(\sqrt{1+x})^m(-\sqrt{1-x})^{k-m} = \sum_{l=0}^{q}\binom{2q}{2l}(1+x)^l(1-x)^{q-l},$$

where the maximum number of times of x is q.

(2) If m is an odd number, let $m = 2l+1, k = 2q$, then

$$\begin{aligned}&\sum_{m=0,m=2l+1}^{k}\binom{k}{m}(\sqrt{1+x})^m(-\sqrt{1-x})^{k-m}\\ =\ & -\sqrt{1-x^2}\sum_{l=0}^{q-1}\binom{2q}{2l+1}(1+x)^l(1-x)^{q-l-1}.\end{aligned}$$

Note that the power series expansion of $(1-x^2)^{\frac{1}{2}}$

$$(1-x^2)^{\frac{1}{2}} = \sum_{n=0}^{\infty}(-1)^n \frac{c(1,n)}{2^n n!} x^{2n}. \tag{8}$$

Applying (8) and comparing the coefficients of x^{2n+k} in (7) we may immediately deduce the identity (3).

(b) If k is an odd number, we discuss m in the following two cases:

(1) If m is an even number, let $m = 2l, k = 2q+1$, then

$$\sum_{m=0,m=2l}^{k} \binom{k}{m} (\sqrt{1+x})^m (-\sqrt{1-x})^{k-m} = -\sqrt{1-x} \sum_{l=0}^{q} \binom{2q+1}{2l} (1+x)^l (1-x)^{q-l}.$$

(2) If m is an odd number, let $m = 2l+1, k = 2q+1$, then

$$\sum_{m=0,m=2l+1}^{k} \binom{k}{m} (\sqrt{1+x})^m (-\sqrt{1-x})^{k-m} = \sqrt{1+x} \sum_{l=0}^{q} \binom{2q+1}{2l+1} (1+x)^l (1-x)^{q-l}.$$

Note that the power series expansion of $(1-x)^{\frac{1}{2}}$ and $(1+x)^{\frac{1}{2}}$,

$$(1-x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot c(1,n)}{2^n n!} x^n, \tag{9}$$

$$(1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{c(1,n)}{2^n n!} x^n. \tag{10}$$

Applying (9) and (10) and comparing the coefficients of x^{2n+k} in (7) we may immediately deduce the identity (4).

Taking k=2 in (7), we have

$$4^{2n}(2-2\sqrt{1-x^2}) = \sum_{n=0}^{\infty} \left(\sum_{a_1+a_2=n} b_{2a_1} b_{2a_2} \right) x^{2n+2}. \tag{11}$$

Applying (8) in (11) we can deduce (5).

Taking k=3 in (7), we have

$$4^{2n}[\sqrt{1-x}(-2x-4) + \sqrt{1+x}(4-2x)] = \sum_{n=0}^{\infty} \left(\sum_{a_1+a_2+a_3=n} b_{2a_1} b_{2a_2} b_{2a_3} \right) x^{2n+3}. \tag{12}$$

Applying (9) and (10) in (12) we can deduce (6).

This proves theorem 1 and corollary 1.

Now we prove theorem 2 and corollary 2. It is clear that

$$f(x) + f(-x) = 2(2 - \sqrt{1-x} - \sqrt{1+x}) = 2x^2 \sum_{n=0}^{\infty} \frac{b_{2n+1} x^{2n}}{4^{2n+1}}.$$

Then from the properties of the power series we have

$$(2-\sqrt{1-x}-\sqrt{1+x})^k = x^{2k}\sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_k=n}\frac{b_{2a_1+1}b_{2a_2+1}\cdots b_{2a_k+1}}{4^{2n+1}}\right)x^{2n}. \tag{13}$$

On the other hand,

$$(2-\sqrt{1-x}-\sqrt{1+x})^k = 2^k\sum_{m=0}^{k}\sum_{l=0}^{m}\binom{k}{m}\binom{m}{l}(-1)^m 2^{k-m}(\sqrt{1-x})^l(\sqrt{1+x})^{m-l}.$$

Note that the power series expansion of $(1-x)^{\frac{l}{2}}$ and $(1+x)^{\frac{m-l}{2}}$

$$(1-x)^{\frac{l}{2}} = \sum_{n=0}^{\infty}(-1)^n\frac{c(l,n)}{2^n n!}x^n, \tag{14}$$

$$(1+x)^{\frac{m-l}{2}} = \sum_{n=0}^{\infty}\frac{c(m-l,n)}{2^n n!}x^n, \tag{15}$$

where

$$c(l,n) = \begin{cases} 1, & \text{if } n=0; \\ \prod_{s=0}^{n-1}(l-2s), & \text{if } n\geq 1. \end{cases}$$

Applying (14) and (15) and comparing the coefficients of x^{2n+2k} in (13) we may immediately deduce theorem 2.

Taking k=2 in (13), we have

$$\begin{aligned} & 4^{2n+1}(-4\sqrt{1-x}-4\sqrt{1+x}+2\sqrt{1-x^2}+6) \\ = \; & x^4\sum_{n=0}^{\infty}\left(\sum_{a_1+a_2=n}b_{2a_1+1}b_{2a_2+1}\right)x^{2n}. \end{aligned} \tag{16}$$

Applying (8), (9) and (10) in (16), we can deduce corollary 2.

This completes the proof of theorem 2 and corollary 2.

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A note on ambiguous probabilistic finite state automata

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Abstract In this paper we discuss the probability of strings generated by various forms of probabilistic finite state automata in terms of extended probability transition function. On the same line, the ambiguity of probabilistic deterministic finite state automata is also discussed.

Keywords Probabilistic finite state automata, extended probability transition function, am -biguous probabilistic finite state automata.

§1. Introduction

As in the classical automata theory, there are deterministic and non-deterministic finite state automata deals with probabilistic processes [5]. Over the years, there are papers analysing the parsing issues of strings generated by a probabilistic finite state automata [1,3,8]. Furthermore the general objects of probabilistic finite state automata and study on their definitions and properties have reached the communities [7]. Probabilistic finite state automata have been introduced to describe distributions over strings and are successfully used in several fields, including pattern recognition, computational biology, speech recognition etc. In this paper, we discuss the ambiguity of probabilistic finite state automata through parsing the strings. This basic idea may have larger influence over the general issues related to probabilities of strings generated by various types of probabilistic finite state automata. The presentation of some of the extensions of probabilistic finite state automata is also dealt through extended probability transition function.

§2. Probabilistic finite state automata

Probabilistic finite state automata (PA) are one of the most widespread backgrounds that knowledges to researchers in computer science [2,6]. An overview of definitions for Probabilistic finite state automata is first established for various generative processes. As in formal language theory there is a key difference between deterministic and nondeterministic finite state machines in probabilistic case. Probabilistic finite state automata admit determinism; the parsing challenges of probabilistic deterministic finite state automata [4] are discussed through extended

probability transition function. It should be noted that results dealing with parsing are also been studied [7]. The ambiguity of deterministic and nondeterministic probabilistic finite state automata is studied and some extensions of the same is recalled using extended probability transition function.

Definition 1. A PA is a 5 tuple $A = (Q, \Sigma, \phi, i, \tau)$, where Q is a set of finite states, Σ is the alphabet, ϕ, i, τ are functions such that

$\phi : Q \times \Sigma \times Q \to [0, 1]$. (Transition probability function).

$i : Q \to R^+$. (Initial state probability function).

$\tau : Q \to R^+$ (Final state probability function).

such that $\sum_{q \in Q} i(q) = 1;\ \forall\ q \in Q,\ \tau(q) + \sum_{a \in \Sigma, q' \in Q} \phi(q, a, q') = 1.$

It is assumed that $\phi(q, a, q') = 0$ for all $(q, a, q') \notin \phi$.

Example 1. As in the formal language theory, the PA can also be represented graphically. Figure 1. is a graphical representation of a PA, in which the functions i, τ and ϕ are defined such that

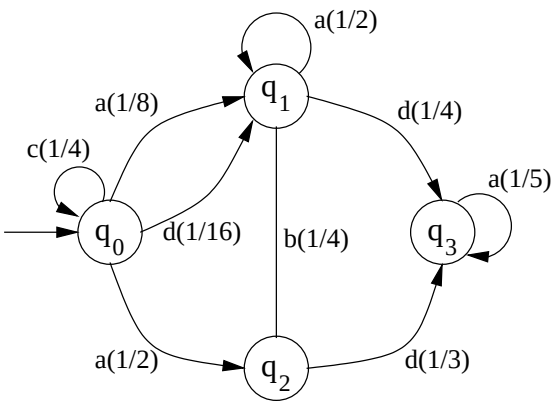

Figure 1.

$i(q_0) = 1,\ i(q) = 0,\ \forall\ q \in Q.$

$\tau(q_0) = 1/16,\ \tau(q_1) = 0,\ \tau(q_2) = 2/3,\ \tau(q_3) = 4/5.$

$\phi(q_0, a, q_1) = \dfrac{1}{8};\ \phi(q_0, c, q_0) = \dfrac{1}{4};\ \ldots$

Definition 2. Let A be a PA and $w = x_1 x_2 \ldots x_n \in \Sigma^*$ and $Q = \{q_0, q_1, \ldots, q_m\}$ with $i(q_0) = 1$. A sequence of the form $q_0 x_1 q_1 x_2 q_2 \ldots q_{n-1} x_n q_n$ such that $\phi(q_i, x_{i+1}, q_{i+1}) > 0$ for all $i = 0, 1, 2, \ldots, n-1$ (not necessary all q_i's are different in the sequence) is called a path of w in PA, and it is denoted by the symbol θ.

A string w is said to be generated by the given PA if there is a path $\theta = q_0 x_1 q_1 x_2 q_2 \ldots q_{n-1} x_n$ q_n such that $\tau(q_n) \neq 0$.

In the above example, for the string $w = aabd$ there is a path $q_0 a q_1 a q_1 b q_2 d q_3$ with $\tau(q_3) = 4/5 \neq 0$. But for $w = ccaaa$, there is a path $\theta = q_0 c q_0 c q_1 a q_1 a q_1 a q_1$ with $\tau(q_1) = 0$. Therefore it is considered that the PA generates the string $aabd$, but not $ccaaa$.

Definition 3. Let $w = xa \in \Sigma^*$ such that $|w| = n$ and $|x| = n - 1$. The function $\hat{\phi}$ defined from $Q \times \Sigma^* \times Q$ to $[0, 1]$ such that $\hat{\phi}(q_0, \epsilon, q_i) = 1$ for all i and $\hat{\phi}(q_0, w, q_n) =$

$\hat{\phi}(q_0, x, q_{n-1})\phi(q_{n-1}, a, q_n)$ is called extended probability transition function.

Definition 4. Let $w = x_1x_2\ldots x_n \in \Sigma^*$ and $\theta = q_0x_1q_1x_2q_2\ldots q_{n-1}x_nq_n$ be a path in a PA A such that $i(q_0) = 1$ and $\tau(q_n) \neq 0$ then we say that A generates w with the probability

$$P_A(w) = i(q_0)\hat{\phi}(q_0, w, q_n)\tau(q_n).$$

In the above example, the string $w = aabd$ is generated by the PA through θ with the probability

$$\begin{aligned} P_A(\theta) &= i(q_0)\phi(q_0, a, q_1)\phi(q_1, a, q_1)\phi(q_1, b, q_2)\phi(q_2, d, q_3)\tau(q_3) \\ &= 1 \times \frac{1}{8} \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{3} \times \frac{1}{5} = \frac{1}{240}. \end{aligned}$$

Theorem 1. If there are two paths $q_0x_1q_1x_2q_2\ldots q_{n-1}x_nq_n$ and $q_0x_1q_1'x_2q_2'\ldots q_{i-1}'x_iq_i'\ldots x_nq_n'$, then

$$P_A(w) = i(q_0)(\hat{\phi}(q_0, x_1, x_2, \ldots x_n, q_n)\tau(q_n) + \phi(q_0, x_1, q_1')(\hat{\phi}(q_1', x_1, x_2, \ldots, x_n, q_n')\tau(q_n')).$$

This result can also be extended to any finite number of paths.

Definition 5. A PA $A = (Q, \Sigma, \phi, i, \tau)$ with $\sum_{i=1}^{s} i(p_i) = 1$, $s > 1$ is called a PA with multiple initial states, if for every such p_i, $i = 1\ldots s$, $\phi(p_i, x, p_j) = 0$ and there exist a $q_i \in Q$ such that $\phi(p_i, x, q_j) > 0$, for some $x \in \Sigma$.

Theorem 2. Let A be a PA and also let $w \in \Sigma^*$ accepted by A through m paths $\theta_{A^1}, \theta_{A^2}, \ldots, \theta_{A^m}$ then

$$P_A(w) = \sum_{i=1}^{m} i(q_{i0})\hat{\phi}(q_{i0}, x_1x_2\ldots x_n, q_{in})\tau(q_{in}).$$

Provided $w = x_1x_2\ldots x_n$,
$\theta_{A^1} = q_{10}x_1q_{11}x_2q_{12}\ldots x_nq_{1n}$,
$\theta_{A^2} = q_{20}x_1q_{21}x_2q_{22}\ldots x_nq_{2n}$,
.....

Suppose if A has only one start state q_0 then $P_A(w) = i(q_0)\sum_{i=1}^{m}\Theta_{A^i}\tau(q_{in})$, where $\Theta_{A^i} = \hat{\phi}(q_{i0}, x_1x_2\ldots x_n, q_{in})$.

Proof. Let $w = x_1x_2\ldots x_n$, $\theta_{A^1} = q_{10}x_1q_{11}x_2q_{12}\ldots x_nq_{1n}$, and $\theta_{A^2} = q_{20}x_1q_{21}x_2q_{22}\ldots x_n q_{2n}$. Then

$$P_A(w) = i(q_{10})\hat{\phi}(q_{10}, w, q_{1n})\tau(q_{1n}) + i(q_{20})\hat{\phi}(q_{20}, w, q_{2n})\tau(q_{2n}). \tag{1}$$

In general, if we denote $\hat{\phi}(q_{r0}, w, q_{rn}) = \Theta_{A^r}$.
Then (1) becomes $P_A(w) = i(q_{10})\Theta_{A^1}\tau(q_{1n}) + i(q_{20})\Theta_{A^2}\tau(q_{2n})$.
Suppose if there are m paths for w,

$$P_A(w) = i(q_{10})\Theta_{A^1}\tau(q_{1n}) + i(q_{20})\Theta_{A^2}\tau(q_{2n}) + \cdots + i(q_{m0})\Theta_{A^m}\tau(q_{mn}).$$

Suppose if there exist $q, q' \in Q$ such that $i(q) = 1$, $i(p) = 0$ and $\tau(q') \neq 0$, $\tau(p) = 0$ for all states p other than q, then $q_{1n} = q_{2n} = \cdots = q_{mn} = q'$ and also

$$P_A(w) = \sum_{i=1}^{m} \Theta_{A^i} \tau(q').$$

Theorem 3. Let $w = x_1 x_2 \ldots x_r^m \ldots x_n \in \Sigma^*$, then

$$P_A(w) = i(q_0)\hat{\phi}(q_0, x_1 x_2 \ldots x_{r-1}, q_{r-1}) m\phi(q_{r-1}, x_r, q_r)\hat{\phi}(q_r, x_{r+1} x_{r+2} \ldots, x_n, q_n)\tau(q_n).$$

In the above theorem if $m \to \infty$, $P_A(w) = 0$.

§3. Ambiguous probabilistic finite state automata

Let w be a string accepted by A through m valid paths. Let $\theta_A(w)$ denote the set of all such m valid paths. Then $P_A(w) = \sum_{i=1}^{m} \Theta_{A^i} \tau(q')$.

If $\sum_w P_A(w) = 1$, then A defines a distribution D on Σ^*. Before discussing conditions for which a PA A defines a probability distribution, we discuss an example in Σ^3. Assume that a coin is tossed 3 times then $\Sigma = \{H, T\}$ and $\Sigma^3 = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$. The PFA A which generates Σ^3 may be given as follows.

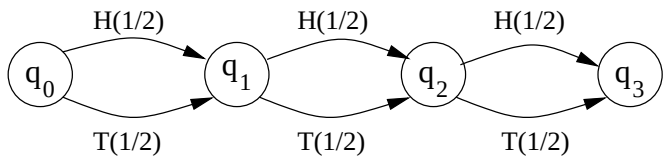

Figure 2.

$$i(q_i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i = 1, 2, 3. \end{cases} \qquad \tau(q_i) = \begin{cases} 1 & \text{if } i = 3 \\ 0 & \text{if } i = 0, 1, 2. \end{cases}$$

Then A defines the probability distribution.

w	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
$P_A(w)$	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8

$$\sum_w P_A(w) = 1.$$

Definition 6. A PA A is said to be ambiguous if there is at least one string which can be generated by A through more than one path. i.e., $|\theta_A(w)| > 1$ for at least one w in an ambiguous PA.

Definition 7. A PA A is called probabilistic deterministic finite state automaton, if

i. there exist $q_0 \in Q$ such that $i(q_0) = 1$.

ii. $\forall\ q \in Q$, $\forall\ a \in \Sigma$, $|\{q' : (q, a, q')\}| \leq 1$.

Theorem 4. No Probabilistic Deterministic finite state automaton (PDFA) is ambiguous.

Proof. Let the PA $A = (Q, \Sigma, \phi, i, \tau)$ be a PDFA. Therefore $\forall\ q \in Q$, $\forall\ a \in \Sigma$, $|\{q' : (q, a, q')\}| \leq 1$. We have to show that $|\theta_A(w)| \leq 1$.

For a string w generated by the PDFA, $\theta_A(w)$ denote the set of all valid paths for w in A.

Suppose assume that $w = x_1x_2 \ldots x_n$ and $q_0x_1q_1x_2q_2 \ldots q_{n-1}x_nq_n$ and $p_0x_1p_1x_2p_2 \ldots x_np_n$ are two valid paths for w in A with $p_0 = q_0$ and $i(p_0) = i(q_0) = 1$, $i(p) = 0$ for all other p.

Consider now the triplets (q_{i-1}, x, q_i) and (p_{i-1}, x_i, p_i). Let q_{i-1} be same as p_{i-1}. Then if q_i and p_i are different, then we have $|\{q' : (q, a, q')\}| = |\{q_i, p_i\}| = 2$. This contradicts the fact that $|\{q' : (q, a, q')\}| \leq 1$. Thus q_i is same as p_i whenever q_{i-1} is same as p_{i-1}.

Hence the paths $q_0x_1q_1x_2q_2 \ldots q_{n-1}x_nq_n$ and $p_0x_1p_1x_2p_2 \ldots x_np_n$ are the same valid paths. Therefore the PDFA is unambiguous.

Theorem 5. Let M be a PA with multiple initial states generating a sub probability space D over the set of finite strings Σ^*, then there is a PA A with single initial state generating the same sub probability space D.

Proof. Let $p_0, p_1, p_2, \ldots, p_k$ are initial states of $M = (Q_M, \Sigma, \phi_M, i_M, \tau_M)$ with initial probabilities $n_1, n_2, \ldots, n_k$ respectively such that $\sum_{i=1}^{k} n_i = 1$. Let $q_1, q_2, \ldots, q_s$ be the other states of M. i.e. Let $Q_M = \{q_1, q_2, \ldots, q_s\} \cup \{p_0, p_1, \ldots, p_k\}$.

Now we define $A = (Q, \Sigma, \phi, i, \tau)$ as follows. $Q = Q_M \cup \{p\}$. We have, in general $P_A(\theta) = i(p_0)\hat{\phi}(p_0, w', q_k)\tau(q_k)$. Where $w' = x'_1x'_2x'_3 \ldots x'_k$.

Suppose if PA is ambiguous, then

$$P_A(\theta) = \sum_{j=1}^{n} i(s_j)\hat{\phi}(s_j, w', s_k)\tau(s_k).$$

We define i such that $i(p) = 1$ and $i(q) = 0$ for all other $q \in Q$ and ϕ includes (p, ϵ, p_i), $i = 1, 2, \ldots, k$ with $\phi_A(p, \epsilon, p_i) = n_i$, $i = 1, 2, \ldots, k$. $P_A(q, a, q') = P_M(q, a, q')$, $\forall\ q, q' \in Q_M$ and define $P_A(\theta) = i(p)\sum_{j=1}^{k} \phi(p, \epsilon, p_{j-1})\hat{\phi}(p_{j-1}, w', q_j)\tau(q_k)$ $\tau_A = \tau_M$ and $\tau_A(p) = 0$.

Then P_A generates exactly the strings generated by P_M with $P_A(\theta) = P_M(\theta)$, $\forall\ \theta \in \theta_A$ and all $\theta \in \theta_M$.

Conclusion

In this paper, we have discussed that the PDFA is not ambiguous. The result will play a significant role in the study of parsing the strings and searching for optimal path for a string in a given PA.

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More about functional Alexandroff topological spaces

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Abstract In the following text we study the product of functional Alexandroff spaces and obtain a theorem on functional Alexandroff topological groups which recognize all functional Alexandroff topologies on a group which made it a topological group, this theorem is parallel to a well-known theorem on Alexandroff topological groups.

Keywords Alexandroff topology, functional Alexandroff topology.

§1. Introduction

A topological space X is called *Alexandroff* if for any $x \in X$ there exists a smallest open neighborhood of x, namely $V(x)$ (in other words the intersection of any nonempty collection of open sets is open). For more details on Alexandroff spaces we refer the interested reader to [2], we should mention here Alexandroff's well-known paper "Diskrete Räume" [1] in this area. If $\alpha : X \to X$ is an arbitrary map then $\{\downarrow^{\alpha} x : x \in X\}$, where $\downarrow^{\alpha} x = \bigcup\{\alpha^{-n}(x) : n \geq 0\}$ $(x \in X)$, is a topological basis on X, the topology generated by this basis is denoted by τ_α and topological space (X, τ_α) is called a *functional Alexandroff* topological space (induced by α), moreover in this case $V(x) = \downarrow^{\alpha} x$ $(x \in X)$. Functional Alexandroff spaces occurs naturally in dynamical systems, for more details on functional Alexandroff spaces we refer the interested reader to [3].

The following remark show the main characters of functional Alexandroff space:

Remark 1.1.[3] (theorem 3.5, main theorem) Let X be an Alexandroff space. X is a functional Alexandroff space if and only if the following statements hold:

$(\mathcal{C}_1)$ $\forall x, y \in X$, $V(x) \subseteq V(y) \vee V(y) \subseteq V(x) \vee V(x) \cap V(y) = \emptyset$;

$(\mathcal{C}_2)$ For $x \in X$ if there exists $y \in X$ with $V(x) \subset V(y)$, then for all $z \in X - \{x\}$ we have $V(x) \neq V(z)$;

$(\mathcal{C}_3)$ For all $x, y \in X$, $\{z \in X : V(x) \subseteq V(z) \subseteq V(y)\}$ is finite.

The following note will be useful:

Note 1.2. Let X be an Alexandroff space. The following statements are equivalent:

- X is a functional Alexandroff space induced by a pointwise periodic function.

- X is a functional Alexandroff space and

$$\forall x, y \in X,\ (V(x) = V(y) \vee V(x) \cap V(y) = \emptyset).$$

- $\forall x, y \in X,\ ((V(x) = V(y) \vee V(x) \cap V(y) = \emptyset) \wedge |V(x)| < \aleph_0)$.
- X is a disjoint union of finite trivial spaces.

In the above case we name X a *pointwise periodic functional Alexandroff space.*

Remark 1.3.[3] (lemma 2.5) Let $\alpha : X \to X$ be an arbitrary map, then (X, τ_α) is:

i. T_0 if and only if α does not have any periodic point except fixed points;

ii. T_1 if and only if $\alpha = \mathrm{id}_x$.

In the following you will see:

- Products of functional Alexandroff topological spaces.
- A theorem on functional Alexandroff topological groups.
- A table on Alexandroff topological groups.

§2. Products of functional Alexandroff topological spaces

It is well known that finite product of Alexandroff spaces are Alexandroff ([2], theorem 2.2), but this is not true for arbitrary products, although with box topology product of Alexandroff spaces are Alexandroff. In this section we will deal with product of functional Alexandroff spaces. The question of this section may be indicated as: If the product of topological spaces is Alexandroff, when will it be functional Alexandroff?

Note 2.1. Let $\Gamma \neq \emptyset$ and for each $t \in \Gamma$, $X_t \neq \emptyset$. The following statements are equivalent:

1. $\prod_{t\in\Gamma} X_t$ is Alexandroff (with product topology).

2. Product and box topologies on $\prod_{t\in\Gamma} X_t$ are the same and for each $t \in \Gamma$, X_t is an Alexandroff space.

3. There exist $t_1, \ldots, t_n \in \Gamma$ such that for each $t \in \Gamma \backslash \{t_1, \ldots, t_n\}$, X_t has trivial topology, moreover for each $t \in \{t_1, \ldots, t_n\}$, X_t is Alexandroff.

Moreover in the above case for all $(x_t)_{t\in\Gamma} \in \prod_{t\in\Gamma} X_t$, we have $V((x_t)_{t\in\Gamma}) = \prod_{t\in\Gamma} V(x_t)$.

The following theorem is the answer of the main question of this section.

Theorem 2.2. Let $\Gamma \neq \emptyset$ and for each $t \in \Gamma$, $|X_t| \geq 2$. The Alexandroff space $\prod_{t\in\Gamma} X_t$ is a functional Alexandroff space if and only if the following statements hold:

1. For all $t \in \Gamma$, X_t is a functional Alexandroff space.

2. One of the following conditions hold:

($\mathcal{S}_1$) There exists $s \in \Gamma$, such that for all $t \in \Gamma - \{s\}$, X_t is discrete.

($\mathcal{S}_2$) For all $t \in \Gamma$, X_t is a poitwise periodic functional Alexandroff space.

3. Γ is finite.

Proof. First let $\prod_{t\in\Gamma} X_t$ be a functional Alexandroff space. For all $t \in \Gamma$, X_t satisfies all three conditions ($\mathcal{C}_1$), ($\mathcal{C}_2$) and ($\mathcal{C}_3$) in remark 1.1, so it is functional Alexandroff. Now we distinguish the following cases:

- First case: There exist $s \in \Gamma$ and $x_s, y_s \in X_s$ such that $V(x_s) \subset V(y_s)$. We claim that for all $t \in \Gamma - \{s\}$, X_t is discrete. Let $r \in \Gamma - \{s\}$ and $x_r \in X_r$ be such that $V(x_r) \neq \{x_r\}$. There exists $y_r \in V(x_r) - \{x_r\}$ and therefore $V(y_r) \subset V(x_r)$ or $V(y_r) = V(x_r)$. If $V(y_r) \subset V(x_r)$ for each $t \in \Gamma - \{s, r\}$ choose $x_t = y_t \in X_t$; Thus $V((x_t)_{t\in\Gamma}) \cap V((y_t)_{t\in\Gamma}) \neq \emptyset$, $V((x_t)_{t\in\Gamma}) \not\subseteq V((y_t)_{t\in\Gamma})$ and $V((y_t)_{t\in\Gamma}) \not\subseteq V((x_t)_{t\in\Gamma})$ which is a contradiction (use $(\mathcal{C}_1)$ in Remark 1.1). Therefore $V(x_r) = V(y_r)$, for all $t \in \Gamma - \{r, s\}$ choose $x_t = y_t \in X_t$, let

$$z_t = \begin{cases} x_t & t \neq r \\ y_t & t = r \end{cases},$$

  we have $V((z_t)_{t\in\Gamma}) = V((x_t)_{t\in\Gamma}) \subset V((y_t)_{t\in\Gamma})$ and $(z_t)_{t\in\Gamma} \neq (x_t)_{t\in\Gamma}$ which is a contradiction according to $(\mathcal{C}_2)$ in the main theorem. The above discussion leads us to the fact that for each $x_r \in X_r$, $V(x_r) = \{x_r\}$ and X_r is discrete. In this case $(\mathcal{S}_1)$ is valid.

- Second case: For all $s \in \Gamma$ and for all $x_s, y_s \in X_s$ we have $V(x_s) = V(y_s)$ or $V(x_s) \cap V(y_s) = \emptyset$. By note 1.2, X_t is a pointwise periodic functional Alexandroff (for all $t \in \Gamma$). Let $\Lambda = \{t \in \Gamma : X_t \text{ is not discrete}\}$. For each $t \in \Lambda$ there exist $x_t, y_t \in X_t$ such that $x_t \neq y_t$ and $V(x_t) = V(y_t)$, for all $t \in \Gamma \backslash \Lambda$ choose $x_t = y_t \in X_t$. Set $B = \{(z_t)_{t\in\Gamma} \in \prod_{t\in\Gamma} X_t : V((z_t)_{t\in\Gamma}) = V((x_t)_{t\in\Gamma})\}$. Since $B \supseteq \{(z_t)_{t\in\Gamma} : z_t \in \{x_t, y_t\}\}$, thus $|B| \geq 2^{|\Lambda|}$, by $(\mathcal{C}_3)$ in the main theorem, B is finite, therefore Λ is finite. In this case $(\mathcal{S}_2)$ is valid.

Using the above cases $\{t \in \Gamma : X_t \text{ is not discrete}\}$ is finite, since by note 2.1, box and product topologies on $\prod_{t\in\Gamma} X_t$ are same thus Γ is finite.

Conversely let $\prod_{t\in\Gamma} X_t$ be an Alexandroff space, which satisfies (1), (2) and (3). Consider $\Gamma = \{t_1, \ldots, t_n\}$ with distinct t_is. First suppose $(\mathcal{S}_1)$ hold, X_{t_1} be a functional Alexandroff space induced by $f_1 : X_{t_1} \to X_{t_1}$ and $X_{t_2}, \ldots, X_{t_n}$ be discrete, for $i \in \{2, \ldots, n\}$ set $f_i := \mathrm{id}_{X_i}$, for $f = (f_1, f_2, \ldots, f_n)$ the functional Alexandroff topology τ_f on $X_{t_1} \times \cdots \times X_{t_n}$ is the same as the original topology on $X_{t_1} \times \cdots \times X_{t_n}$, and $X_{t_1} \times \cdots \times X_{t_n}$ is a functional Alexandroff space. Now suppose $(\mathcal{S}_2)$ hold, by note 1.2, $X_{t_1} \times \cdots \times X_{t_n}$ is a functional Alexandroff space.

The following theorem closes this section by answering an other question arisen from theorem 2.2: Let for each $t \in \Gamma$, X_t be a functional Alexandroff space induced by map f_t, when $\tau_{\prod_{t\in\Gamma} f_t}$ and product topology on $\prod_{t\in\Gamma} X_t$ are the same?

Theorem 2.3. Let $\Gamma \neq \emptyset$ and for each $t \in \Gamma$, $|X_t| \geq 2$ and X_t be a functional Alexandroff space induced by map $f_t : X_t \to X_t$. The following statements are equivalent:

1. $\tau_{\prod_{\alpha\in\Gamma} f_\alpha}$ and product topology on $\prod_{\alpha\in\Gamma} X_\alpha$ are the same.
2. $\tau_{\prod_{\alpha\in\Gamma} f_\alpha}$ and box topology on $\prod_{\alpha\in\Gamma} X_\alpha$ are the same.
3. Γ is finite and one of the following statements hold:

a. There exists $s \in \Gamma$ such that for all $t \in \Gamma - \{s\}$, $f_t = \mathrm{id}_{X_t}$.

b. For all $t \in \Gamma$, X_t is a pointwise periodic functional Alexandroff space and for all distinct $s, t \in \Gamma$ we have $\gcd(|V(x_t)|, |V(x_s)|) = 1$ $(\forall x_t \in X_t, \forall x_s \in X_s)$.

Proof. By theorem 2.2, (1) and (2) are equivalent. Now suppose (1) and (2) hold. By theorem 2.2, Γ is finite. Moreover by theorem 2.2, $(\mathcal{S}_1)$ or $(\mathcal{S}_2)$, indicated in theorem 2.2, hold. If $(\mathcal{S}_1)$ holds, then there exists $s \in \Gamma$ such that for all $t \in \Gamma - \{s\}$, X_t is discrete, i.e., for all $t \in \Gamma - \{s\}$, $f_t = \mathrm{id}_{X_t}$, by remark 1.3 and (a) holds. If $(\mathcal{S}_2)$ holds, then for all $t \in \Gamma$, X_t is a poitwise periodic functional Alexandroff space, in addition if $\Gamma = \{t_1, \ldots, t_n\}$ with distinct t_is and $(x_1, \ldots, x_n) \in X_{t_1} \times \cdots \times X_{t_n}$ then period of $(x_1, \ldots, x_n)$ under $(f_{t_1}, \ldots, f_{t_n})$ equals to $|V(x_1, \ldots, x_n)| = |V(x_1) \times \cdots \times V(x_n)| = |V(x_1)| \cdots |V(x_n)|$ =(period of x_1 under f_{t_1})$\cdots$ (period of x_n under f_{t_n}); Which leads to gcd((period of x_i under f_{t_i}), (period of x_j under f_{t_j})) $= 1$ for $i \neq j$, i.e., $\gcd(|V(x_i)|, |V(x_j)|) = 1$ (for $i \neq j$) and (b) holds.

Conversely if (3) holds, then by theorem 2.2, $\prod_{t \in \Gamma} X_t$ is a functional Alexandroff space. Clearly (a) leads to (2), suppose (b) and let $\Gamma = \{t_1, \ldots, t_n\}$ with distinct t_is and $(x_1, \ldots, x_n) \in X_{t_1} \times \cdots \times X_{t_n}$. By $\gcd(|V(x_i)|, |V(x_j)|) = 1$ $(1 \leq i < j \leq n)$ we have $|\downarrow^{(f_{t_1}, \ldots, f_{t_n})} (x_1, \ldots, x_n)| = |V(x_1) \times \cdots \times V(x_n)| (< \aleph_0)$. Since we have $\downarrow^{(f_{t_1}, \ldots, f_{t_n})} (x_1, \ldots, x_n) \subseteq V(x_1) \times \cdots \times V(x_n)$, thus $\downarrow^{(f_{t_1}, \ldots, f_{t_n})} (x_1, \ldots, x_n) = V(x_1) \times \cdots \times V(x_n)$, which leads to (2), since $V(x_1) \times \cdots \times V(x_n)$ is the smallest open neighborhood of $(x_1, \ldots, x_n)$ in $X_{t_1} \times \cdots X_{t_n}$ in box topology and $\downarrow^{(f_{t_1}, \ldots, f_{t_n})} (x_1, \ldots, x_n)$ is the smallest open neighborhood of $(x_1, \ldots, x_n)$ in $\tau_{(f_{t_1}, \ldots, f_{t_n})}$.

§3. A theorem on functional Alexandroff topological gro -ups

It is well known that there exists a one to one correspondence between Alexandroff topologies on group G which made G a topological group and its normal subgroups ([4], theorem 4), moreover if for each normal subgroup N of G, T_N denotes the Alexandroff topology on G obtained from N, then T_N is the topology with minimal basis $\{gN : g \in G\}$. We have the following parallel theorem:

Theorem 3.1. There exists a one to one correspondence between functional Alexandroff topologies on group G which made G a topological group and its finite normal subgroups. Moreover in this case G is a pointwise periodic functional Alexandroff space.

Proof. Let the topological group G be a functional Alexandroff space. Since in any topol -ogical group, the intersection of all open neighborhoods of e is a normal subgroup, thus $V(e)$ is a normal subgroup of G such that $V(g) = gV(e)$ $(g \in G)$, moreover for each $g \in V(e)$ we have $V(g) = V(e)$. By remark 1.1, $\{g \in G : V(g) = V(e)\}$ is finite, therefore $V(e)$ is finite. Since $V(e)$ is finite and for all $g \in G$ we have $|V(g)| = |V(e)|$, thus G is a pointwise periodic functional Alexandroff space, by note 1.2.

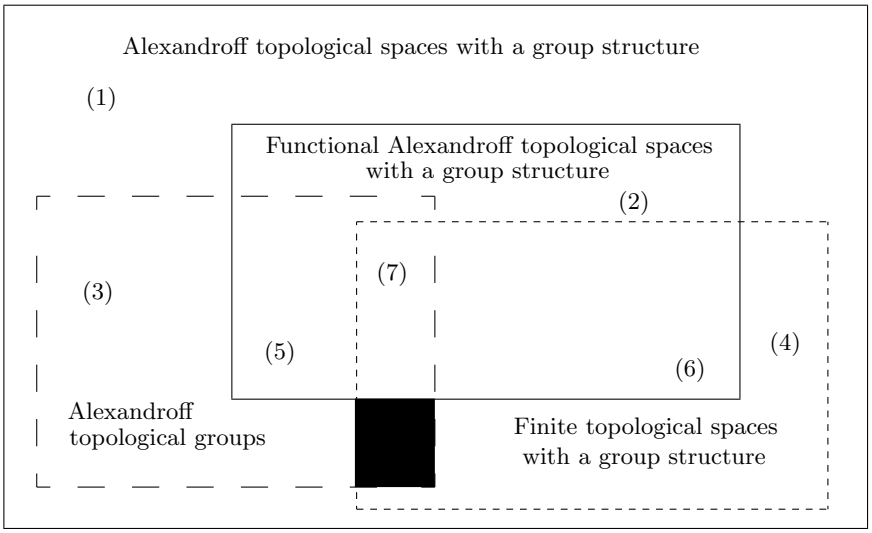
In fact $\varphi : \{(G, \tau) : (G, \tau)$ is a functional Alexandroff space and a topological group$\} \to \{N : N$ is a finite normal subgroup of $G\}$ with $\varphi(G, \tau) = V(e)$, is a bijection.

Corollary 3.2. Let G be a torsion free topological group. G is functional Alexandroff if

and only if G is discrete (since in this case G does not have any finite normal subgroup but $\{e\}$).

§4. A table on Alexandroff topological groups

Using theorem 3.1, we find out that any finite Alexandroff topological group is functional Alexandroff. The following table shows interaction between topological spaces with a group structure which are Alexandroff. Between subcategories of Alexandroff spaces with a group structure categories of finite topological spaces (with a group structure), functional Alexandroff topological spaces (with a group structure), and Alexandroff topological groups has been considered:



In the above table number (i) indicated in each area states that there exists a counterexample belongs to that area in the following counterexamples in item (i). In the following counterexamples $\mathbb{Z}$ and $\mathbb{Z}_3$ are considered under usual addition, with topological basis $\{V(x) : x \in G\}$.

1. $G = \mathbb{Z}$, $V(0) = \{0\}$, $V(n) = \{m : m \geq |n| \vee m \leq -|n|\} \cup \{0\}$ $(n \in \mathbb{Z} - \{0\})$.

2. $G = \mathbb{Z}$, $V(n) = \{m \in \mathbb{Z} : m < n\} \cup \{n\}$ $(n \in \mathbb{Z})$.

3. $G = \mathbb{Z}$, $V(n) = n + 3\mathbb{Z}$ $(n \in \mathbb{Z})$.

4. $G = \mathbb{Z}_3$, $V(1) = V(2) = \{1, 2\}$, $V(0) = \mathbb{Z}_3$.

5. $G = \mathbb{Z}$, $V(n) = \{n\}$ $(n \in \mathbb{Z})$.

6. $G = \mathbb{Z}_3$, $V(0) = \{0\}$, $V(1) = \{0, 1\}$, $V(2) = \mathbb{Z}_3$.

7. $G = \mathbb{Z}_3$, $V(n) = \{n\}$ $(n \in \mathbb{Z}_3)$.

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Weak consistency of wavelet estimator for ρ-mixing errors[1]

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Abstract This paper is concerned with the estimation of a semi-parametric regression model that is frequently used in statistical modeling. The wavelet technique is developed to estimate the unknown parameter and non-parameter. The weak consistencies of the estimators are obtained under the ρ-mixing condition.

Keywords Semi-parametric regression model, wavelet estimator, ρ-mixing, consistency.

§1. Introduction

Consider a semi-parametric regression model

$$y_i = x_i\beta + g(t_i) + e_i, i = 1, 2, \cdots, n \tag{1}$$

where $x_i \in R^1$, $t_i \in [0, 1]$, $\{(x_i, t_i), 1 \le i \le n\}$ is deterministic design point sequence, β is an unknown regression parameter, $g(\cdot)$ is an unknown Borel function; The unobserved process $\{e_i, 1 \le i \le n\}$ is a stationary ρ-mixing sequence and satisfies $Ee_i = 0,\ i = 1, 2, \cdots, n$.

In recent years, the parametric or non-parametric estimators in the semi-parametric or non-parametric regression model have been widely studied in the literature when errors are a stationary mixing sequence (example [1]-[4]). But up to now, the discussion of the wavelet estimators in the semi-parametric regression model, whose errors are a stationary ρ-mixing sequence, has been scarcely seen.

Wavelets techniques, due to their ability to adapt to local features of curves, have recently received much attention from mathematicians, engineers and statisticians. Many authors have applied wavelet procedures to estimate nonparametric and semi-parametric models ([5]-[8]).

In this article, we establish weak consistencies of the wavelet estimators in the semi-parametric regression model with ρ-mixing errors, which enrich existing estimation theories and methods for semi-parametric regression models.

Suppose that $\varphi(\cdot)$ is given scaling function in Schwartz space with order q. A multiresolution analysis of $L^2(R)$ consists of an increasing sequence of closed subspace $\{V_m\}$, where $L^2(R)$

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is the set of square integral functions over real line. The associated integral kernel of V_m is given by

$$E_m(t,s) = 2^m E_0(2^m t, 2^m s) = 2^m \sum_{k\in Z} \varphi(2^m t - k)\varphi(2^m s - k).$$

When β is known, we define the estimator of $g(\cdot)$

$$\hat{g}_0(t,\beta) = \sum_{i=1}^{n} (y_i - x_i\beta) \int_{A_i} E_m(t,s)ds,$$

where $A_i = [s_{i-1}, s_i]$ is a partition of interval $[\,0,1]$ with $t_i \in A_i,\ 1 \le i \le n$. Then, we minimize the following equation $\min\limits_{\beta\in R} \sum\limits_{i=1}^{n} (y_i - x_i\beta - \hat{g}_0(t,\beta))^2$. Let its resolution be $\hat{\beta}_n$, we have

$$\hat{\beta}_n = \tilde{S}_n^{-2} \sum_{i=1}^{n} \tilde{x}_i \tilde{y}_i, \tag{2}$$

where $\tilde{x}_i = x_i - \sum\limits_{j=1}^{n} x_j \int_{A_j} E_m(t_i,s)ds$, $\tilde{y}_i = y_i - \sum\limits_{j=1}^{n} y_j \int_{A_j} E_m(t_i,s)ds$, $\tilde{S}_n^2 = \sum\limits_{i=1}^{n} \tilde{x}_i^2$.

Finally, we can define the estimator of $g(\cdot)$

$$\hat{g}(t) \triangleq \hat{g}_0(t,\hat{\beta}_n) = \sum_{i=1}^{n} (y_i - x_i\hat{\beta}_n) \int_{A_i} E_m(t,s)ds. \tag{3}$$

§2. Assumption and lemmas

In what follows, we will use C and $C_i,\ i \ge 1$ to denote positive constants whose values are unimportant and may vary. We begin the following assumptions required to derive the large sample property of the estimators in section 1.

§2.1. Basic assumption

(A1) $g(\cdot)$ satisfies the Lipschitz condition of order γ and $g(\cdot) \in H^v (v > 1/2)$;

(A2) φ has compactly supported set and is a q-regular function;

(A3) φ satisfies $|\hat{\varphi}(\xi) - 1| = O(\xi)$ as $\xi \to \infty$, where $\hat{\varphi}$ denotes the Fourier transformation of φ;

(A4) $\max\limits_{1\le i\le n} (s_i - s_{i-1}) = O(n^{-1}),\ \max\limits_{1\le i\le n} |\tilde{x}_i| = O(2^m)$;

(A5) $C_1 \le \tilde{S}_n^2/n \le C_2$, where n is large enough;

(A6) $\left|\sum\limits_{i=1}^{n} x_i \int_{A_i} E_m(t,s)ds\right| = \lambda,\ t \in [\,0,1]$, where λ is a constant that depends only on t.

The above conditions are mild and easily satisfied. In order to prove the main results, we present the following several lemmas.

§2.2. Lemmas

Lemma 1.[8] Let $\varphi(\cdot) \in S_q$. Under basic assumptions of (A1)-(A3), if for each integer $k \ge 1,\ \exists c_k > 0$, such that

$$(i)\,|E_0(t,s)| \le \frac{c_k}{1+|t-s|^k},\ |E_m(t,s)| \le \frac{2^m c_k}{1+2^m\,|t-s|^k};\qquad (ii)\sup_{t,m}\int_0^1 |E_m(t,s)|ds \le C.$$

Lemma 2.[3] Under basic assumption (A1)-(A4), we have

$$(i)\left|\int_{A_i} E_m(t,s)ds\right| = O(\frac{2^m}{n});\quad (ii)\sum_i^n (\int_{A_i} E_m(t,s)ds)^2 = O(\frac{2^m}{n}).$$

Lemma 3.[3] Under basic assumption (A1)-(A5), we have

$$\sup_t\left|\sum_{i=1}^n g(t_i)\int_{A_i} E_m(t,s)ds - g(t)\right| = O(n^{-\gamma}) + O(\tau_m),\ n\to\infty,$$

where $\tau_m = \begin{cases} (2^m)^{-\alpha+1/2}, & (1/2<\alpha<3/2),\\ \sqrt{m}/2^m, & (\alpha=3/2),\\ 1/2^m, & (\alpha>3/2).\end{cases}$

§3. Main results and proofs

Now we establish the large sample property of the estimators described in section 1.

Theorem 1. Under basic assumption (A1)-(A6), and $\sum\limits_{i=1}^{\infty}\rho(i)<\infty, Ee_i=0,\ Ee_i^2=\sigma^2$, $\dfrac{2^m}{n}\to 0$, we have

$$\hat\beta_n - \beta = o_p(1), \tag{4}$$

$$\sup_t |\hat g(t) - g(t)| = o_p(1). \tag{5}$$

Proof. It is obvious that

$$\begin{aligned}\hat\beta_n - \beta &= \tilde S_n^{-2}\sum_{i=1}^n \tilde x_i\tilde e_i + \sum_{i=1}^n \tilde x_i\tilde g_i\\ &= \tilde S_n^{-2}[\sum_{i=1}^n \tilde x_i e_i - \sum_{i=1}^n \tilde x_i\sum_{j=1}^n e_j\int_{A_j} E_m(t_i,s)ds + \sum_{i=1}^n \tilde x_i\tilde g_i]\\ &\triangleq B_{1n}+B_{2n}+B_{3n}.\end{aligned} \tag{6}$$

where $\tilde e_i = e_i - \sum\limits_{j=1}^n e_j\int_{A_j} E_m(t_i,s)ds,\ \tilde g_i = g_i - \sum\limits_{j=1}^n g_j\int_{A_j} E_m(t_i,s)ds.$

Since

$$\begin{aligned}
EB_{1n}^2 &= \sum_{i=1}^{n} (\tilde{S}_n^{-2}\tilde{x}_i e_i)^2 + 2\sum_{1\le i<j\le n} E[(\tilde{S}_n^{-2}\tilde{x}_i e_i)\cdot(\tilde{S}_n^{-2}\tilde{x}_j e_j)] \\
&\le \sigma^2 \sum_{i=1}^{n} (\tilde{S}_n^{-2}\tilde{x}_i)^2 + 4\sum_{i<j}\rho(j-i)E[(\tilde{S}_n^{-2}\tilde{x}_i e_i)^2]^{\frac{1}{2}}\cdot E[(\tilde{S}_n^{-2}\tilde{x}_j e_j)^2]^{\frac{1}{2}} \\
&\le \sigma^2 \sum_{i=1}^{n} (\tilde{S}_n^{-2}\tilde{x}_i)^2 + 2\sigma^2 \sum_{k=1}^{n-1}\rho(k)\sum_{i=1}^{n-k}[(\tilde{S}_n^{-2}\tilde{x}_i)^2 + (\tilde{S}_n^{-2}\tilde{x}_{k+i})^2] \\
&\le \sigma^2 \sum_{i=1}^{n} (\tilde{S}_n^{-2}\tilde{x}_i)^2 + 2\sigma^2 \sum_{k=1}^{n-1}\rho(k)\cdot 2\sum_{i=1}^{n}(\tilde{S}_n^{-2}\tilde{x}_i)^2 \\
&= [1+4\sum_{i=1}^{n}\rho(i)]\cdot\sigma^2\sum_{i=1}^{n}(\tilde{S}_n^{-2}\tilde{x}_i)^2 \le C\cdot\sum_{i=1}^{n}(\tilde{S}_n^{-2}\tilde{x}_i)^2 \\
&\le C\cdot\max_i(\tilde{S}_n^{-2}\tilde{x}_i)\cdot\sum_{i=1}^{n}(n^{-1}\tilde{x}_i)\cdot(n\tilde{S}_n^{-2}) \le C\cdot\frac{2^m}{n}\to 0.
\end{aligned}$$

Hence, by Chebyshev inequality, we have for $\forall\varepsilon>0, P(|B_{1n}|\ge\varepsilon)\le\frac{EB_{1n}^2}{\varepsilon^2}\to 0$. Thus

$$B_{1n} = o_p(1). \tag{7}$$

Similarly to the proof of (7), and note that

$$\sum_{i=1}^{n}\tilde{S}_n^{-2}\tilde{x}_i\int_{A_j}E_m(t_i,s)ds \le \sum_{i=1}^{n}\tilde{S}_n^{-2}\tilde{x}_i\cdot\max_i\int_{A_j}E_m(t_i,s)ds \le C\cdot\max_i\int_{A_j}E_m(t_i,s)ds.$$

We obtain

$$B_{2n} = o_p(1). \tag{8}$$

We terminate the proof of (4) with (6), (7), (8) and

$$|B_{3n}| \le \tilde{S}_n^{-2}\sum_{i=1}^{n}|\tilde{x}_i|\cdot\max_{1\le i\le n}|\tilde{g}_i| = O(n^{-\gamma}) + O(\tau_m).$$

Now, consider (5). It is easy to see the following decompositions

$$\begin{aligned}
\sup_t|\hat{g}(t)-g(t)| &\le \sup_t|\hat{g}_0(t,\beta)-g(t)| + \sup_t\left|\sum_{i=1}^{n}x_i(\beta-\hat{\beta}_n)\textstyle\int_{A_i}E_m(t,s)ds\right| \\
&\le \sup_t\left|\sum_{i=1}^{n}g(t_i)\textstyle\int_{A_i}E_m(t,s)ds - g(t)\right| + \sup_t\left|\sum_{i=1}^{n}e_i\textstyle\int_{A_i}E_m(t,s)ds\right| \\
&\quad + \left|\beta-\hat{\beta}_n\right|\cdot\sup_t\left|\sum_{i=1}^{n}x_i\textstyle\int_{A_i}E_m(t,s)ds\right| \\
&\triangleq T_1+T_2+T_3.
\end{aligned} \tag{9}$$

By lemma 3, we have

$$T_1 = O(n^{-\gamma}) + O(\tau_m). \tag{10}$$

The proof for $T_2 = o_p(1)$ is quite similar to that of (7) and hence it is omitted for the sake of brevity. And because of assumption (A6), we have $\sum_{i=1}^{n}x_i\int_{A_i}E_m(t,s)ds = O_p(1)$. Hence, following from (4), we have

$$T_3 = o_p(1). \tag{11}$$

Thus, we have shown (5).

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Involute-evolute curves in Galilean space G_3

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Abstract In this paper, definition of involute-evolute curve couple in Galilean space is given and some well-known theorems for the involute-evolute curves are obtained in 3-dimensional Galilean space.

Keywords Galilean space, involute-evolute curve.

§1. Introduction and preliminaries

C. Boyer discovered involutes while trying to build a more accurate clock [1]. Later, H. H. Hacısalihoḡlu [2] gave the relations Frenet apparatus of involute-evolute curve couple in the space $\mathbb{E}^3$. A. Turgut [3] examined involute-evolute curve couple in $\mathbb{R}^n$. At the beginning of the twentieth century, Cayley-Klein discussed Galilean geometry which is one of the nine geometries of projective space. After that, the studies with related to the curvature theory were maintained [4,5,6] and A. O. Öḡrenmiş [7] and et al. studied the properties of the curves in the Galilean space were studied. In this paper, we define involute-evolute curves couple and give some theorems and conclusions, which are known from the classical differential geometry, in the three dimensional Galilean space G_3. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

The Galilean space G_3 is a Cayley-Klein space equipped with the projective metric of signature (0,0,+,+), as in E. Molnar's paper [8]. The absolute figure of the Galilean Geometry consist of an ordered triple $\{w, f, I\}$, where w is the ideal (absolute) plane, f is the line (absolute line) in w and I is the fixed elliptic involution of points of f [6]. In the non-homogeneous coordinates the similarity group H_8 has the form

$$\begin{aligned} \overline{x} &= a_{11} + a_{12}x, \\ \overline{y} &= a_{21} + a_{22}x + a_{23}y\cos\varphi + a_{23}z\sin\varphi, \\ \overline{z} &= a_{31} + a_{32}x - a_{23}y\sin\varphi + a_{23}z\cos\varphi. \end{aligned} \tag{1}$$

Where a_{ij} and φ are real numbers [5].

In what follows the coefficients a_{12} and a_{23} will play the special role. In particular, for $a_{12} = a_{23} = 1$, (1) defines the group $B_6 \subset H_8$ of isometries of Galilean space G_3.

In G_3 there are four classes of lines:

i) (proper) non-isotropic lines- they don't meet the absolute line f.

ii) (proper) isotropic lines- lines that don't belong to the plane w but meet the absolute line f.

iii) unproper non-isotropic lines-all lines of w but f.

iv) the absolute line f.

Planes x =constant are Euclidean and so is the plane w. Other planes are isotropic.

If a curve C of the class C^r $(r \geq 3)$ is given by the parametrization

$$r = r(x, y(x), z(x)) \tag{2}$$

then x is a Galilean invariant the arc length on C.

The curvature is

$$\kappa = \sqrt{y''(x)^2 + z''(x)^2} \tag{3}$$

and torsion is

$$\tau = \tfrac{1}{\kappa^2} \det(r'(x), r''(x), r'''(x)). \tag{4}$$

The orthonormal trihedron is defined

$$\begin{aligned} T(s) &= \alpha'(s) = (1, y'(s), z'(s)) \\ N(s) &= \tfrac{1}{\kappa(s)}(0, y''(s), z''(s)) \\ B(s) &= \tfrac{1}{\kappa(s)}(0, -z''(s), y''(s)). \end{aligned} \tag{5}$$

The vectors T, N, B are called the vectors of tangent, principal normal and binormal line of α, respectively. For their derivatives the following Frenet formulas hold [9]

$$\begin{aligned} T'(s) &= \kappa(s)N(s) \\ N'(s) &= \tau(s)B(s) \\ B'(s) &= -\tau(s)N(s). \end{aligned} \tag{6}$$

Galilean scalar product can be written as

$$\langle u_1, u_2 \rangle = \begin{cases} x_1 x_2 \ , & if \ \ x_1 \neq 0 \ \vee \ x_2 \neq 0, \\ y_1 y_2 + z_1 z_2 \ , & if \ \ x_1 = 0 \ \wedge \ x_2 = 0. \end{cases} \tag{7}$$

Where $u_1 = (x_1, y_1, z_1)$ and $u_2 = (x_2, y_2, z_2)$. It leaves invariant the Galilean norm of the vector $u = (x, y, z)$ defined by

$$\|u\| = \begin{cases} x \ , & x \neq 0, \\ \sqrt{y^2 + z^2} \ , & x = 0. \end{cases} \tag{8}$$

§2. Involute-evolute curves in Galilean space

In this section, we give a definition of involute-evolute curve and obtain some theorems about these curves in G_3.

Definition 2.1. Let α and α^* be two curves in the Galilean space G_3. The curve α^* is called involute of the curve α if the tangent vector of the curve α at the point $\alpha(s)$ passes through the tangent vector of the curve α^* at the point $\alpha^*(s)$ and

$$\langle \ T, T^* \rangle = 0,$$

where $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ are Frenet frames of α and α^*, respectively. Also, the curve α is called the evolute of the curve α^*.

This definition suffices to the define this curve mate as (see Figure 1.)

$$\alpha^* = \alpha + \lambda T.$$

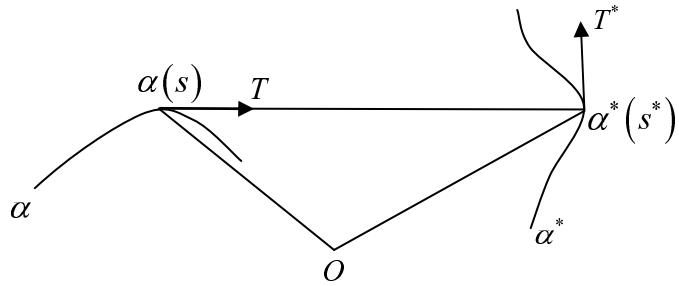

Figure 1. Involute-evolute curves

Theorem 2.2. Let α and α^* be two curves in the Galilean space G_3. If the curve α^* is an involute of the curve α, then the distance between the curves α and α^* is $\lambda(s)$, where $\lambda(s) = |c - s|$.

Proof. From definition of involute-evolute curve couple, we know

$$\alpha^*(s) = \alpha(s) + \lambda(s)T(s). \tag{9}$$

Differentiating both sides of the equation (9) with respect to s and use the Frenet formulas, we obtain

$$T^*(s) = T(s) + \frac{d\lambda}{ds}T(s) + \lambda(s)\kappa(s)N(s).$$

Since the curve α^* is involute of α, $\langle T, T^* \rangle = 0$.

Then we have

$$\frac{d\lambda}{ds} + 1 = 0. \tag{10}$$

From the last equation, we easily get

$$\lambda(s) = c - s. \tag{11}$$

Where c is constant. Thus, the equation (9) can be written as

$$\alpha^*(s) - \alpha(s) = (c - s)T(s). \tag{12}$$

Taking the norm of the equation (12), we reach

$$\|\alpha^*(s) - \alpha(s)\| = |c - s|. \tag{13}$$

This completes the proof.

Theorem 2.3. Let α and α^* be two curves in Galilean space G_3. κ, τ and κ^*, τ^* be the curvature functions of α and α^*, respectively. If α is evolute of α^* then there is a relationship

$$\kappa^* = \frac{\tau}{(c - s)\kappa},$$

where c is constant and s is arc length parameter of α.

Proof. Let Frenet frames be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ at the points $\alpha(s)$ and $\alpha^*(s)$, respectively. Differentiating both sides of equation (12) with respect to s and using Frenet formulas, we have following equation

$$T^*(s)\frac{ds^*}{ds} = (c-s)\kappa(s)N(s), \tag{14}$$

where s and s^* are the arc length parameter of the curves α and α^*, respectively. Taking the norm of the equation (14), we reach

$$\frac{ds^*}{ds} = (c-s)\kappa(s) \tag{15}$$

and

$$T^* = N. \tag{16}$$

By taking the derivative of equation (16) and using the Frenet formulas and equation (15), we obtain

$$\kappa^* N^* = \frac{\tau}{(c-s)\kappa}B. \tag{17}$$

From the last equation, we get

$$\kappa^* = \frac{\tau}{(c-s)\kappa}.$$

Theorem 2.4. Let α be the non-planar evolute of curve α^*, then α is a helix.

Proof. Under assumption s and s^* are arc length parameter of the curves α and α^*, respectively. We take the derivative of the following equation with respect to s

$$\alpha^*(s) = \alpha(s) + \lambda(s)T(s).$$

We obtain that

$$T^*\frac{ds^*}{ds} = \lambda\kappa N.$$

$\{T^*, N\}$ are linearly dependent. We may define function as

$$f = \langle\, T, T^* \wedge N^* \rangle$$

and take the derivative of the function f with respect to s, we obtain

$$f' = -\tau \langle\, T, N^* \rangle\,. \tag{18}$$

From the equation (18) and the scalar product in Galilean space, we have

$$f' = 0.$$

That is,

$$f = const.$$

The velocity vector of the curve α always composes a constant angle with the normal of the plane which consist of α^*. Then the non-planar evolute of the curve α^* is a helix.

Theorem 2.5. Let the curves β and γ be two evolutes of α in the Galilean space G_3. If the points P_1 and P_2 correspond to the point P of α, then the angle $P_1\hat{P}P_2$ is constant.

Proof . Let's assume that the curves β and γ be two evolutes of α (see Figure 2.). And let the Frenet vectors of the curves α, β and γ be $\{T, N, B\}$, $\{T^*, N^*, B^*\}$ and $\left\{\tilde{T}, \tilde{N}, \tilde{B}\right\}$, respectively.

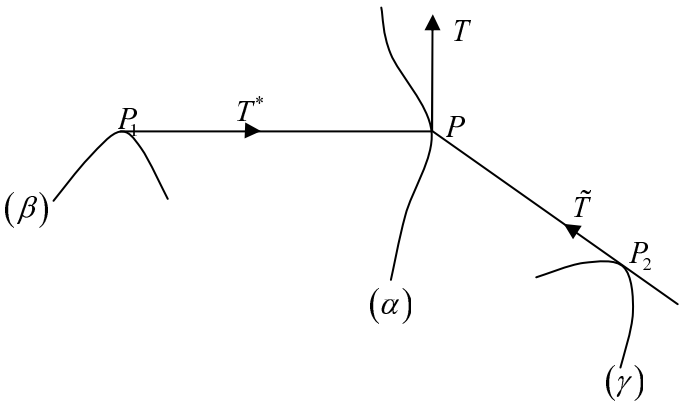

Figure 2. Evolutes of α curve

Following the same way in the proof of the theorem 2.4, it is easily seen that $\{T, N^*\}$ and $\left\{T, \tilde{N}\right\}$ are linearly dependent.

Thus,

$$\langle\, T, T^* \rangle = 0 \tag{19}$$

and

$$\left\langle T, \tilde{T} \right\rangle = 0. \tag{20}$$

When θ is an angle between tangent vector T^* and $\tilde{T}$, we define a function f. That is,

$$f(s) = \left\langle T^*, \tilde{T} \right\rangle. \tag{21}$$

Then, differentiating equation (21) with respect to s, we have

$$f'(s) = \kappa^* \frac{ds^*}{ds} \left\langle N^*, \tilde{T} \right\rangle + \tilde{\kappa} \frac{d\tilde{s}}{ds} \left\langle T^*, \tilde{N} \right\rangle, \tag{22}$$

where s, s^* and $\tilde{s}$ are arc length parameter of the curves α, β and γ, respectively. Also, κ, κ^* and $\widetilde{\kappa}$ are the curvatures of the curves α, β and γ, respectively. Considering the equations (19), (20) and (22), we get

$$f'(s) = 0.$$

This means that

$$f = const.$$

So, θ is constant. The proof is completed.

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S_2 near-rings

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Abstract A near-ring N is said to be S_1 if for every $a \in N$, there exists $x \in N^* = N - \{0\}$ such that $axa = xa$. Closely following this, we introduce in this paper the concept of S_2 near-rings. A near-ring N is said to be an S_2 near-ring if, for every $a \in N$, there exists $x \in N^*$ such that $axa = ax$. Further by generalizing this, we introduce strong S_2 near-rings i.e., near-rings in which $aba = ab$ for all $a, b \in N$. We discuss some of their properties, obtain a characterisation and also a structure theorem.

Keywords S_2 near-rings, strong S_2 near-rings, zero divisors.

§1. Introduction

Throughout this paper N stands for a right near-ring $(N, +, \cdot)$ and '0' denotes the identity element of $(N, +)$. A subgroup $(M, +)$ of $(N, +)$ is called an $N-$subgroup if $NM \subset M$. N is called weak commutative if $abc = acb$ for all $a, b, c \in N$ [1]. N is said to be regular if for every $a \in N$, there exists $b \in N$ such that $a = aba$. N is called an S near-ring or an S' near-ring according as $x \in Nx$ or $x \in xN$ for all $x \in N$. N is called a strong S_1 near-ring [2] if $N^* = N_{S_1}(a)$ for all $a \in N$ where $N^* = N - \{0\}$ and $N_{S_1}(a) = \{x \in N^*/axa = xa\}$. For basic concepts and terms used but left undefined in this paper, we refer to Pilz [1]. $N_0 = \{a \in N/a0 = 0\}$ is called the zero symmetric part of N and N is called zero symmetric if $N = N_0$. $N_c = \{a \in N/aa' = a$ for all $a' \in N\}$ is called the constant part of N and N is called constant if $N = N_c$. $N_d = \{n \in N/n(a + a') = na + na'$ for all $a, a' \in N\}$ is the set of all distributive elements of N and N is called distributive if $N = N_d$. For $a \in N$, $(0 : a) = \{n \in N/na = 0\}$ is the annihilator of a. N is said to have IFP (i.e. Insertion of Factors Property) if for $x, y \in N$, $xy = 0$ implies $xny = 0$ for all $n \in N$. N has Property(P_4) if for all ideals I of N, $xy \in I$ implies $yx \in I$ for all $x, y \in N$ [1].

We freely make use of the following results.

Theorem 1.1.[1] Every near-ring is isomorphic to a subdirect product of subdirectly irreducible near-rings.

Theorem 1.2.[4] If a near-ring N has IFP and if $xy = 0$ implies $yx = 0$ for $x, y \in N$ then we say that N has $(*, IFP)$.

Theorem 1.3.[1] A near-ring N has strong IFP if for all ideals I of N, $ab \in I$ implies $anb \in I$ for all $a, b, n \in N$.

Theorem 1.4.[1] A zero symmetric near-ring N has IFP if and only if $(0 : n)$ is an ideal for all $n \in N$.

Theorem 1.5.[2] A near-ring N is a strong S_1 near-ring if and only if $aba = ba$ for all $a, b \in N$.

§2. S_2 near-rings

In this section, we define S_2 near-rings, give examples and prove certain elementary properties of such near-rings.

Definition 2.1. A near-ring N is said to be an S_2 near-ring if for every $a \in N$, there exists $x \in N^*$ such that $axa = ax$.

Examples 2.2.

(a) Every Boolean near-ring is an S_2 near-ring.

(b) Consider the near-ring $(Z_6, +, \cdot)$ where $(Z_6, +)$ is the group of integers modulo '6' and '$\cdot$' is defined as follows [1]. The multiplication table for $(N, \cdot)$ is given below:

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	3	1	3	1	1
2	0	0	2	0	2	2
3	3	3	3	3	3	3
4	0	0	4	0	4	4
5	3	3	5	3	5	5

One can check that $(N, +, \cdot)$ is an S_2 near-ring. It is worth noting that it is neither zero symmetric nor constant.

We furnish below a characterisation of S_2 near-rings.

Theorem 2.3. Let N be a weak commutative near-ring without non-zero zero divisors. Then N is an S_2 near-ring if and only if N is Boolean.

Proof. For the 'only if' part, let $a \in N$. Since N is an S_2 near-ring, there exists $x \in N^*$ such that $axa = ax$. Since N is weak commutative, $aax = ax$. This implies $(a^2 - a)x = 0$. Since N has no non-zero zero divisors, we get $a^2 - a = 0 \Rightarrow a^2 = a$. Thus N is Boolean. The proof of 'if' part is obvious.

Proposition 2.4. Let N be an S_2 near-ring. If N is regular, then, for every $a \in N$, there exist $x \in N^*$ and $y \in N$ such that $axya = axa$.

Proof. The result is obvious when $a = 0$. Now, let $a \in N^*$. Since N is an S_2 near-ring, there exists $x \in N^*$ such that $axa = ax$. Since N is regular, there exists $y \in N$ such that $a = aya$. Now $axya = (ax)ya = (axa)ya = ax(aya) = axa$.

We furnish below a characterisation of regular near-rings.

Corollary 2.5. Let N be a distributive S_2 near-ring without non-zero zero divisors. Then N is regular if and only if N is Boolean.

Proof. For the 'only if' part, we observe that when $a = 0$, clearly $a^2 = a$. Now, let $a \in N^*$. By proposition 2.4, there exist $x \in N^*$ and $y \in N$ such that $aya = a$ and $axya = axa$. Since N is distributive, this implies that $ax(ya - a) = 0$. Since N has no non-zero zero divisors, we get $ya - a = 0$. That is $ya = a$. Now $a^2 = a(ya) = a$ and so N is Boolean. The proof of 'if' part is obvious.

§3. Strong S_2-near-rings

In this section, we define strong S_2 near-rings and obtain its properties.

Definition 3.1. A near-ring N is called a strong S_2 near-ring, if $aba = ab$ for all $a, b \in N$.

Examples 3.2.

(a) Every constant near-ring is a strong S_2 near-ring.

(b) We consider the near-ring $(N, +, .)$ where $(N, +)$ is the Klein's four group {0, a, b, c } and '.' is defined as follows [1]:

.	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	0
c	a	a	a	a

This is a strong S_2 near-ring.

Remark 3.3. Clearly every strong S_2 near-ring is an S_2 near-ring, but converse is not true. For, consider the near-ring $(N, +, \cdot)$ where $(N, +)$ is the Klein's four group { 0, a, b, c } and '.' is defined as follows [1]:

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	0	a

This is an S_2 near-ring, but not a strong S_2 near-ring as bcb $\neq$ bc.

Remark 3.4. We observe that, when N is a commutative near-ring, the concepts of strong S_1 and strong S_2 near-rings coincide. It immediately follows from the definition 3.1 that when N is a strong S_2 near-ring, every ideal and every $N-$subgroup of N is also so.

One can easily verify the following.

Theorem 3.5. Homomorphic image of a strong S_2 near-ring is also a strong S_2 near-ring.

Theorem 3.6. Every strong S_2 near-ring is isomorphic to a subdirect product of subdirectly irreducible strong S_2 near-rings.

Proof. By theorem 1.1, N is isomorphic to a subdirect product of subdirectly irreducible near-rings N_i's, say, and each N_i is a homomorphic image of N under the usual projection map π_i. The desired result now follows from theorem 3.5.

Theorem 3.7. Let N be a strong S_2 near-ring. Then the following are equivalent.

(i) N is an S near-ring.

(ii) N is an S' near-ring.

(iii) N is Boolean.

(iv) N is regular.

Proof. (i) $\Rightarrow$ (ii) Let $x \in N$. Since N is an S near-ring, there exists $y \in N$ such that $x = yx$. This implies that $x = yx = (yx)y = xy \in xN$. Thus N is an S' near-ring.

(ii) $\Rightarrow$ (iii) Let $x \in N$. Since N is an S' near-ring, $x \in xN$, there exists $y \in N$ such that $x = xy$. Now $x^2 = (xy)x = xyx = xy = x$. That is $x^2 = x$ for all $x \in N$ and so N is Boolean.

(iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) are obvious.

Theorem 3.8. Let N be a zero symmetric strong S_2 near-ring. Then the following are true.

(i) N has (*, IFP).

(ii) N has strong IFP.

(iii) N has property (P_4).

(iv) For every $a \in N$, $(0 : a)$ is an ideal of N.

Proof. (i) Let $x, y \in N$. Suppose $xy = 0$. Now $yx = yxy = y(xy) = y0 = 0$. Now, for any $n \in N$, $xny = (xn)y = (xnx)y = xn(xy) = xn(0) = 0$. That is $xny = 0$. From these N has (*, IFP).

(ii) Let $x, y, n \in N$ and let I be an ideal of N. Since N is zero symmetric, $NI \subset I$. Suppose $xy \in I$. Now $xny = (xn)y = (xnx)y = xn(xy) \in NI$. Consequently $xny \in I$ and (ii) follows.

(iii) Let $x, y \in N$ and let I be an ideal of N. Suppose $xy \in I$. As in (ii), $NI \subset I$. Now $yx = yxy = y(xy) \in NI$. Therefore $yx \in I$ and (iii) follows.

(iv) Follows from (i) and theorem 1.5.

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Some clopen sets in uniform topology on BE-algebras

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Abstract In this paper, we consider a collection of filters of a BE-algebra A. We use the concept of congruence relation with respect to filters to construct a uniformity which induces a topology on A and prove it is natural for BE-algebras to be topological BE-algebras with respect to this topology. We study the properties of this topology regarding different filters.

Keywords Uniformity, BE-algebra, filter, topological BE-algebra.

§1. Introduction

The study of BCK/BCI-algebras was initiated by K. Iseki as a generalization of the concept of set-theoretic difference and propositional calculus ([1],[2]). In [6], J. Neggeres and H. S. Kim introduced the notion of d-algebra which is a generalization of BCK-algebras. Moreover, Y. B. Jun, E. H. Roh and H. S. Kim [3] introduced a new notion, called BH-algebra, which is a generalization of BCK/BCI-algebras. Recently, as another generalization of BCK-algebras, the notion of BE-algebras was introduced by H. S. Kim and Y. H. Kim [5]. In this paper we consider a collection of filters and use congruence relation with respect to filters to define a uniformity and make the BE-algebra into a uniform topological space with the desired subset as the open sets.

Towards our goal, we renew some needed algebraic notions in §2. Then consider the uniformity based on congruence relations with respect to given collection of filters and construct the induced topology by this uniformity in §3. In the last sections we study the properties of these topologies such as compactness regarding different filters.

§2. Periliminiaries

Definition 2.1.[5] A BE-algebra is an algebra $(A, *, 1)$ with a binary operation $*$, and a constant 1 such that:

(BE1) $a * a = 1$, $\forall a \in A$;
(BE2) $a * 1 = 1$, $\forall a \in A$;
(BE3) $1 * a = a$, $\forall a \in A$;
(BE4) $a * (b * c) = b * (a * c)$, $\forall a, b, c \in A$.

We introduce the relation $\leq$ by setting $a \leq b$ if and only if $a * b = 1$, for any $a, b \in A$.

Definition 2.2.[5] A filter of a BE-algebra A is a nonempty subset F of A such that for all $a, b \in A$, we have

(F1) $1 \in F$;

(F2) $a * b \in F, a \in F$ then $b \in F$.

Let A be a BE-algebra and K a non-empty subset of A. Denote by $\sim_K$ the relation on A given by

$$a \sim_K b \text{ if and only if } a * b \in K \text{ and } b * a \in K.$$

Theorem 2.3.[7] If K be a filter of a BE-algebra A. Then relation $\sim_K$ is an congruence relation on A.

§3. Uniformity in BE-algebra

From now on $(A, *, 1)$ is a BE-algebra.

Let X be a nonempty set and U, V be any subset of $X \times X$. Define

$U \circ V = \{(x, y) \in X \times X \mid (z, y) \in U \text{ and } (x, z) \in V, \text{ for some } z \in X\}$,

$U^{-1} = \{(x, y) \in X \times X \mid (y, x) \in U\}$,

$\Delta = \{(x, x) \in X \times X \mid x \in X\}$.

Definition 3.1.[4] By a uniformity on X we shall mean a nonempty collection $\mathcal{K}$ of subsets of $X \times X$ which satisfies the following conditions:

(U_1) $\Delta \subseteq U$ for any $U \in \mathcal{K}$;

(U_2) if $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{K}$;

(U_3) if $U \in \mathcal{K}$, then there exist a $V \in \mathcal{K}$such that $V \circ V \subseteq U$;

(U_4) if $U, V \in \mathcal{K}$, then $U \cap V \in \mathcal{K}$;

(U_5) if $U \in \mathcal{K}$, and $U \subseteq V \subseteq X \times X$ then $V \in \mathcal{K}$.

The pair $(X, \mathcal{K})$ is called a *uniform structure* (uniform space).

Theorem 3.2. Let Λ be an arbitrary family of filters of A which is closed under intersection. If $U_F = \{(x, y) \in A \times A \mid x \sim_F y\}$ and $\mathcal{K}^* = \{U_F \mid F \in \Lambda\}$, then $\mathcal{K}^*$ satisfies the conditions (U_1)-(U_4).

Proof. (U_1): Since F is a filter of A then we have $x \sim_F x$ for any $x \in A$, hence $\Delta \subseteq U_F$, for all $U_F \in \mathcal{K}^*$.

(U_2): For any $U_F \in \mathcal{K}^*$, we have $(x, y) \in (U_F)^{-1} \Leftrightarrow (y, x) \in U_F \Leftrightarrow y \sim_F x \Leftrightarrow x \sim_F y \Leftrightarrow (x, y) \in U_F$.

(U_3): For any $U_F \in \mathcal{K}^*$, the transitivity of $\sim_F$ implies that $U_F \circ U_F \subseteq U_F$.

(U_4): For any $U_F, U_J \in \mathcal{K}^*$, we claim that $U_F \cap U_J = U_{F \cap J}$. Let $(x, y) \in U_F \cap U_J$. Then $x \sim_F y$ and $x \sim_J y$. Hence $x * y \in F$, $y * x \in F$ and $x * y \in J$, $y * x \in J$. Then $x \sim_{F \cap J} y$ and hence $(x, y) \in U_{F \cap J}$. Conversely, let $(x, y) \in U_{F \cap J}$. Then $x \sim_{F \cap J} y$, hence $x * y \in F \cap J$ and $y * x \in F \cap J$. Then $x * y \in F$, $y * x \in F$, $x * y \in J$ and $y * x \in J$. Therefore $x \sim_F y$ and $x \sim_J y$. Then $(x, y) \in U_F \cap U_J$. So $U_F \cap U_J = U_{F \cap J}$. Since $F, J \in \Lambda$, then $F \cap J \in \Lambda$, $U_F \cap U_J \in \mathcal{K}^*$.

Theorem 3.3. Let $\mathcal{K} = \{U \subseteq A \times A \mid U_F \subseteq U \text{ for some } U_F \in \mathcal{K}^*\}$. Then $\mathcal{K}$ satisfies a uniformity on A and the pair $(A, \mathcal{K})$ is a uniform structure.

Proof. By theorem 3.2, the collection $\mathcal{K}$ satisfies the conditions $(U_1)-(U_4)$. It suffices to show that $\mathcal{K}$ satisfies (U_5). Let $U \in \mathcal{K}$ and $U \subseteq V \subseteq A \times A$. Then there exists a $U_F \subseteq U \subseteq V$, which means that $V \in \mathcal{K}$. This proves the theorem.

Let $x \in A$ and $U \in \mathcal{K}$. Define

$$U[x] := \{y \in A \mid (x, y) \in U\}.$$

Theorem 3.4. Given a BE-algebra A, then

$$T = \{G \subseteq A \mid \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G\}$$

is a topology on A.

Proof. It is clear that $\emptyset$ and the set A belong to T. Also from the very nature of that definition, it is clear that T is closed under arbitrary union. Finally to show that T is closed under finite intersection, let $G, H \in T$ and suppose $x \in G \cap H$. Then there exist U and V $\in \mathcal{K}$ such that $U[x] \subseteq$G and $V[x] \subseteq H$. Let $W = U \cap V$, then W $\in \mathcal{K}$. Also $W[x] \subseteq U[x] \cap V[x]$ and so $W[x] \subseteq G \cap H$ and so $G \cap H \in T$. Thus T is topology on A.

Note that for any x in A, $U[x]$ is an open neighborhood of x.

Lemma 3.5. Let F be a filter of A. Then $F = \{1\}$ if and only if $U_F = U_{\{1\}}$.

Proof. Since $F \neq \{1\}$, there exist $z \in F$ such that $z \neq 1$. By $(BE3)$, $1 * z = z \in F$ and by $(BE2)$, $z * 1 = 1 \in F$. Hence $1 \in U_F[z]$ and then $(z, 1) \in U_F$. On other hand since $z \neq 1$, $(z, 1) \notin U_{\{1\}}$. Therefore if $U_F = U_{\{1\}}$ then $F = \{1\}$. Conversely is clear.

Definition 3.6. Let $(A, \mathcal{K})$ be a uniform structure. Then the topology T is called the uniform topology on A induced by $\mathcal{K}$.

Example 3.7. Let $A = \{a, b, c, 1\}$. Define $*$ as follow:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Easily we can check that $(A, *, 1)$ is a BE-algebra [6]. It is easy to see that $F = \{c, a, 1\}$ is a filter and let $\Lambda = \{F\}$. Therefore as theorem 3.2, we construct

$$\begin{aligned} K^* &= \{\{U_F\}\} \\ &= \{\{(x, y) \mid x \sim_F y\}\} \\ &= \{(1,1), (a,1), (1,a), (1,c), (c,1), (a,a), (a,c), (c,a), (b,b), (b,d), (d,b), (c,c), (d,d)\}. \end{aligned}$$

We can check that $(A, \mathcal{K})$ is a uniform space, where $\mathcal{K} = \{U \mid U_F \subseteq U\}$. Open neighborhoods are:

$$U_F[a] = \{1, a, c\},$$
$$U_F[b] = \{b, d\},$$
$$U_F[c] = \{1, a, c\},$$
$$U_F[d] = \{b, d\},$$
$$U_F[1] = \{1, a, c\}.$$

From above we get that

$$T = \{\{1, a, c\}, \{b, d\}, \{1, a, b, c, d\}, \emptyset\ \}.$$

Thus (A, T) is a uniform topological space.

Proposition 3.8. Topological space (A, T) is completely regular.

Proof. See theorem 14.2.9 of [4].

§4. Topological property of space (A, T)

Let A be a BE-algebra and C, D subsets of A. Then we define $C * D$ as follows:

$$C * D = \{x * y \mid x \in C, y \in D\}.$$

Let A be a BE- algebra and T a topology defined on the set A. Then we say that the pair (A, T) is a topological BE-algebra if the BE-algebraic operation $*$ is continuous with respect to T. The continuity of the BE-algebraic operation $*$ is equivalent to having the following properties satisfied:

Let O be an open set and $a, b \in A$ such that $a * b \in O$. Then there are O_1 and O_2 such that $a \in O_1$, $b \in O_2$ and $O_1 * O_2 \subseteq O$.

Theorem 4.1. The pair (A, T) is a topological BE-algebra.

Proof. Assume that $x * y \in G$, with $x, y \in A$ and G an open subset of A. Then there exist $U \in \mathcal{K}$, $U[x * y] \subseteq G$ and a filter F of A such that $U_F \subseteq U$. We claim that the following relation holds:

$$U_F[x] * U_F[y] \subseteq U_F[x * y].$$

Indeed for $h \in U_F[x]$ and $k \in U_F[y]$ we get $x \sim_F h$ and $y \sim_F k$. Hence by theorem 2.3, $x * y \sim_F h * k$. From that $(x * y, h * k) \in U_F \subseteq U$. Hence $h * k \in U_F[x * y] \subseteq U[x * y]$. Then $h * k \in G$. Thus the condition is verified.

Theorem 4.2.[5] Let X be a set and $\mathcal{S} \subset \mathcal{P}(X \times X)$ be a family such that for every $U \in \mathcal{S}$ the following conditions hold:

(a) $\Delta \subseteq U$;

(b) U^{-1} contains a member of $\mathcal{S}$, and

(c) there exists a $V \in \mathcal{S}$, such that $V \circ V \subseteq U$. Then there exists a unique uniformity $\mathcal{U}$, for which $\mathcal{S}$ is a subbase.

Theorem 4.3. If we let $\mathcal{B} = \{U_F \mid F \text{ is a filter of } A\}$. Then $\mathcal{B}$ is a subbase for a uniformity of A, we denote this topology by S.

Proof. Since $\sim_F$ is an equivalence relation, then it is clear that $\mathcal{B}$ satisfies the axioms of theorem 4.2.

We say that topology σ is finer than τ if $\tau \subseteq \sigma$ as subsets of the power set. Then we have:

Corollary 4.4. Topology S is finer than T.

Theorem 4.5. Any filter in the collection Λ is a clopen subset of A.

Proof. Let F be a filter of A in Λ and $y \in F^c$. Then $y \in U_F[y]$ and we get $F^c \subseteq \bigcup\{U_F[y] \mid y \in F^c\}$. We claim that, $U_F[y] \subseteq F^c$, for all $y \in F^c$. Let $z \in U_F[y]$, then $z \sim_F y$. Hence $z * y \in F$. If $z \in F$ then, $y \in F$, that is a contradiction. So $z \in F^c$ and we get $\bigcup\{U_F[y] \mid y \in F^c\} \subseteq F^c$. Hence $F^c = \bigcup\{U_F[y] \mid y \in F^c\}$ and since $U_F[y]$ is open for all $y \in A$, F is a closed subset. We show that $F = \bigcup\{U_F[y] \mid y \in F\}$. If $y \in F$ then $y \in U_F[y]$ and we get $F \subseteq \bigcup\{U_F[y] \mid y \in F\}$. Let $y \in F$, if $z \in U_F[y]$ then $z \sim_F y$ and so $y * z \in F$. Since $y \in F$ hence $z \in F$ and we get $\bigcup\{U_F[y] \mid y \in F\} \subseteq F$. So F is also an open subset of A.

Theorem 4.6. For any $x \in A$ and $F \in \Lambda$, $U_F[x]$ is a clopen subset of A.

Proof. We show that $(U_F[x])^c$ is open. Let $y \in (U_F[x])^c$, then $x*y \in F^c$ or $y*x \in F^c$. Let $y*x \in F^c$. Hence by theorem 4.1 and the proof of theorem 4.5, $(U_F[y]*U_F[x]) \subseteq U_F[y*x] \subseteq F^c$. We claim that: $U_F[y] \subseteq (U_F[x])^c$. Let $z \in U_F[y]$, then $z * x \in (U_F[y] * U_F[x])$. So $z * x \in F^c$ then we get $z \in (U_F[x])^c$. Hence $U_F[x]$ is closed. It is clear that $U_F[x]$ is open. So $U_F[x]$ is clopen subset of A.

A topological space X is connected if and only if has only X and $\emptyset$ as clopen subsets. Therefore we have

Corollary 4.7. The space (A, T) is not a connected space.

§5. Some connection between uniform topology and filters

We denote the uniform topology obtained by an arbitrary family Λ, by T_Λ and if $\Lambda = \{F\}$, we denote it by T_F.

Theorem 5.1. $T_\Lambda = T_J$, where $J = \bigcap\{F \mid F \in \Lambda\}$.

Proof. Let $\mathcal{K}$ and $\mathcal{K}^*$ be as in theorem 3.2 and 3.3. Now consider $\Lambda_0 = \{J\}$, define: $(\mathcal{K}_0)^* = \{U_J\}$ and $\mathcal{K}_0 = \{U \mid U_J \subseteq U\}$.

Let $G \in T_\Lambda$. So for all $x \in G$, there exist $U \in \mathcal{K}$ such that $U[x] \subseteq G$. From $J \subseteq F$ we get that $U_J \subseteq U_F$, for all filter F of A. Since $U \in \mathcal{K}$, there exist $F \in \Lambda$ such that $U_F \subseteq U$. Hence $U_J[x] \subseteq U_F[x] \subseteq G$. Since $U_J \in \mathcal{K}_0$, $G \in T_J$. So $T_\Lambda \subseteq T_J$.

Conversely, let $H \in T_J$ then for all $x \in H$, there exist $U \in \mathcal{K}_0$ such that $U[x] \subseteq H$. So $U_J[x] \subseteq H$ and sine Λ is closed under intersection, $J \in \Lambda$. Then we get $U_J \in \mathcal{K}$ and so $H \in T_\Lambda$. Thus $T_J \subseteq T_\Lambda$.

Corollary 5.2. Let F and J be filters of A and $F \subseteq J$. Then J is clopen in topological space (A, T_F).

Proof. Consider $\Lambda = \{F, J\}$. Then by theorem 5.1, $T_\Lambda = T_F$ and therefore J is clopen in topological space (A, T_F).

Theorem 5.3. Let F and J be filters of A. Then $T_F \subseteq T_J$ if and only if $J \subseteq F$.

Proof. Let $J \subseteq F$. Consider:

$\Lambda_1 = \{F\}$, ${\mathcal{K}_1}^* = \{U_F\}$, $\mathcal{K}_1 = \{U \mid U_F \subseteq U\}$ and

$\Lambda_2 = \{J\}$, $\mathcal{K}_2{}^* = \{U_J\}$, $\mathcal{K}_2 = \{U \mid U_J \subseteq U\}$.

Let $G \in T_F$. Then for all $x \in G$, there exist $U \in \mathcal{K}_1$ such that $U[x] \subseteq G$. Since $J \subseteq F$, then $U_J \subseteq U_F$ and since $U_F[x] \subseteq G$, we get $U_J[x] \subseteq G$. $U_J \in \mathcal{K}_2$ and so $G \in T_J$.

Conversely, let $T_F \subseteq T_J$. In contrary let $a \in J \setminus F$, since $F \in T_F$, by hypothesis we get that $F \in T_J$. Then for all $x \in F$, there exist $U \in \mathcal{K}_2$ such that $U[x] \subseteq F$, and so $U_J[x] \subseteq F$. Then $U_J[1] \subseteq F$. We have $a * 1 = 1 \in J$ and $1 * a = a \in J$. Then $a \sim_J 1$, so $a \in U_J[1]$, thus $a \in F$. Which is a contradiction.

Corollary 5.4. Let F and J be filters of A. Then $F = J$ if and only if $T_F = T_J$.

Theorem 5.5. Let F be a filter of A. Then the following conditions are equivalent:

(1) Topological space (A, T_F) is compact;

(2) Topological space (A, T_F) is totally bounded;

(3) there exists $P = \{x_1, x_2, \ldots, x_n\} \subseteq A$ such that for all $a \in A$ there exists $x_i \in P$ where $a * x_i \in F$ and $x_i * a \in F$.

Proof. $(1 \Rightarrow 2)$: it is clear by theorem 14.3.8. of [5].

$(2 \Rightarrow 3)$: Let $U_F \in \mathcal{K}$ since (A, T_F) is totally bounded, then there exists $x_1, x_2, \ldots, x_n \in F$ such that $A = \bigcup_{i=1}^{n} U_F[x_i]$. Now let $a \in A$ then there exist x_i such that $a \in U_F[x_i]$, therefore $a * x_i \in F$ and $x_i * a \in F$.

$(3 \Rightarrow 1)$: For all $a \in A$ by hypothesis there exists $x_i \in P$ where $a * x_i \in F$ and $x_i * a \in F$. Hence $a \in U_F[x_i]$, thus $A = \bigcup_{i=1}^{n} U_F[x_i]$. Now let $A = \bigcup_{\alpha \in I} O_\alpha$, where each O_α is an open set of A, then for any $x_i \in A$ there exists $\alpha_i \in I$ such that $x_i \in O_{\alpha_i}$, since O_{α_i} is an open set then $U_F[x_i] \subseteq O_{\alpha_i}$, so $A = \bigcup_{i=1}^{n} U_F[x_i] \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}$, therefore $A = \bigcup_{i=1}^{n} O_{\alpha_i}$ which means that (A, T_F) is compact.

Theorem 5.6. If F is a filter of A such that F^c is a finite set. Then topological space (A, T_F) is compact.

Proof. Let $A = \bigcup_{\alpha \in I} O_\alpha$, where each O_α is an open set of A. Let $F^c = \{x_1, x_2, \ldots, x_n\}$. Then there exists $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that $1 \in O_\alpha$, $x_1 \in O_{\alpha_1}, x_2 \in O_{\alpha_2}, \ldots, x_n \in O_{\alpha_n}$. Then $U_F[1] \subseteq O_\alpha$, but $U_F[1] = F$. Hence $A = O_{\alpha_1} \bigcup O_{\alpha_2} \bigcup \ldots \bigcup O_{\alpha_n}$.

Theorem 5.7. If F is a filter of A. Then F is a compact set in topological space (A, T_F).

Proof. Let $F \subseteq \bigcup_{\alpha \in I} O_\alpha$, where each O_α is an open set of A. Since $1 \in F$, then there exists $\alpha \in I$ such that $1 \in O_\alpha$. Then $F = U_F[1] \subseteq O_\alpha$. Hence F is compact.

Theorem 5.8. If F is a filter of A. Then $U_F[x]$ is a compact set in topological space (A, T_F), for all $x \in A$.

Proof. Let $U_F[x] \subseteq \bigcup_{\alpha \in I} O_\alpha$, where each O_α is an open set of A. Since $x \in U_F[x]$, then there exists $\alpha \in I$ such that $x \in O_\alpha$. Then $U_F[x] \subseteq O_\alpha$. Hence $U_F[x]$ is compact.

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Testing for comparison of two expectation functions of non-parametric regression

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Abstract In this study, we introduce a procedure for testing the equality of two regression functions, the hypothesis: $H_o : m_1 = m_2$ vs H_1: $m_1 \neq m_2$. The test statistic is based on the comparison of two estimators of the distribution of the error in each population, and the estimator of error in each population is $\frac{\bar{Y_j} - \hat{m_j}(X_{ij})}{\hat{\sigma_j}(X_{ij})/\sqrt{n_j}}$. Kuiper type statistic are considered. In the simulation study, a bootstrap mechanism is used to approximate the critical value in practice.

Keywords Nonparametric regression, bootstrap procedure, comparison expectation function.

§1. Introduction and preliminaries

Regression analysis can be divided into two parts : 1) parametric regression and 2) non-parametric regression. The comparison of two or more groups is an important problem in statistical inference. This comparison can be performed through the regression curves in non-parametric regression context. Let (X_{ij}, Y_{ij}), $i = 1, \ldots, n_j, j = 1, 2$ be two independent random vectors, and assume the following non-parametric regression model

$$Y_{ij} = m_j(X_{ij}) + \sigma_j(X_{ij})\varepsilon_{ij}, i = 1, \ldots, n_j, j = 1, 2, \tag{1}$$

where $m_j(X_{ij}) = E(Y_{ij})|(X_{ij})$ is the unknown regression function, $\sigma_j(X_{ij})$ is the conditional variance function $\sigma_j^2(X_{ij}) = Var(Y_{ij})|(X_{ij})$, ε_{ij} is the error variable with distribution $F_{\varepsilon j}$, and let $n = n_1 + n_2$. In this paper, we are interested in comparison the equality of regression functions, the hypothesis $H_o : m_1 = m_2$ vs $H_1 : m_1 \neq m_2$, and the test statistic is the Kuiper type statistics: $U_{KU} = \sum_{j=1}^{2}[sup_y|\hat{U}_j(y)| - inf_y|\hat{U}_j(y)|]$. The problem of testing for comparison expectation functions of non-parametric regression or the problem of testing for the equality of non-parametric regression curves has been treated in the literature during the last eighteen years. The first reference about comparison of regression curves assume some restriction on the model. Fixed design and homoscedastic errors are common assumption in several references. W. Hardle and J. S. Marron [4] analyzed semi-parametric models by comparing nonparametric regression functions under the assumption of fixed equal designs ($i.e., n_1 = n_2, \ldots, = n_k$). P. Hall and J. D. Hart [3] designed two-sided tests for $H_o : m_1 = m_2$, and discussed the non-parametric homoscedastic ($i.e., \sigma_i^2(t) = \sigma_i^2, i = 1, \ldots, k$) models in the case of equal design

points. E. C. King, J. D. Hart and T. E. Wehrly [5] also presented a test based on the difference between two curve estimators from kernel smoothers when the design points are fixed and equal. M. A. Delgado [2] investigated the test statistics for testing the equality of k regression function when the design point are fixed and equal. S. Young and A. W. Bowman [13] motivated the test statistic by the classical one-way analysis of variance. K. B. Kulasekera [7] presented three tests with common fixed design points. The first two tests are based on estimators Quasi-Residuals technique while the last test based on estimators of the variance of the error distributions. L. H. Koul and A. Schick [6] discussed the problem of testing the equality of non-parametric regression curves one-side alternatives when design points are common and distinct design points. A. Munk and H. Dette [8] considered the problem of testing for the equality of two curves with heteroscedastic errors and fixed design. The testing procedure is based on the estimation of L^2-distance of difference between two curves. N. Neumeyer and H. Dette [9] considered an empirical process approach in complete non-parametric, heteroscedastic and random designed setup to compare two regression curves. Their test can detect local alternatives which converge to the null hypothesis at the rate of order $n^{-1/2}$. The idea of our testing procedure come from [10]. Namely, if two empirical distributions of the errors are equal, there are evidence for the equality of the regression curves. On the contrary, if these two empirical distributions are different, there are evidence for the inequality of the regression curves. In this paper, the estimator of the error in j population (ε_{ij}) for $j = 1, 2$ is $\frac{\bar{Y_j} - \hat{m}_j(X_{ij})}{\hat{\sigma_j}(X_{ij})/\sqrt{n_j}}$, $E(\varepsilon_{ij}|X_{ij} = 0), \mathrm{Var}(\varepsilon_{\mathrm{ij}}) = 1, \mathrm{and} \frac{\bar{\mathrm{Y}}_{\mathrm{j}} - \hat{\mathrm{m}}(\mathrm{X}_{\mathrm{ij}})}{\hat{\sigma}_{\mathrm{j}}(\mathrm{X}_{\mathrm{ij}})/\sqrt{\mathrm{n}_{\mathrm{j}}}}$ is the estimator of the error under the null hypothesis (H_o), where $\bar{Y_j}$ is the average of Y in j population, $\bar{Y_j} = \frac{\sum_{i=1}^{n_j} Y_{ij}}{n_j}$, $\hat{m}_j(X_{ij})$ is the estimator of regression function $m_j(X_{ij})$, $\hat{m}(X_{ij})$ is the estimator of regression function $m(X_{ij})$ under the null hypothesis, $\hat{\sigma_j}(X_{ij})$ is the estimator of variance $\sigma_j(X_{ij})$, and n_j is sample size in population j respectively. In addition, in this paper the test statistic can detect local alternatives which converge to the null hypothesis at the rate of order $n^{-1/2}$.

§2. Definition and assumptions

In this section, we introduced the definition and assumptions for proofs of the main results. Consider (X_{ij}, Y_{ij}), $i = 1, \ldots, n_j, j = 1, 2$ be an independent and identically distribution, and satisfying the non-parametric regression model (1). Let $\hat{m}_j(X_{ij})$ is the estimator of regression function $m_j(X_{ij})$, $\hat{m}_j(x) = \sum_{i=1}^{n_j} W_{ij}^{(j)}(x, h)Y_{ij}$, $W_{ij}^{(j)}(x, h) = \frac{K((x - X_{ij})/h)}{\sum_{i=1}^{n_j} K((x - X_{ij})/h)}$ is Nadaraya - Watson - type weight, K is a known kernel, h is an appropriate bandwidth, $\hat{\sigma}_j^2(x) = \sum_{i=1}^{n_j} W_{ij}^{(j)}(x, h)Y_{ij}^2 - \hat{m}_j^2(x)$, and $\hat{m}(x) = \sum_{j=1}^{k} \sum_{i=1}^{n_j} W_{ij}^{(j)}(x, h)Y_{ij}$ be an estimator of the common regression function $m(x) = m_1(x) = m_2(x)$ under the null hypothesis H_o. For $j = 1, 2$, let $\hat{\varepsilon}_{ij} = \frac{\bar{Y_j} - \hat{m}_j(X_{ij})}{\hat{\sigma_j}(X_{ij})/\sqrt{n_j}}$ is the estimator of the error in j population, $\hat{F}_{\varepsilon j}(y)$ is the estimator of the error distribution function in j population $F_{\varepsilon j}(y)$: $\hat{F}_{\varepsilon j}(y) = P(\hat{\varepsilon}_{ij} \leq y) = \frac{1}{n_j} \sum_{i=1}^{n_j} I[\frac{\bar{Y_j} - \hat{m}_j(X_{ij})}{\hat{\sigma_j}(X_{ij})/\sqrt{n_j}} \leq y]$, I(.) is an indicator function, and probability density function of

$F_{\varepsilon j}(y)$ is $f_{\varepsilon j}(y)$. We used the notation $f_{\varepsilon j}(y) = F'_{\varepsilon j}(y)$. Let $\hat{\varepsilon}_{ij0} = \dfrac{\bar{Y}_j - \hat{m}(X_{ij})}{\hat{\sigma}_j(X_{ij})/\sqrt{n_j}}$ is an estimator of the error under the null hypothesis H_o, $\hat{F}_{\varepsilon 0}(y)$ is the estimator of the error distribution function in j population $F_{\varepsilon 0}(y)$: $\hat{F}_{\varepsilon 0}(y) = P(\hat{\varepsilon}_{ij0} \leq y) = \dfrac{1}{n_j}\sum_{i=1}^{n_j} I[\dfrac{\bar{Y}_j - \hat{m}(X_{ij})}{\hat{\sigma}_j(X_{ij})/\sqrt{n_j}} \leq y]$, I(.) is an indicator function, and probability density function of $F_{\varepsilon 0}(y)$ is $f_{\varepsilon 0}(y)$. We used the notation $f_{\varepsilon 0}(y) = F'_{\varepsilon 0}(y)$. For simplicity, we work with the same bandwidth h to estimate $\hat{m}$, $\hat{m}_j$, $\hat{\sigma}_j$. In addition, in this paper we perform the comparison between these two estimators of distribution of error in each population using the 2-dimensional process, $\hat{U}(y) = (\hat{U}_1(y), \hat{U}_2(y))$, $-\infty < y < \infty$, for $j = 1, 2$, $\hat{U}_j(y) = n_j^{1/2}(\hat{F}_{\varepsilon 0}(y) - \hat{F}_{\varepsilon j}(y))$. More precisely, we will use the Kuiper test statistic $U_{KU} = \sum_{j=1}^{2}[sup_y|\hat{U}_j(y)| - inf_y|\hat{U}_j(y)|]$. The assumptions we need for proofs of the main results are listed below for convenient reference.

A1 : $F_j(x) = P(X_j \leq x)$ and $F_j(y|x) = P(Y_j \leq y|X_j = x)$ be a distribution of independent variable X and the conditional distribution of response given X respectively. The probability density functions of $F_j(x)$ and $F_j(y|x)$ will be denoted respectively by $f_j(x)$ and $f_j(y|x)$. We used the notation $F_j^{'}(x) = f_j(x)$ and $F_j^{'}(y|x) = f_j(y|x)$.

A2 : For $j = 1, 2$,

(i) $F_{\varepsilon j}(y)$ is an empirical distribution of error of j population,

$F_{\varepsilon j}(y) = P(\varepsilon_{ij} \leq y) = \dfrac{1}{n_j}\sum_{i=1}^{n_j} I[\dfrac{\bar{Y}_j - m_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{n_j}} \leq y]$ and $f_{\varepsilon j}(y) = F'_{\varepsilon j}(y)$. (ii) $F_{\varepsilon 0}(y)$ is an empirical distribution of error under the null hypothesis (H_0).

$F_{\varepsilon 0}(y) = P(\varepsilon_{ij0} \leq y) = \dfrac{1}{n_j}\sum_{i=1}^{n_j} I[\dfrac{\bar{Y}_j - m(X_{ij})}{\sigma_j(X_{ij})/\sqrt{n_j}} \leq y]$, and $f_{\varepsilon 0}(y) = F'_{\varepsilon 0}(y)$.

A3 : For $j = 1, 2$,

(i) X_j is a continuous variable with compact support $x \in \mathbb{R}_x$, and density $f_j(x)$.

(ii) $f_j(x)$, $m_j(x)$, and $\sigma_j(x)$ are twice continuously differentiable.

(iii) $inf_{x\in\mathbb{R}_x} f_j(x) > 0$ and $inf_{x\in\mathbb{R}_x}\sigma_j(x) > 0$.

A4 : For $j = 1, 2$,

(i) $\dfrac{n_j}{n} \to p_j > 0$, as $n \to \infty$,

(ii) $nh^4 \to 0$, and $n_j h^{3+2\delta}(logh^{-1}) \to \infty$, for some $\delta > 0$, as $h \to 0$.

A5 : K is a symmetric density function.

(i) $\int u^2 K(u)du < \infty$.

(ii) $\int K(u)du = 1$.

(iii) $\int uK(u)du = 0$.

A6 : For $j = 1, 2$,

(i) $F_j(y|x)$ is the continuous in (x, y) and differentiable with respect to y;

(ii) $F_j(y|x)$ is the continuous in (x, y) and differentiable with respect to x.

§3. Main results

In this section, we derive the test statistic for finding the distribution of the test statistic.

Theorme 3.1. $F_{\varepsilon 0}(y) = F_{\varepsilon j}(y)$, $-\infty < y < \infty$, for all $j = 1, 2$ if and only if $m(x) = m_1(x) = m_2(x)$ for all $x \in \mathbb{R}_x$.

Proof. Assume $F_{\varepsilon 0}(y) = F_{\varepsilon j}(y)$. This implies that two empirical distributions of the errors are equal, there are evidence for the equality of the 1st moment and the 2nd moment of the distribution. Consider the 1st moment : $E(\frac{\bar{Y}_j - m(X_{ij})}{\sigma_j(X_{ij})/\sqrt{n_j}}) = E(\frac{\bar{Y}_j - m_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{n_j}}) = 0$, then $m(x) = m_j(x)$. In the same way, the 2nd moment have originate from the 1^{st}moment, then $m(x) = m_j(x)$. Namely, $m(x) = m_1(x) = m_2(x)$. Conversely, assume $m(x) = m_1(x) = m_2(x)$. Claim that $F_{\varepsilon 0}(y) = F_{\varepsilon j}(y)$. Consider the 1st moment : From $m(x) = m_j(x)$, then

$$(\frac{\bar{Y}_j - m(X_{ij})}{\sigma_j(X_{ij})/\sqrt{n_j}}) = (\frac{\bar{Y}_j - m_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{n_j}}),\ E(\frac{\bar{Y}_j - m(X_{ij})}{\sigma_j(X_{ij})/\sqrt{n_j}}) = E(\frac{\bar{Y}_j - m_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{n_j}}) = 0.$$

Consider the 2nd moment : From $m(x) = m_j(x)$, then $(\frac{\bar{Y}_j - m(X_{ij})}{\sigma_j(X_{ij})/\sqrt{n_j}}) = (\frac{\bar{Y}_j - m_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{n_j}})$, and $E(\frac{\bar{Y}_j - m(X_{ij})}{\sigma_j(X_{ij})/\sqrt{n_j}})^2 = E(\frac{\bar{Y}_j - m_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{n_j}})^2$. From the 1st moment and the 2nd moment, if $m(x) = m_1(x) = m_2(x)$ then $F_{\varepsilon 0}(y) = F_{\varepsilon j}(y)$.

Theorem 3.2. Assume (A1) - (A6). Under the null hypothesis (H_o) , for $j = 1, 2$,

$$\hat{F}_{\varepsilon 0}(y) - \hat{F}_{\varepsilon j}(y) = f_{\varepsilon j}(y) \sum_{r=1}^{2} p_r \{\frac{1}{n_r} \sum_{i=1}^{n_r} \frac{Y_{ir} - m(X_{ir})}{\sigma_j(X_{ir})/\sqrt{n_j}} (\frac{f_{Xj}(X_{ir})}{f_{mix}(X_{ir})} - \frac{I(r=j)}{p_j})\} + o_p(n)^{-1/2}.$$

Proof. From the theorem 1 in [1], we get the following representation:

$$\hat{F}_{\varepsilon 0}(y) - F_{\varepsilon j}(y) = f_{\varepsilon j}(y) \sum_{r=1}^{2} p_r (\frac{1}{n_r} \sum_{i=1}^{n_r} \frac{Y_{ir} - m(X_{ir})}{\sigma_j(X_{ir})/\sqrt{n_j}} (\frac{f_{Xj}(X_{ir})}{f_{mix}(X_{ir})} - \frac{I(r=j)}{p_j})) + o_p(n)^{-1/2},$$

uniformly in y.

Theorem 3.3. Assume (A1) - (A6). Under the null hypothesis (H_o), 2 - dimensions of $\hat{U}(y) = (\hat{U}_1(y), \hat{U}_2(y))$ converges weakly to $U(y) = (U_1(y), U_2(y))$, where $U_j(y) = f_{\varepsilon j}(y) p_j \sum_{r=1}^{2} \frac{Y_r - m(X_r)}{\sigma_j(X_r)} (\frac{f_{Xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j)}{p_j})$ are normal random variables with mean zero and covariance structure :

$$Cov(U_j(y), U_{j'}(y)) = f_{\varepsilon j}(y) f_{\varepsilon j'}(y) p_j p_{j'} \sum_{r=1}^{2} E[\frac{(Y_r - m(X_r))^2}{\sigma_j(X_r)\sigma_{j'}(X_r)} (\frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j)}{p_j})(\frac{f_{xj'}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j')}{p'_j})],$$ j=1, 2.

Proof. By Cramer - Wold device in R. J. Serfling [11] , and theorem 1 in J. C. Pardo-Fernandez [10] showed that the weak convergence of multidimensional process is weak convergence of linear combination in each component. Assume b_j is any real number, $\hat{U}(y) = (\hat{U}_1(y), \hat{U}_2(y))^t$, where $\hat{U}_j(y) = n_j^{1/2}(\hat{F}_{\varepsilon 0}(y) - \hat{F}_{\varepsilon j}(y))$, $-\infty < y < \infty$ for $j = 1, 2$. Linear combination of $\hat{U}(y)$ is easy to prove that $\hat{Z}(y) = \sum_{j=1}^{2} b_j n_j^{1/2}(\hat{F}_{\varepsilon 0}(y) - \hat{F}_{\varepsilon j}(y))$, from theorem 2,

$$\hat{Z}(y) = \sum_{r=1}^{2} \sum_{i=1}^{n_r} \frac{Y_{ir} - m(X_{ir})}{\sigma_r(X_{ir})} [\sum_{j=1}^{2} b_j p_j f_{\varepsilon j}(y) \frac{\sigma_r(X_{ir})}{\sigma_j(X_{ir})} \frac{f_{Xj}(X_{ir})}{f_{mix}(X_{ir})} - b_r f_{\varepsilon r}(y)] + o_p(n)^{-1/2},$$

$$\hat{Z}(y) = \sum_{r=1}^{2} \sum_{i=1}^{n_r} \Psi_r(X_{ir}, Y_{ir}, y) + o_p(n)^{-1/2}, \text{ then } \hat{Z}_r(y) = \sum_{i=1}^{n_r} \Psi_r(X_{ir}, Y_{ir}, y),$$

where

$$\Psi_r(u,v,y) = \frac{v-m(u)}{\sigma_r(u)}[\sum_{j=1}^{2} b_j p_j f_{\varepsilon j}(y)\frac{\sigma_r(u)}{\sigma_j(u)}\frac{f_{Xj}(u)}{f_{mix}(u)} - b_r f_{\varepsilon r}(y)], r = 1,2. \tag{2}$$

From (2). Given a collection $\mathcal{F}_r$ of measurable function, the empirical measure induces a map from $\mathcal{F}_r$ to R given by : $\mathcal{F}_r$=$\{(u,v) \to \Psi_r(u,v,y), -\infty < y < \infty\}$, where $\mathcal{F}_r$ is - indexed process. A. W. Van der Vaart and J. A. Wellner [12], and J. C. Pardo-Fernandez [10] introduced for any class of function $\mathcal{G}_1$ and $\mathcal{G}_2$, define $\mathcal{G}_1 + \mathcal{G}_2 = \{g_1 + g_2, g_1 \in \mathcal{G}_1, g_2 \in \mathcal{G}_2\}$. The notation of class $\mathcal{F}_r$ can be written as $\mathcal{F}_r = \sum_{j=1}^{k+1} \mathcal{F}_{rj}$, for $j = 1,2$. The empirical measure induces a map form $\mathcal{F}_{rj}$ to R given by : $\mathcal{F}_{rj} = \{(u,v) \to b_j p_j f_{\varepsilon j}(y)\frac{\sigma_r(u)}{\sigma_j(u)}\frac{f_{Xj}(u)}{f_{mix}(u)}\frac{v-m(u)}{\sigma_r(u)},\ -\infty < y < \infty\}$, and the empirical measure induces a map form $\mathcal{F}_{r,(k+1)}$ to R given by: $\mathcal{F}_{r,k+1} = \{(u,v) \to -b_r f_{\varepsilon r}(y)\frac{v-m(u)}{\sigma_r(u)}, -\infty < y < \infty\}$. Let M be such that $sup_{y,j=1,2}|f_{\varepsilon j}(y)| < M$. Then $N_{[\,]}(\delta, \mathcal{F}_{rj}, L_2(P)) \leq 2M\delta^{-1}$ if $\delta < 2M$, and $N_{[\,]}(\delta, \mathcal{F}_{rj}, L_2(P)) = 1$ if $\delta > 2M$. Where $N_{[\,]}$ is a bracketing number, and the bracketing number $N_{[\,]}(\delta, \mathcal{F}_{rj}, L_2(P))$ is the minimum number of δ-brackets needed to cover $\mathcal{F}_r$, P is the measure of probability corresponding to the joint distribution of (X_r, Y_r), and $L_2(P)$ is L_2 - norm. Therefore,

$$N_{[\,]}(\delta, \mathcal{F}_r, L_2(P)) \leq \Pi_{j=1}^{k+1} N_{[\,]}(\delta, \mathcal{F}_{rj}, L_2(P)), \tag{3}$$

and, by taking logarithms, $logN_{[\,]}(\delta, \mathcal{F}_r, L_2(P)) \leq \sum_{j=1}^{k} logN_{[\,]}(\delta, \mathcal{F}_{rj}, L_2(P))$. Now,

$$\int_0^\infty \sqrt{logN_{[\,]}(\delta, \mathcal{F}_{rj}, L_2(P))}d\delta \leq \sum_{j=1}^{k+1}\int_0^{2M} \sqrt{logN_{[\,]}(\delta, \mathcal{F}_{rj}, L_2(P))}d\delta. \tag{4}$$

From (4), can be conclude that $\int_0^\infty \sqrt{logN_{[\,]}(\delta, \mathcal{F}_{rj}, L_2(P))}d\delta$ is finite, therefore, the class of function $\mathcal{F}_r$ is Donsker class or P-Donsker class, from Donsker-Theorem, if $\mathcal{F}_r$ is Donsker class, it follows that the limit process $\{Gf : f \in \mathcal{F}_r\}$ must be a zero-mean Gaussian process with covariance function $E(Gf_1)(Gf_2)$, from this study $\mathcal{F}_r$- indexed process is $\hat{Z}_r(y)$, then weak convergence of process $\mathcal{F}_r$ must be a zero-mean process with covariance function $Cov(Z_r(y), Z_r(y^{'})) = E[\Psi(X_r, Y_r, y)\Psi(X_r, Y_r, y^{'})]$. From Cramer - Wold device $\hat{U}_r(y)$ is $\hat{Z}_r(y)$ where $b_r = 1$, then

$$\hat{U}_r(y) = f_{\varepsilon j}(y)p_j\sum_{r=1}^{2}\frac{Y_r - m(X_r)}{\sigma_j(X_r)}[\frac{f_{Xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j)}{p_j}] + o_p(n)^{-1/2}. \tag{5}$$

From (5). $\hat{U}(y) = (\hat{U}_1(y)), \hat{U}_2(y)$ converge to $U(y) = (U_1(y), U_2(y))$, where

$$U_j(y) = f_{\varepsilon j}(y)p_j\sum_{r=1}^{2}\frac{Y_r - m(X_r)}{\sigma_j(X_r)}(\frac{f_{Xj}(X_i)}{f_{mix}(X_r)} - \frac{I(r=j)}{p_j})$$

are normal random variables with mean zero and covariance structure :
$Cov(U_j(y), U_{j'}(y)) = f_{\varepsilon j}(y)f_{\varepsilon j'}(y)p_j p_{j'}\sum_{r=1}^{2}E[\frac{(Y_r - m(x_r))^2}{\sigma_j(X_r)\sigma_{j'}(X_r)}(\frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j)}{p_j})(\frac{f_{xj'}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j')}{p'_j})]$.

Corollary 3.4. Assume (A1)-(A6). Then, under the null hypothesis (H_0),

$$U_{KU} \overrightarrow{d} \sum_{j=1}^{2}[sup_y|U_j(y)| - inf_y|U_j(y)|],$$

where $U_{KU} = \sum_{j=1}^{2}[sup_y|\hat{U}_j(y)| - inf_y|\hat{U}_j(y)|]$.

Proof. From theorem 3 and Continuous Mapping Theorem, $U_{KU} \overrightarrow{d} \sum_{j=1}^{2}[sup_y|U_j(y)| - inf_y|U_j(y)|]$, where $U_j(y)$ are the normal random variables with mean zero and covariance structure:
$Cov(U_j(y), U_{j'}(y)) = f_{\varepsilon j}(y)f_{\varepsilon j'}(y)p_jp_{j'}\sum_{r=1}^{2}E[\frac{(Y_r - m(X_r))^2}{\sigma_j(X_r)\sigma_{j'}(X_r)}(\frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j)}{p_j})(\frac{f_{xj'}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j')}{p'_j})]$. Then, $|U_j(y)|$ are a half-normal distribution with mean
$\sqrt{f_{\varepsilon j}(y)f_{\varepsilon j'}(y)p_jp_{j'}\sum_{r=1}^{2}E[\frac{(Y_r - m(X_r))^2}{\sigma_j(X_r)\sigma_{j'}(X_r)}(\frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j)}{p_j})(\frac{f_{xj'}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j')}{p'_j})]}\sqrt{2/\pi}$,
and covariance structure:
$Cov(U_j(y), U_{j'}(y)) = f_{\varepsilon j}(y)f_{\varepsilon j'}(y)p_jp_{j'}\sum_{r=1}^{2}E[\frac{(Y_r - m(X_r))^2}{\sigma_j(X_r)\sigma_{j'}(X_r)}(\frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j)}{p_j})(\frac{f_{xj'}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j')}{p'_j})](1 - \frac{2}{\pi})$.

§4. Bootstrap and simulations

§4.1 Bootstrap procedure

Asymptotic distribution of test statistics U_{KU} given in corollary 3.4 are complicated. In practical applications, the critical values of the test statistic U_{KU} can be approximated by bootstrap procedure. The bootstrap procedure can be described in the following step:

4.1.1. Assume bootstrap replication b= 1,..., B (B = 200), for $j = 1, 2$, $i = 1, \ldots, n_j$, let $\varepsilon^*_{ij,b}$ be an independent and identically distribution sample from the distribution of $(1 - h_j^2)^{1/2}V_j + h_jZ$, where V_j is the distribution of ε_{ij}, $Z \sim N(0, 1)$, $h_j = cn_j^{-3/10}$ and c=1 [10].

4.1.2. For $j = 1, 2$, $i = 1, \ldots, n_j$, the new response under the null hypothesis $Y^*_{ij,b}$, b= 1, ..., B, defined as follow : $Y^*_{ij,b} = \hat{m}_j(X_{ij}) + \hat{\sigma}_j(X_{ij})\varepsilon^*_{ij,b}$.

4.1.3. For $j = 1, 2$, $i = 1, \ldots, n_j$, calculate the test statistic U_{KU} from bootstrap sample $X_{ij}, Y^*_{ij,b}$.

4.1.4. In step 4.1.3, let $U^*_{KU,(b)}$ be order statistic of $U^*_{KU,(1)}, \ldots, U^*_{KU,(B)}$ form 200 bootstrap replications respectively, then $U^*_{KU,(1-\alpha)B}$ approximates the $(1 - \alpha)$ - quantiles of distribution of the test statistic U_{KU} under the null hypothesis.

4.1.5. The test statistic U_{KU} iterates 500 trails and show the proportion of rejections.

§4.2 Simulations

4.2.1. Testing the equality of two regression functions, they are assumed in three types:

1) linear function $Y = ax + b$;

2) exponential function $Y = exp(ax) + b$ and

3) trigonometric function $Y = sin(2\pi x) + b$, where a and b are constants. The models for regression function are six models:

(1) linear function $m_1(x) = m_2(x)$, where $a_1 = a_2 = 2$, $b_1 = b_2 = 0$;

(2) exponential function $m_1(x) = m_2(x)$, where $a_1 = a_2 = 2$, $b_1 = b_2 = 0$;

(3) trigonometric function $m_1(x) = m_2(x)$, where $a_1 = a_2 = 2$, $b_1 = b_2 = 0$;

(4) linear function $m_1(x) = 2x$, $m_2(x) = 2x + 2$;

(5) exponential function $m_1(x) = exp(2x)$, $m_2(x) = exp(2x) + 2$;

(6) trigonometric function $m_1(x) = sin(2\pi x)$, $m_2(x) = sin(2\pi x) + 2$.

4.2.2. The variance of two regression function are $\sigma_1^2(x) = 0.25, \sigma_2^2(x) = 0.50$.

4.2.3. Assume the error distribution are $\varepsilon_{1n1} \sim N(0,1)$ and $\varepsilon_{2n2} \sim N(0,1)$.

4.2.4. In all cases, the covariates X_1, X_2 are uniform distribution on the interval [0,1].

4.2.5. The Kernel Function is Epanechnikov $K(u) = 0.75(1 - U^2)I(|u| < 1)$.

4.2.6. In this paper, we consider bandwidth of the form $h_j = cn_j^{-3/10}$, where c =1.

4.2.7. In each population, the samples sizes are determined $(n_1, n_2) = (20, 20), (50, 50)$ and $(100, 100)$.

§5. Results

From simulation study, we show six simulated examples with two regression functions. The model for regression function are three types: linear function, exponential function, and trigonometric function. Model (1)-(3) correspond to the null hypothesis, and model (4)-(6) correspond to the alternative hypothesis. Table 1 shows the proportion of rejections under the null hypothesis in 500 trails for sample sizes $(n_1, n_2) = (20, 20), (50, 50)$ and $(100, 100)$, B=200 bootstrap replications. The significance level are 0.05 and 0.10. Table 2 shows the proportion of rejections under the alternative hypothesis in 500 trails for sample sizes $(n_1, n_2) = (20, 20), (50, 50)$ and $(100, 100)$, B=200 bootstrap replications. The significance level are 0.05 and 0.10. In the simulation study we find out that the level is well - approximated in $(n_1, n_2) =$ (100,100) and regression function in linear form are good in the most situation. The power of the test based on U_{KU} is well - approximated in $(n_1, n_2) =$ (100,100). The behavior of the power of the test based on U_{KU} is good for models in linear form, exponential form and trigonometric form respectively.

Table 1. Rejection probabilities under the null hypothesis-model (1) to (3).

(n_1, n_2)	regression function	$U^1_{KU} : \alpha = 0.05$	$U^1_{KU} : \alpha = 0.10$
(20,20)	1	0.042	0.080
	2	0.030	0.074
	3	0.034	0.076
(50,50)	1	0.052	0.090
	2	0.054	0.084
	3	0.060	0.080
(100,100)	1	0.050	0.100
	2	0.052	0.102
	3	0.054	0.106

Table 2. Rejection probabilities under the alternative hypothesis-models (4) to (6).

(n_1, n_2)	regression function	$U^1_{KU} : \alpha = 0.05$	$U^1_{KU} : \alpha = 0.10$
(20,20)	4	0.910	0.930
	5	0.890	0.930
	6	0.384	0.450
(50,50)	4	0.930	0.965
	5	0.926	0.960
	6	0.400	0.520
(100,100)	4	1.000	1.000
	5	0.970	0.980
	6	0.546	0.644

§6. Conclusion and discussion

In this study, we introduce a procedure for testing the equality of two expectation function of non-parametric regression, the hypothesis $H_o : m_1 = m_2$ vs $H_1 : m_1 \neq m_2$. The result was

found that, under the null hypothesis (H_o), for $j = 1, 2$, $\hat{U}(y) = (\hat{U}_1(y), \hat{U}_2(y))$ converges weakly to $U(y) = (U_1(y), U_2(y))$, where $U_j(y) = f_{\varepsilon j}(y)p_j \sum_{r=1}^{2} \frac{Y_r - m(X_r)}{\sigma_j(X_r)}(\frac{f_{Xj}(X_i)}{f_{mix}(X_r)} - \frac{I(r=j)}{p_j})$ are normal random variables with mean zero and covariance structure:
$Cov(U_j(y), U_{j'}(y)) = f_{\varepsilon j}(y)f_{\varepsilon j'}(y)p_j p_{j'} \sum_{r=1}^{2} E[\frac{(Y_r - m(X_r))^2}{\sigma_j(X_r)\sigma_{j'}(X_r)}(\frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j)}{p_j})(\frac{f_{xj'}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j')}{p'_j})]$ and the test statistic $U_{KU} \overrightarrow{d} \sum_{j=1}^{2}[sup_y|U_j(y)| - inf_y|U_j(y)|]$, where $|U_j(y)|$ are half-normal distribution with mean:

$$\sqrt{f_{\varepsilon j}(y)f_{\varepsilon j'}(y)p_j p_{j'} \sum_{r=1}^{2} E[\frac{(Y_r - m(X_r))^2}{\sigma_j(X_r)\sigma_{j'}(X_r)}(\frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j)}{p_j})(\frac{f_{xj'}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j')}{p'_j})]\sqrt{2/\pi}},$$

and covariance structure:
$Cov(U_j(y), U_{j'}(y)) = f_{\varepsilon j}(y)f_{\varepsilon j'}(y)p_j p_{j'} \sum_{r=1}^{2} E[\frac{(Y_r - m(X_r))^2}{\sigma_j(X_r)\sigma_{j'}(X_r)}(\frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j)}{p_j})(\frac{f_{xj'}(X_r)}{f_{mix}(X_r)} - \frac{I(r=j')}{p'_j})](1 - \frac{2}{\pi})$

The result from the simulation study showed that under the null hypothesis, the regression function in linear form are well approximated in all situation, especially for the large sample sizes. The power of the test is well - approximated for the large sample sizes, and the regression function in linear form provided the highest power of the test.

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The existence of solution for $p(x)$-Laplacian equation with no flux boundary [1]

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Abstract Using the principle least action, we investigate the existence solution for $p(x)$-Lapla -cian equation with no flux boundary condition in a bounded domain $\Omega \subset R^N$, where no flux boundary condition is given in the following:

$$\begin{cases} u = constant, & x \in \partial\Omega, \\ \int_{\partial\Omega} |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \eta} ds = 0. \end{cases}$$

Keywords Variable exponent Sobolev spaces, $p(x)$-Laplacian, no flux boundary, the princi -ple least action.

§1.Introduction

In recent years there has been an increasing interest in the study of various mathematics problem with variable exponent, see the papers [1,2,3,5]. The existence of solutions of $p(x)$-Laplacian Dirichlet problems has been studied in many papers (see e.g. [7, 8, 9, 12, 14, 15]). The aim of the present paper is to study the existence of solutions of $p(x)-$Laplacian equation with no flux boundary. Where no flux boundary condition is given in the following:

$$\begin{cases} u = constant, & x \in \partial\Omega, \\ \int_{\partial\Omega} |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \eta} ds = 0. \end{cases}$$

Throughout the paper, Ω will be bounded domain in R^N, $p \in C(\bar{\Omega})$ and

$$1 < p^- := \inf_{x\in\Omega} p(x) \le p^+ := \sup_{x\in\Omega} p(x) < \infty.$$

Consider the existence of solution of the following problem

$$\begin{cases} -\triangle_{p(x)} u + f(x,u) = 0, & x \in \Omega, \\ u(x) = constant, & x \in \Omega, \\ \int_{\partial\Omega} |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \eta} ds = 0. \end{cases} \tag{1}$$

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When a term of $|u|^{p(x)-2}u$ is involved in $p(x)-$Laplacian equation, we can apply variation method to obtain the existence of solution and multiplicity easily. In this paper, we may use the principle least action to get the existence of solution when f satisfies some appropriate conditions.

§2. Preliminaries

Let Ω be an open subset of $\mathbb{R}^N$. On the basic properties of the space $W^{1,p(x)}(\Omega)$ we refer to [4, 13]. In the following we display some facts which we will use later.

Denote by $S(\Omega)$ the set of all measurable real functions defined on Ω, and elements in $S(\Omega)$ that equal to each other almost everywhere are considered as one element. Denote $L^{\infty}_{+}(\Omega) = \{p \in L^{\infty}(\Omega) : \operatorname{ess\,inf}_{\Omega} \mathrm{p(x)} := \mathrm{p}_{-} \geq 1\}$.

For $p \in L^{\infty}_{+}(\Omega)$, define

$$L^{p(x)}(\Omega) = \{u \in S(\Omega) : \int_{\Omega} |u|^{p(x)} \mathrm{d}x < \infty\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_{\Omega} |u/\lambda|^{p(x)} \mathrm{d}x \leq 1\};$$

and define

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Define

$$p^* = \begin{cases} \frac{Np(x)}{N-p(x)}, \ if \ p(x) < N, \\ \infty, \ \ if \ p(x) \geq N. \end{cases}$$

Hereafter, we always assume that $p(x)$ is continuous and $p_{-} > 1$.

Proposition 2.1.[4,13] The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, all are separable and reflexive Banach spaces.

Proposition 2.2.[4,10,13] The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p^o(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p^o(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p^o(x)}(\Omega)$, the Hölder inequality holds:

$$\int_{\Omega} |uv| \mathrm{d}x \leq 2|u|_{p(x)}|v|_{p^o(x)}. \tag{2}$$

Remark 2.3. In the right of (2), the constants 2 is suitable, but not the best. The best constant is given in [10] denoted by $d_{(p_-,p_+)}$ which only depends on p_- and p_+ when $p(x)$ is given and $d_{(p_-,p_+)}$ is smaller than $\frac{1}{p_-} + \frac{1}{p_+}$.

Proposition 2.4.[13] (Theorem 1.3.) Set $\rho(u) = \int_{\Omega} |u(x)|^{p(x)} \mathrm{d}x$. For $u, u_k \in L^{p(x)}(\Omega)$, we have:

(i) $|u|_{p(x)} < 1 \ (= 1; > 1) \iff \rho(u) < 1 \ (= 1; > 1)$;

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$; $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$;

(iii) $\lim_{k\to\infty} |u_k|_{p(x)} = 0\ (=\infty) \iff \lim_{k\to\infty} \rho(u_k) = 0\ (=\infty)$.

Proposition 2.5.[11] (Theorem 1.1.) If p: $\Omega \to \mathbb{R}$ is Lipschitz continuous and $p_+ < N$, then for $q \in L_+^\infty(\Omega)$ with $p(x) \leq q(x) \leq p^*(x)$, there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Proposition 2.6.[6] (Proposition 2.4.) Ω is bounded. Assume that the boundary of Ω possesses the cone property and $p \in C(\bar{\Omega})$. If $q \in C(\bar{\Omega})$ and $1 \leq q(x) < p^*(x)$ for $x \in \bar{\Omega}$, then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

In this paper we use space $X := \{u \in W^{1,p(x)}(\Omega) : u|_{\partial\Omega} = \text{constant}\} = \mathrm{W}_0^{1,\mathrm{p(x)}}(\Omega) \oplus \mathrm{R}$. X is a subspace of $W^{1,p(x)}(\Omega)$. The space X is also separable and reflexive Banach space and with the equivalent norm

$$\|u\| = \|u\|_X = |\bar{u}| + |\nabla\tilde{u}|_{p(x)}, u \in X.$$

where $u = \bar{u} + \tilde{u}, \bar{u} \in R, \tilde{u} \in W_0^{1,p(x)}(\Omega)$.

Definition 2.7. We call $u \in X$ is a weak solution of (1), if u satisfies

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v dx + \int_\Omega |u|^{p(x)-2} uv dx = \int_\Omega f(x,u)v dx,\ \forall v \in X.$$

In this paper, we always suppose f satisfies the following basic assumption:

Basic assumption. Suppose f satisfies *Carathéodory*, and

$$(F_0)\quad |f(x,t)| \leq \alpha + \beta|t|^{q(x)-1}, 1 < q(x) < P^*(x).$$

Consider the following function:

$$I(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx + \int_\Omega F(x,u)dx, u \in X,$$

where $F(x,t) = \int_0^t f(x,s)ds$. We obtain $I \in C^1(X,R)$.

The main result of this paper is given by the following theorem.

Theorem 2.8. Suppose (F_0) holds, I has a bounded minimizing sequence. Then I has a minimizer.

Proof. Set $\{u_n\}$ is a minimizing sequence of I, namely, $I(u_n) \to \inf_{n\to\infty} I(u)$, $\|u_n\|$ is bounded. Suppose $u_n \rightharpoonup u_0$ in X, obviously, I is weakly lower semicontinuous, thus we have $\inf_{u\in X} = \lim_{n\to\infty} I(u_n) \geq I(u_0)$, it follows that $I(u_0) = \inf_{u\in X} I(u)$.

§3. Main result

Theorem 3.1. Suppose $\alpha \in R$, and $0 \leq \alpha < p^- - 1$, for any $x \in \Omega$, $t \in R$. And the following assumptions hold:

$$|f(x,t)| \leq C_1 + C_2|t|^\alpha. \tag{3}$$

$$|t|^{-\frac{\alpha p^-}{p^- - 1}} \int_\Omega F(x,t)dx \to +\infty \quad if\ |t| \to \infty. \tag{4}$$

Then I has a minimizer.

Proof. Clearly, I is weakly lower semicontinuous. We shall prove I is coercive.

$$I(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega (F(x,u(x)) - F(x,\bar{u}))dx + \int_\Omega F(x,\bar{u})dx.$$

where

$$\begin{aligned}
\int_\Omega (F(x,u(x)) - F(x,\bar{u}))dx &= |\int_\Omega (\int_0^1 f(x,\bar{u}+s\tilde{u}(x))\tilde{u}(x)ds)dx| \\
&\leq \int_\Omega (C_1 + C_3|\bar{u}|^\alpha + C_3|\tilde{u}(x)|^\alpha)|\tilde{u}(x)|dx \\
&\leq \int_\Omega C_1|\tilde{u}(x)|dx + \int_\Omega C_3|\bar{u}|^\alpha|\tilde{u}(x)|dx + \int_\Omega C_3|\tilde{u}(x)|^{\alpha+1}dx \\
&\triangleq \mathbf{I} + \mathbf{II} + \mathbf{III}.
\end{aligned}$$

We next estimate forms $\mathbf{I}, \mathbf{II}, \mathbf{III}$. Where

$$\mathbf{I} = C_4|\tilde{u}|_{L^1(\Omega)} \leq C_5|\tilde{u}|_{L^{p(x)}(\Omega)} \leq C_6|\nabla\tilde{u}|_{L^{p(x)}(\Omega)}. \tag{5}$$

$$\mathbf{III} = C_3|\tilde{u}|^{\alpha+1}_{L^{\alpha+1}} \leq C_7|\tilde{u}|^{\alpha+1}_{L^{p(x)}(\Omega)} \leq C_8|\nabla\tilde{u}|^{\alpha+1}_{L^{p(x)}(\Omega)}. \tag{6}$$

According to Young inequality, we deduce

$$\begin{aligned}
\mathbf{II} &= C_3\int_\Omega |\bar{u}|^\alpha|\tilde{u}(x)|dx \\
&\leq C_3\int_\Omega (\varepsilon|\tilde{u}(x)|^{p^-} + C_9(\varepsilon)|\bar{u}|^{\frac{\alpha p^-}{p^- - 1}})dx \\
&\leq \varepsilon C_{10}|\nabla\tilde{u}|^{p^-}_{L^{p(x)}(\Omega)} + C_3C_9(\varepsilon)|\Omega||\bar{u}|^{\frac{\alpha p^-}{p^- - 1}}.
\end{aligned}$$

Fix $\varepsilon > 0$ so small that $\varepsilon C_{10} \leq \frac{1}{2p^+}$. we conclude that

$$\mathbf{II} \leq \frac{1}{2p^+}|\nabla\tilde{u}|^{p^-}_{L^{p(x)}(\Omega)} + C_{11}|\bar{u}|^{\frac{\alpha p^-}{p^- - 1}}. \tag{7}$$

By (3), (4) and (5),we have

$$\begin{aligned}
I(u) &\geq \frac{1}{p^+}|\nabla\tilde{u}|^{p^-}_{L^{p(x)}(\Omega)} - \frac{1}{p^+} - C_6|\nabla\tilde{u}|_{L^{p(x)}(\Omega)} - \frac{1}{2p^+}|\nabla\tilde{u}|^{p^-}_{L^{p(x)}(\Omega)} - C_{11}|\bar{u}|^{\frac{\alpha p^-}{p^- - 1}} \\
&\quad - C_8|\nabla\tilde{u}|^{\alpha+1}_{L^{p(x)}(\Omega)} + \int_\Omega F(x,\bar{u})dx \\
&\geq C_{10}|\nabla\tilde{u}|^{p^-}_{L^{p(x)}(\Omega)} - C_{12} + \int_\Omega F(x,\bar{u})dx - C_{11}|\bar{u}|^{\frac{\alpha p^-}{p^- - 1}} \\
&\geq C_{10}|\nabla\tilde{u}|^{p^-}_{L^{p(x)}(\Omega)} - C_{12} + |\bar{u}|^{\frac{\alpha p^-}{p^- - 1}}(|\bar{u}|^{\frac{\alpha p^-}{p^- - 1}}(\int_\Omega F(x,\bar{u})dx - C_{11})).
\end{aligned} \tag{8}$$

Using (1) and (6) we get I is coercive, thus I has a minimizer.

Given $\alpha = 0$ in theorem 3.1, we have

Corollary 3.2. Suppose f satisfies

$$|f(x,t)| \leq C_1. \tag{9}$$

$$\int_\Omega F(x,t)dx \to \infty,\ if\ |t| \to \infty. \tag{10}$$

Then I has a minimizer.

Theorem 3.3. Suppose $F(x,t)$ is convex and $\int_\Omega F(x,t)dx \to \infty$ when $|t| \to \infty$. Then I has a minimizer.

Proof. Set $g(t) = \int_\Omega F(x,t)dx$, for any $t \in R$. Obviously $g : R \to R$ is a differentiable and convex function, and when $|t| \to \infty$, $g(t) \to +\infty$. We deduce g has a minimizer t_*, thus $g'(t_*) = 0$. Namely $\int_\Omega f(x,t_*)dx = 0$. Suppose $\{u_n\}$ is a minimizing sequence of I, according to the proposition of $F(x,t)$, we have

$$\begin{aligned} I(u_n) &\geq \frac{1}{p^+}\int_\Omega |\nabla \tilde{u}_n|^{p(x)}dx + \int_\Omega (F(x,t_*) + f(x,t_*))(u_n(x) - t_*)dx \\ &\geq \frac{1}{p^+}\int_\Omega |\nabla \tilde{u}_n|^{p(x)}dx - C_1 + \int_\Omega f(x,t_*)\tilde{u}_n(x)dx \\ &\geq \frac{1}{p^+}\int_\Omega |\nabla \tilde{u}_n|^{p(x)}dx - C_1 - C_2|\nabla \tilde{u}_n|_{L^{p(x)}(\Omega)}. \end{aligned} \tag{11}$$

We derive that $|\nabla \tilde{u}_n|_{L^{p(x)}(\Omega)}$ is bounded from (9). Next we proof $|\bar{u}_n|_{L^{p(x)}(\Omega)}$ is also bounded. Owing now to the convexity of F, we deduce that

$$F(x, \frac{\bar{u}_n}{2}) = F(x, \frac{u_n(x) - \tilde{u}_n(x)}{2}) \leq \frac{1}{2}(F(x,u_n(x)) + F(x,-\tilde{u}_n(x))). \tag{12}$$

From (10), we have

$$\begin{aligned} I(u_n) &\geq \int_\Omega F(x,u_n)dx \geq 2\int_\Omega F(x,\frac{\bar{u}_n}{2})dx - \int_\Omega F(X,-\tilde{u}_n(x))dx \\ &\geq 2\int_\Omega f(x,\frac{\bar{u}_n}{2})dx - \int_\Omega (C_{12} + C_{13}|\bar{u}_n(x)|^{q(x)})dx \\ &\geq 2\int_\Omega F(x,\frac{\bar{u}_n}{2})dx - C_{12}|\Omega| - C_{14}|\nabla \tilde{u}_n|^{p^+}_{L^{p(x)}(\Omega)} - C_{15} \\ &\geq 2\int_\Omega F(x,\frac{\bar{u}_n}{2})dx - C_{16}. \end{aligned} \tag{13}$$

We get that $|\bar{u}_n|_{L^{p(x)}(\Omega)}$ is bounded, thus $\|u_n\|$ bounded. Consequently, I has a minimizer.

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The rainbow k-connectivity of random graphs

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Abstract A path in an edge-colored graph (not necessarily a proper coloring) is a rainbow path if no two edges of it are colored the same. For an l-connected graph G and $1 \leq k \leq l$, the rainbow k-connectivity $rc_k(G)$ of G is the minimum integer j for which there exists a j-edge-coloring of G such that every two different vertices in G are connected by k independent rainbow paths. We prove that, in a random graph $G(n,p)$, $p = \sqrt{\ln n/n}$ is a sharp threshold function for the property $rc_k(G(n,p)) \leq 2$ when $k = O(\ln \ln n)$.

Keywords Rainbow connectivity, rainbow connection, graph coloring, random graph.

§1. Introduction

In this note, we use [1] for graph theory terminology not described here and consider only undirected simple graphs. Let G be a nontrivial connected graph on which is defined an edge-coloring $f : E(G) \to \{1, 2, \cdots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. A path in G is a rainbow path if no two edges of it are colored the same. The graph G is rainbow-connected (with respect to f) if every two different vertices in G are connected by a rainbow path [3]. The minimum integer k for which there exists a k-coloring of the edges of G that results in a rainbow-connected graph is called the rainbow connection number $rc(G)$ of G.

For two different vertices in a connected graph G, a rainbow geodesic in G is a rainbow path which has the length of a shortest path between them in G [3]. The graph G is strongly rainbow-connected if every two different vertices in G are connected by a rainbow geodesic. The minimum k for which there exists a k-coloring of the edges of G such that G is strongly rainbow-connected is the strong rainbow connection number $src(G)$ of G. Clearly, $diam(G) \leq rc(G) \leq src(G)$ for every connected graph G, where $diam(G)$ denotes the diameter of G. Also note that $rc(G) = 1$ if and only if G is a complete graph.

The authors of [4] introduce a concept of rainbow connectivity recently. Suppose that G is an l-connected graph for $l \geq 1$. For $1 \leq k \leq l$, the rainbow k-connectivity $rc_k(G)$ of G is the minimum integer i for which there exists an i-coloring of the edges of G such that for every two different vertices of G, there is at least k independent rainbow paths between them. Note that Menger's Theorem justifies the definition (see e.g. [1] pp.52) and $rc(G) = rc_1(G)$. Moreover, $rc_k(G) \leq rc_j(G)$ for $1 \leq k \leq j \leq l$.

Observe that $rc_k(G)$ is a monotonic property in the sense that if we add an edge to G we cannot increase its rainbow connectivity. Hence, it is natural to explore the random graph setting [2,7,8]. Motivating this idea, in this paper we mainly consider the rainbow connectivity in Erdős-Rényi random graph model $G(n,p)$ with n vertices and edge probability $p \in [0,1]$. A function $f(n)$ is called a sharp threshold function for a graph property $\mathcal{A}$ if there are two positive constants C and c such that $G(n,p)$ satisfies $\mathcal{A}$ almost surely for $p \geq Cf(n)$ and $G(n,p)$ almost surely does not satisfy $\mathcal{A}$ for $p \leq cf(n)$. A remarkable feature of random graphs is that all monotone graph properties have sharp thresholds (see e.g. [5, 6]).

All the logarithms in this paper have natural base. Let $\delta(G)$ be the minimum degree of graph G. We establish the following results:

Theorem 1.1. Let $k \in \mathbb{N}$ and $k = O(\ln\ln n)$. Then, any non-complete graph G on n vertices with $\delta(G) \geq n/2 + (3/2\ln 2)\ln n + O(\ln\ln n)$ has $rc_k(G) = 2$.

Theorem 1.2. Let $k \in \mathbb{N}$ and $k = O(\ln\ln n)$. Then, $p = \sqrt{\ln n/n}$ is a sharp threshold function for the property $rc_k(G(n,p)) \leq 2$.

The rest of the paper is organized as follows. In section 2, we provide the proof of theorem 1.1 and 1.2. In section 3, we give a remark on the strong rainbow connection number $src(G)$ in a random graph.

§2. Proofs

In this section, we show theorem 1.1 and theorem 1.2 with similar reasoning of [2]. Theorem 1.1 is a deterministic result which is proved in a probabilistic argument.

Proof of Theorem 1.1. For a graph G on n vertices and $\delta(G) \geq n/2 + (3/2\ln 2)\ln n + O(\ln\ln n)$, we randomly color the edges with two colors, red and blue. We will show that with positive probability, such a random coloring makes G rainbow k-connected for all $k = O(\ln\ln n)$.

Given two vertices x and y in G, the minimum degree requirement forces x and y to have more than $(3/\ln 2)\ln n + O(\ln\ln n)$ common neighbors whether x and y are adjacent or not. Hence, the graph G is $\big[(3/\ln 2)\ln n + O(\ln\ln n)\big]$-connected by Menger's Theorem [1]. Thus, we can well define rainbow connectivity $rc_k(G)$ when $k = O(\ln\ln n)$.

Now, for each such common neighbor z, the probability that the path x, z, y is not a rainbow path is $1/2$. Since the paths corresponding to distinct common neighbors are independent, the probability that there are at least $(3/\ln 2)\ln n$ paths of them being not rainbow path is bounded above by

$$\left(\frac{1}{2}\right)^{\frac{3\ln n}{\ln 2}} \cdot \binom{\frac{3\ln n}{\ln 2} + O(\ln\ln n)}{O(\ln\ln n)} \leq \frac{1}{n^3} \cdot n = n^{-2} \tag{1}$$

involving the useful bound $\binom{n}{m} \leq (en/m)^m$, valid for all positive n and m.

Since there are $\binom{n}{2}$ pairs x, y to consider, it follows from the union bound and (1) that with probability more than $1/2$, each pair of vertices in G are connected by $O(\ln\ln n)$ independent rainbow paths.

Proof of Theorem 1.2. Recall the definition of sharp threshold function in section 1. The proof consists of two parts.

For the first part of the theorem, we want to prove that for a sufficiently large constant C, the random graph $G(n,p)$ with $p = C\sqrt{\ln n/n}$ almost surely has $rc_k(G(n,p)) \leq 2$. From the proof of theorem 1.1, we only need to show that almost surely any two vertices in $G(n,p)$ have at least $(3/\ln 2)\ln n + O(\ln\ln n)$ common neighbors.

Fix a pair of vertices x, y, and the probability that z is a common neighbor of them is $C^2 \ln n/n$. Let random variable X represents the number of common neighbors of x and y. Consequently, we obtain $EX = (n-2)(C^2 \ln n/n)$. By using the Chernoff bound (e.g. [7] pp.26), for large enough C, we have

$$P\big(X < (3/\ln 2)\ln n + O(\ln\ln n)\big) \leq P\Big(X < EX - \frac{C^2 \ln n}{4}\Big) \leq e^{-\frac{C^2 \ln n}{32}} = o(n^{-2}).$$

Since there are $\binom{n}{2}$ pairs of vertices in $G(n,p)$, the union bound readily yields the result.

For the other direction, it suffices to show that for a sufficiently small constant c, the random graph $G(n,p)$ with $p = c\sqrt{\ln n/n}$ almost surely has $diam(G(n,p)) \geq 3$. Fix a set X of $n^{1/6}$ vertices, and let Y be the remaining $n - n^{1/6}$ vertices. Let $\mathcal{A}$ be the event that X induces an independent set. Let $\mathcal{B}$ be the event that there exists a pair of vertices $x, y \in X$ with no common neighbor in Y. Therefore, it suffices to prove that

(i) $P(\mathcal{A}) \to 1$;

(ii)$P(\mathcal{B}) \to 1$, as $n \to \infty$.

For (i): For c sufficiently small we obtain

$$\begin{aligned} P(\mathcal{A}) &= (1-p)^{\binom{n^{1/6}}{2}} = (1 - c\sqrt{\ln n/n})^{\binom{n^{1/6}}{2}} \\ &\sim e^{-\frac{c\sqrt{\ln n}}{2n^{1/6}}} \to 1, \end{aligned}$$

as $n \to \infty$.

For (ii): For a pair $x, y \in X$, the probability that x, y have a common neighbor in Y is shown to be given by

$$1 - \Big(1 - \frac{c^2 \ln n}{n}\Big)^{n - n^{1/6}} \sim (1 - n^{-c^2}).$$

Since the vertex set X can be divided into $n^{1/6}/2$ pairs, the probability that all $n^{1/6}/2$ pairs have a common neighbor is

$$1 - P(\mathcal{B}) = \Big(1 - \Big(1 - \frac{c^2 \ln n}{n}\Big)^{n - n^{1/6}}\Big)^{n^{1/6}/2} \sim \big(1 - n^{-c^2}\big)^{n^{1/6}/2} \sim e^{-\frac{n^{1/6}}{2n^{c^2}}}. \tag{2}$$

For sufficiently small c, the right hand side of (2) tends to zero, which thus completes the proof.

§3. A remark on $src(G(n,p))$

We give a remark on the strong rainbow connection number $src(G(n,p))$ of random graph $G(n,p)$.

Observe that $src(G)$ is also a monotonic property in the sense that if we add an edge to G we cannot increase its strong rainbow connection number. We may establish the following result regarding the threshold function of $src(G(n,p))$.

Theorem 3.1. $p = \sqrt{\ln n/n}$ is a sharp threshold function for the property $src(G(n,p)) \leq 2$.

Proof. Recall that $diam(G) \leq rc_1(G) \leq src(G)$. The proof of theorem 1.1 and 1.2 in conjunction with proposition 1.1 in [3] yields the result directly.

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The power-serieswise Armendariz rings with nilpotent subsets

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Abstract In this note, a new kind of power-serieswise Armendariz ring with nilpotent subsets was obtained. Necessary and sufficient condition for some rings without identity to be power-serieswise Armendariz are also showed.

Keywords Armendariz ring, power-serieswise Armendariz ring, ring with nilpotent subsets.

§1. Introduction

Throughout this paper R denotes an associative ring. M. B. Rege and S. Chhawchharia introduced the notion of an Armendariz ring. They defined a ring R to be an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. The name "Armendariz ring" was chosen because E. Armendariz had noted that a reduced ring satisfies this condition. After that, more comprehensive study of this kind of ring was carried out (see Anderson and Camillo, 1998; Hirano, 2002; Huh et al., 2002; Kim and Lee, 2000; Lee and Wong, 2003; Rege and Chhawchharia, 1997). Various known works on Armendariz rings deal with the Armendariz property of some extensions of an Armendariz ring R, such as $R[x]$, $R[x]/(x^n)$, the trivial extensions of $R \propto M$ where M is a bimodule over R, and the classical quotient ring $Q(R)$ (where R satisfies the right Ore condition). Some Armendariz subrings of the matrix ring were studied in Lee and Zhou (2004), Wang (2006) and Yan (2003). Recently, the concept of Armendariz ring was connected with power series ring by Kim et al. (2006). They defined a ring R to be a power-serieswise Armendariz ring if $a_ib_j = 0$ for all i, j whenever power series $f(x) = \sum_{i=0}^{\infty} a_ix^i$, $g(x) = \sum_{j=0}^{\infty} b_jx^j$ in $R[[x]]$ satisfy $f(x)g(x) = 0$. They gave two kinds of power-serieswise Armendariz rings, one is the reduced ring and the other is not. The structure of the set of nilpotent elements in Armendariz rings and the concept of nil-Armendariz as a generalization were introduced by Antoine (2008). In this note we will give a new kind of power-serieswise Armendariz ring which is not reduced. The connection and difference between the three kinds of power-serieswise Armendariz rings are also showed. Some rings which are not Armendariz when they have identity are also considered. Necessary and sufficient condition for this ring without identity to be power-serieswise Armendariz are obtained.

§2. A new kind of power-serieswise Armendariz ring

Reduced ring is one of the earliest and most important Armendariz ring. With the development of study, the Armendariz rings with nilpotent subsets catched more and more one's attention. The connection between power-serieswise Armendariz ring and ring with nilpotent subsets is showed below:

Proposition 2.1. Let R be a ring. If $R^2 = 0$, then R is power-serieswise Armendariz.

Proposition 2.2. Let R be a power-serieswise Armendariz ring, then one of the following statements holds:

1) R is reduced;

2) R is not reduced and has a non-zero ideal I with $I^2 = 0$.

Proof. If R is not reduced, let $I = \{a \in R | \, a^2 = 0\}$, Anderson and Camillo (1998) showed I is a non-zero ideal of R and $I^2 = 0$.

Kim et al. (2006) proposition 3.7 showed that: R be a ring and J be an ideal of R such that every element in $R \setminus J$ is regular and $J^2 = 0$, then R is a power-serieswise Armendariz ring with nilpotent subsets. We give an equivalent characterization of this ring.

Proposition 2.3. Let R be a ring. Then R satisfies Kim et al. (2006) proposition 3.7 if and only if there is an ideal I of R satisfies:

(1) $I^2 = 0$;

(2) For non-zero elements x_1, x_2, x_3, x_4 in R, if $x_1x_2 = 0$, $x_1x_4 + x_3x_2 = 0$ and $x_3x_4 \in I$, then x_1, x_2, x_3, x_4 are all in I.

Proof. Suppose R satisfies Kim et al. (2006) proposition 3.7, let $I = J$, where J is the ideal in Kim et al. (2006) proposition 3.7. Then (1) holds. For non-zero elements x_1, x_2, x_3, x_4 in R, if $x_1x_2 = 0$, then x_1 and x_2 are in I. If $x_3x_4 = 0$, then x_3 and x_4 are in I. Assume to the contrary that $x_3x_4 \neq 0$. If $x_3x_4x_3 = 0$, then $x_3 \in I$ and $x_1x_4 + x_3x_2 = x_1x_4 = 0$, so $x_4 \in I$, which is a contradiction. If $x_3x_4x_3 \neq 0$, then $x_3x_4x_3x_4 = 0$ implies $x_4 \in I$. Then $x_1x_4 + x_3x_2 = x_3x_2 = 0$ implies $x_3 \in I$, which is a contradiction.

Conversely, let $J = I$, then $J^2 = 0$. We show that $R \setminus J$ is regular. For a in $R \setminus J$, assume to the contrary there is $0 \neq b$ in R such that $ab = 0$. Let $x_1 = a$, $x_2 = b$, $x_3 = -a$, $x_4 = b$, then (2) implies that a and b are in J, which is a contradiction. By the same method we obtain a is not a right divisor of zero.

We now give another kind of power-serieswise Armendariz ring:

Proposition 2.4. If a non-empty subset A of R satisfies:

1) $A^2 = 0$;

2) If $ab = 0$ for a, b in R, then at least one of the following statements holds:

   i) $a \in A$ and $aR = 0$;

   ii) $b \in A$ and $Rb = 0$.

then R is a power-serieswise Armendariz ring.

Proof. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=0}^{\infty} b_j x^j$ be in $R[[x]]$ and $fg = 0$, then $a_0b_0 = 0$. By hypothesis,

if $a_0 \in A$, $b_0 \notin A$, then $a_0 b_j = 0$ and

$$0 = f(x)g(x) = (\sum_{i=1}^{\infty} a_i x^i)(\sum_{j=0}^{\infty} b_j x^j).$$

if $a_0 \notin A$, $b_0 \in A$, then $a_i b_0 = 0$ and

$$0 = f(x)g(x) = (\sum_{i=0}^{\infty} a_i x^i)(\sum_{j=1}^{\infty} b_j x^j).$$

if $a_0 \in A$, $b_0 \in A$, then $a_0 b_j = 0$, $a_i b_0 = 0$ and

$$0 = f(x)g(x) = (\sum_{i=1}^{\infty} a_i x^i)(\sum_{j=1}^{\infty} b_j x^j).$$

By the same method we obtain $a_i b_j = 0$ for all i, j.

We write $M_n(R)$ and $T_n(R)$ for the $n \times n$ matrix ring and the $n \times n$ upper triangular matrix ring over R. The $n \times n$ identity matrix is denoted by I_n.

Example 2.5. Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\} \subseteq T_2(\mathbb{Z}_2)$. Then R is a subring of $T_2(\mathbb{Z}_2)$. Let $A = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$, then A is the nonempty subset of R in proposition 2.4, so R is power-serieswise Armendariz. R is not reduced and the only non-zero ideal of R is A. $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in R \backslash A$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$ imply that R doesn't satisfy Kim et al. (2006) proposition 3.7.

Example 2.6. $\mathbb{Z}_4$ is not reduced and the nonempty nilpotent subsets are $\{[0]\}$ and $I = \{[0], [2]\}$. [2][2]=0 and [2][3] $\neq 0$ imply that $\mathbb{Z}_4$ doesn't satisfy proposition 2.4. But I is the non-zero ideal of R in Kim et al. (2006) proposition 3.7.

The two examples above can be applied to show the difference between reduced ring, the ring in Proposition 2.4 and the ring in Kim et al (2006) proposition 3.7. We now consider the connection between them. It is clear that the only nonempty nilpotent subset in a reduced ring is $\{0\}$. A reduced ring satisfied proposition 2.4 is the ring without divisors; A reduced ring R satisfied Kim et al. (2006) proposition 3.7 is also the ring without divisors; A ring R satisfied both proposition 2.3 and proposition 2.4 is either the ring without zero divisors or the ring with $R^2 = 0$.

§3. Power-serieswise Armendariz extension rings

In this section, we study the power-serieswise Armendariz property of some extensions of the ring R with $R^2 = 0$. Kim and Lee (2000) example 1 showed that if R has identity, then $T_2(R)$ are not Armendariz (hence not power-serieswise Armendariz). But for a ring without identity, we have:

Proposition 3.1. Let R be a ring without identity, then $T_2(R)$ is power-serieswise Armendariz if and only if $R^2 = 0$.

Proof. Suppose that $T_2(R)$ is power-serieswise Armendariz. Assume to the contrary that $R^2 \neq 0$. Then there exists a, b in R such that $ab \neq 0$.

Let $f(x) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix} x$, $g(x) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix} x$. Then $f(x)$ and $g(x)$ are in $T_2(R)[x]$ and $f(x)g(x) = 0$. But $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix} \neq 0$, which is a contradiction, therefore $R^2 = 0$.

Conversely, suppose that $R^2 = 0$. Then $(T_2(R))^2 = 0$ and the result follows from proposition 2.1.

Some notation we will use below can be found in Lee and Zhou (2004). They showed that for a ring R with identity, $B_n^e(R)$ is not an Armendariz ring for $n = 2k \geq 2$, $B_n^o(R)$ is not an Armendariz ring for $n = 2k + 1 \geq 3$, $B_n(R)$ is not an Armendariz ring for $n \geq 2$. We have:

Proposition 3.2. Let R be a ring without identity. The following hold:

(1) For $n = 2k \geq 2$, $B_n^e(R)$ is a power-serieswise Armendariz ring if and only if $R^2 = 0$.

(2) For $n = 2k+1 \geq 3$, $B_n^o(R)$ is a power-serieswise Armendariz ring if and only if $R^2 = 0$.

(3) For $n \geq 2$, $B_n(R)$ is a power-serieswise Armendariz ring if and only if $R^2 = 0$.

Proof. Suppose $R^2 = 0$ then the result is clear.

Conversely, assume to the contrary that $R^2 \neq 0$, then there exists a, b in R such that $ab \neq 0$ and

$$[aE_{1,k} + (aE_{1,k} - aE_{1,k+1})x][bE_{k+1,n} + (bE_{k,n} + bE_{k+1,n})x] = 0,$$

$$[aE_{1,k+1} + (aE_{1,k+1} - aE_{1,k+2})x][bE_{k+2,n} + (bE_{k+1,n} + bE_{k+2,n})x] = 0.$$

So (1) and (2) hold. (3) can be implied by (1) and (2).

Lee and Zhou (2004) theorem 1.4 showed that if R is a reduced ring, then $A_n(R)$ is an Armendariz ring for every $n = 2k + 1 \geq 3$. We have:

Theorem 3.3. Let R be a ring without identity. The following hold:

(1) For $n = 2k \geq 4$ and positive integer i, j satisfied $k \geq i \geq 1$, $k \geq j \geq i$ and $i \neq 1$ when $j = k$, any subring of $M_n(R)$ containing $A_n(R) + RE_{ij}$ is power-serieswise Armendariz if and only if $R^2 = 0$.

For $n = 2k+1 \geq 3$ and positive integer i, j satisfied $k+1 \geq i \geq 1$, $k+1 \geq j \geq i$ and $i \neq 1$ when $j = k+1$, any subring of $M_n(R)$ containing $A_n(R)+RE_{ij}$ is power-serieswise Armendariz if and only if $R^2 = 0$.

(2) If $n \geq 3$ is an even number, let $n = 2k$; If $n \geq 3$ is an odd number, let $n = 2k + 1$. For positive integer i, j satisfied $n \geq i \geq k + 1$, $n \geq j \geq i$ and $i \neq k + 1$ when $j = n$, any subring of $M_n(R)$ containing $A_n(R) + RE_{ij}$ is power-serieswise Armendariz if and only if $R^2 = 0$.

Proof. (1) Let a, b be in R. When $i = 1$, $n = 2k \geq 4$, we have

$$[aE_{1j} + (aE_{1j} - aV^{k-1})x][bE_{kn} + (bE_{kn} + bE_{jn})x] = 0.$$

When $i = 1$, $n = 2k + 1 \geq 3$, we have

$$[aE_{1j} + (aE_{1j} - aE_{1,k+1})x][bE_{k+1,n} + (bE_{k+1,n} + bE_{jn})x] = 0.$$

When $i > 1$, we have

$$[(aV^{j-i} - aE_{ij}) + (aV^{j-i} - aE_{ij} - aV^{j-1})x][bE_{jn} + (bE_{jn} + bE_{j-i+1,n})x] = 0.$$

Then the result is clear.

(2) When $j = n$, we have

$$[aE_{1,k+1} + (aE_{1,k+1} - aE_{1i})x][bE_{in} + (bE_{in} + bE_{k+1,n})x] = 0.$$

When $j < n$, we have

$$[aE_{1i} + (aE_{1i} - aE_{1,n-j+i})x][(bV^{j-i} - bE_{ij}) + (bV^{j-i} - bE_{ij} + bV^{n-i})x)] = 0.$$

Then the result is clear.

Corollary 3.4. For every $n \geq 2$, $T_n(R)$ is power-serieswise Armendariz if and only if $R^2 = 0$.

Corollary 3.5. For every $n \geq 2$, $M_n(R)$ is power-serieswise Armendariz if and only if $R^2 = 0$.

Let R be a ring, $R_n = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} | a,\, a_{ij} \in R \right\}$.

Kim and Lee (2000) showed that R_1, R_2 and R_3 are Armendariz when R is reduced. If $n \geq 4$, $|R| \geq 2$ and R have identity, then R_4 is not Armendariz. We have:

Corollary 3.6. Let R be a ring without identity. For every $n \geq 4$, R_n is a power-serieswise Armendariz ring if and only if $R^2 = 0$.

Let R be a ring, $S = R \oplus R$. Define $\alpha : S \to S$ by $\alpha((a, b)) = (b, a)$, then α is an automorphism of S. Kim and Lee (2000) showed that the skew polynomial ring $S[x; \alpha]$ is not Armendariz when R has identity, we have:

Proposition 3.7. Let R be a ring without identity. $S[x; \alpha]$ is power-serieswise Armendariz if and only if $R^2 = 0$.

Proof. Suppose $R^2 = 0$, then $(S[x; \alpha])^2 = 0$. The result is clear by proposition 2.1.

Conversely, suppose that $S[x; \alpha]$ is power-serieswise Armendariz. Assume to the contrary that $ab \neq 0$ for some a, b in R. Let $f(y) = (a, 0) + [(a, 0)x]y$, $g(y) = (0, -b) + [(b, 0)x]y$. Then $f(y)$, $g(y)$ are in $S[x; \alpha][y]$ and $f(y)g(y) = 0$, but $[(a, 0)x](0, -b) \neq 0$, which is a contradiction.

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$\nu-$open mappings

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Abstract The aim of the paper is to study basic properties of $\nu-$open mappings and inter-relations with other mappings.

Keywords $\nu-$open mappings.

§1. Introduction

Mappings place an important role in the study of modern mathematics, especially in Topology and Functional Analysis. Open mappings are one such mappings which are studied for different types of open sets by various mathematicians for the past many years. In this paper we tried to study a new variety of open maps called $\nu-$open maps. Throughout the paper X, Y means a topological spaces (X, τ) and (Y, σ) unless otherwise mentioned without any separation axioms defined in it.

§2. Preliminaries

Definition 2.1. $A \subset X$ is called

(i) pre-open [5] if $A \subseteq (\overline{A})^o$ and pre-closed if $\overline{(A^o)} \subseteq A$;

(ii) semi-open [7] if $A \subseteq \overline{(A^o)}$ and semi-closed if $(\overline{A})^o \subseteq A$;

(iii) semipre-open [β-open] [2] if $A \subseteq \overline{((\overline{A})^o)}$ and semipre-closed [β-closed] if $(\overline{(A^o)})^o \subseteq A$;

(iv) α-open [10] if $A \subseteq (\overline{(A^o)})^o$ and α-closed if $\overline{((\overline{A})^o)} \subseteq A$;

(v) regular open [6] if $A = (\overline{A})^o$ and regular closed if $A = \overline{(A^o)}$;

(vi) ν-open [7] if there exists a regular open set $U \ni U \subseteq A \subseteq \overline{U}$;

(vii) regular $\alpha-$closed [rα-closed] [10] if there exists a regular closed set $U \ni \alpha(U)^o \subseteq A \subseteq U$;

(viii) generalized closed [10] [resp: regular generalized closed] [6]; {(briefly g-closed; rg-closed;)} if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is open [resp: regular open].

Note 1. From the above definition we have the following implication diagram.

$$\begin{array}{ccccc} & \nearrow r\alpha-\text{open} \to \nu-\textbf{open} \searrow & & & \\ \text{Regular open} & & & \to \text{semi open} \to \beta-\text{open} \\ & \searrow \text{open} \to \alpha-\text{open} \nearrow & & & \\ & \searrow & & & \\ & \text{pre-open} & & & \end{array}$$

Definition 2.2. A function $f\colon X \to Y$ is said to be

1. continuous [resp: pre-continuous; semi-continuous; $\beta-$continuous; $\alpha-$continuous; nearly-continuous; $\nu-$continuous; r$\alpha-$continuous] if the inverse image of every open set is open [resp: pre-open; semi-open; $\beta-$open; $\alpha-$open; regular-open; $\nu-$open; r$\alpha-$open];

2. irresolute [resp: pre-irresolute; $\beta-$irresolute; $\alpha-$irresolute; nearly-irresolute; $\nu-$irresolute; r$\alpha-$irresolute] if the inverse image of every semi-open [resp: pre-open; $\beta-$open; $\alpha-$open; regular-open; $\nu-$open; r$\alpha-$open] set is semi-open [resp: pre-open; $\beta-$open; $\alpha-$open; regular-open; $\nu-$open; r$\alpha-$open];

3. open [resp: pre-open; semi-open; $\beta-$open; $\alpha-$open; nearly-open; r$\alpha-$open] if the image of every open set is open [resp: pre-open; semi-open; $\beta-$open; $\alpha-$open; regular-open; r$\alpha-$open];

4. g-continuous [resp: rg-continuous] if the inverse image of every closed set is g-closed [resp: rg-closed].

Definition 2.3. X is said to be $T_{\frac{1}{2}}$ $[r - T_{\frac{1}{2}}]$ if every [regular-] generalized closed set is [regular-] closed.

§3. $\nu-$open mappings

Definition 3.1. A function $f\colon X \to Y$ is said to be $\nu-$open if image of every open set in X is $\nu-$open in Y.

Theorem 3.1.

(i) Every r-open map is $\nu-$open but not conversely.

(ii) Every r-open map is r$\alpha-$open but not conversely.

(iii) Every rα-open map is $\nu-$open but not conversely.

(iv) Every $\nu-$open map is semi-open but not conversely.

(v) Every $\nu-$open map is $\beta-$open but not conversely.

(vi) Every r-open map is open but not conversely.

(vii) Every r-open map is semi-open but not conversely.

Proof. (i) f is r-open $\Rightarrow$ image of every open set is r-open $\Rightarrow$ image of every open set is $\nu-$open [since every r-open set is $\nu-$open] $\Rightarrow f$ is $\nu-$open.

Similarly we can prove the remaining parts using definition 2.1 and note 1.

Example 1. Let $X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, X\}$ and let $f\colon X \to Y$ is identity map. Then f is open, semi-open and $\nu-$open but not r-open and r$\alpha-$open.

Example 2. Let $X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f\colon X \to Y$ is identity map. Then f is semi open but not $\nu-$open.

Example 3. In example 2 above, f is $\beta-$open but not $\nu-$open.

Example 4. Let $X = Y = \{a, b, c, d\}; \tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f\colon X \to Y$ is identity map. Then f is open but not r-open and $\nu-$open.

Example 5. Let $X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, X\}$. Let $f\colon X \to Y$ is identity map. Then f is not open, semi-open, r-open and $\nu-$open.

Theorem 3.2.

(i) If RαO(Y) = $RO(Y)$, then f is r$\alpha-$open iff f is r-open.

(ii) If RαO(Y) = νO(Y), then f is r$\alpha-$open iff f is $\nu-$open.

(iii) If νO(Y) = $RO(Y)$, then f is r-open iff f is $\nu-$open.

(iv) If νO(Y) = αO(Y), then f is $\alpha-$open iff f is $\nu-$open.

Corollary 3.1.

(i) Every rα-open map is semi-open and hence $\beta-$open but not conversely.

(ii) Every r-open map is $\beta-$open but not conversely.

Note 2.

(i) open map and $\nu-$open map are independent to each other.

(ii) $\alpha-$open map and $\nu-$open map are independent to each other.

(iii) pre-open map and $\nu-$open map are independent to each other.

Example 6. In example 2 above, f is open; pre-open and $\alpha-$open but not $\nu-$open.

Example 7. Let $X = Y = \{a, b, c\}; \tau = \{\phi, \{a, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f\colon X \to Y$ is identity map. Then f is $\nu-$open but not open; pre-open and $\alpha-$open.

Example 8. Let $X = Y = \{a, b, c\}$; $\tau = \{\phi, \{a, b\}, X\}$, and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\}$. $f\colon (X, \tau) \to (Y, \sigma)$ be defined by f(a) = c; f(b) = b; f(c) = a; is not $\nu-$open and r-open.

Example 9. Let $X = Y = \{a, b, c\}$; $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, and $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. $f\colon (X, \tau) \to (Y, \sigma)$ be the identity map then f is not $\nu-$open and r-open.

Note 3. We have the following implication diagram among the open maps.

$$\begin{array}{ccccc}
 & \nearrow r\alpha\text{o-map} \to \boldsymbol{\nu\text{o-map}} \searrow & & & \\
\text{ro-map} & & & \to \text{so-map} \to \beta\text{o-map} \\
 & \searrow \text{open map} \to \alpha\text{o-map} \nearrow & & & \\
 & \searrow & & & \\
 & \text{po-map} & & &
\end{array}$$

Theorem 3.3.

(i) If f is open and g is $\nu-$open then $g \circ f$ is $\nu-$open.

(ii) If f is open and g is r-open then $g \circ f$ is $\nu-$open.

(iii) If f and g are r-open then $g \circ f$ is $\nu-$open.

(iv) If f is r-open and g is $\nu-$open then $g \circ f$ is $\nu-$open.

Proof. (i) Let A be open set in $X \Rightarrow f(A)$ is open in $Y \Rightarrow g(f(A))$ is $\nu-$open in $Z \Rightarrow g \circ f(A)$ is $\nu-$open in $Z \Rightarrow g \circ f$ is $\nu-$open.

Similarly we can prove the remaining parts and so omitted.

Corollary 3.2.

(i) If f is open and g is $\nu-$open [r-open] then $g \circ f$ is semi-open and hence $\beta-$open.

(ii) If f and g are r-open then $g \circ f$ is semi-open and hence $\beta-$open.

(iii) If f is r-open and g is $\nu-$open then $g \circ f$ is semi-open and hence $\beta-$open.

Theorem 3.4. If $f\colon X \to Y$ is $\nu-$open, then $f(A^o) \subset \nu(f(A))^o$.

Proof. Let $A \subset X$ and $f\colon X \to Y$ is $\nu-$open gives $f(A^o)$ is $\nu-$open in Y and $f(A^o) \subset f(A)$ which in turn gives

$$\nu(f(A^o))^o \subset \nu(f(A))^o. \tag{1}$$

Since $f(A^o)$ is $\nu-$open in Y,

$$\nu(f(A)^o)^o = f(A^o). \tag{2}$$

Combaining (1) and (2) we have $f(A^o) \subset \nu(f(A))^o$ for every subset A of X.

Remark. Converse is not true in general, as shown by the following:

Example 10. Let $X = Y = \{a, b, c\}$; $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$, and

$$\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\}.$$

$f\colon (X, \tau) \to (Y, \sigma)$ be the identity map then $f(A^o) \subset \nu(f(A))^o$ for every subset A of X but f is not $\nu-$open. Since $f(\{a, b\}) = \{a, b\}$ is not $\nu-$open.

Corollary 3.3. If $f\colon X \to Y$ is r-open, then $f(A^o) \subset \nu(f(A))^o$.

Theorem 3.5. If $f\colon X \to Y$ is $\nu-$open and $A \subset X$ is open, then f(A) is τ_ν-open in Y.

Proof. Let $A \subset X$ and $f : X \to Y$ is $\nu-$open implies $f(A^o) \subset \nu(f(A))^o$ which in turn implies $\nu(f(A))^o \supset f(A)$, since $f(A) = f(A^o)$. But $\nu(f(A))^o \subset f(A)$. Combaining we get $f(A) = \nu(f(A))^o$. Therefore f (A) is τ_ν-open in Y.

Corollary 3.4. If $f\colon X \to Y$ is r-open, then f (A) is τ_ν-open in Y if A is r-open set in X.

Theorem 3.6. If $\nu(A)^o = r(A)^o$ for every $A \subset Y$, then the following are equivalent:

(i) $f\colon X \to Y$ is $\nu-$open map.

(ii) $f(A^o) \subset \nu(f(A)^o)$.

Proof. $(i) \Rightarrow (ii)$ follows from theorem 3.4.

$(ii) \Rightarrow (i)$ Let A be any open set in X, then $f(A) = f(A^o) \supset \nu(f(A))^o$ by hypothesis. We have $f(A) \subset \nu(f(A))^o$. Combaining we get $f(A) = \nu(f(A))^o = r(f(A))^o$ [by given condition] which implies f(A) is r-open and hence $\nu-$open. Thus f is $\nu-$open.

Theorem 3.7. $f : X \to Y$ is $\nu-$open iff for each subset S of Y and each open set U containing $f^{-1}(S)$, there is a $\nu-$open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof. Assume f is $\nu-$open, $S \subset Y$ and U an open set of X containing $f^{-1}(S)$, then f $(X - U)$ is $\nu-$open in Y and $V = Y - f(X - U)$ is $\nu-$open in Y. $f^{-1}(S) \subset U$ implies $S \subset V$ and $f^{-1}(V) = X - f^{-1}(f(X - U)) \subset X - (X - U) = U$.

Conversely let F be open in X, then $f^{-1}(f(F^c)) \subset F^c$. By hypothesis, $\exists\ V \in \nu O(Y) \ni f(F^c) \subset V$ and $f^{-1}(V) \subset F^c$ and so $F \subset (f^{-1}(V))^c$. Hence $V^c \subset f$ (F) $\subset f[(f^{-1}(V))^c] \subset V^c$ implies $f(F) \subset V^c$, which implies f (F) $= V^c$. Thus f (F) is $\nu-$open in Y and therefore f is $\nu-$open.

Remark. Composition of two $\nu-$open maps is not $\nu-$open in general.

Theorem 3.8. Let X, Y, Z be topological spaces and every $\nu-$open set is open [r-open] in Y, then the composition of two $\nu-$open maps is $\nu-$open.

Proof. Let A be open in $X \Rightarrow f$ (A) is $\nu-$open in $Y \Rightarrow f$ (A) is open in Y [by assumption] $\Rightarrow g(f$ (A)) is $\nu-$open in $Z \Rightarrow g \circ f(A)$ is $\nu-$open in $Z \Rightarrow g \circ f$ is $\nu-$open.

Theorem 3.9. If $f\colon X \to Y$ is g-open; $g : Y \to Z$ is $\nu-$open [r-open] and Y is $T_{\frac{1}{2}}[r - T_{\frac{1}{2}}]$, then $g \circ f$ is $\nu-$open.

Proof. (i) Let A be open in $X \Rightarrow f$ (A) is g-open in $Y \Rightarrow f$ (A) is open in Y [since Y is $T_{\frac{1}{2}}] \Rightarrow g(f(\mathrm{A}))$ is $\nu-$open in $Z \Rightarrow g \circ f\,(A)$ is $\nu-$open in $Z \Rightarrow g \circ f$ is $\nu-$open.

(ii) Since every g-open set is rg-open, this part follows from the above case.

Corollary 3.5. If $f\colon X \to Y$ is g-open; $g : Y \to Z$ is $\nu-$open [r-open] and Y is $T_{\frac{1}{2}}\{r - T_{\frac{1}{2}}\}$, then $g \circ f$ is semi-open and hence $\beta-$open.

Theorem 3.10. If $f\colon X \to Y$ is rg-open; $g : Y \to Z$ is $\nu-$open [r-open] and Y is $r - T_{\frac{1}{2}}$, then $g \circ f$ is $\nu-$open.

Proof. Let A be open in $X \Rightarrow f(\mathrm{A})$ is rg-open in $Y \Rightarrow f$ (A) is r-open in Y [since Y is $r - T_{\frac{1}{2}}] \Rightarrow f$ (A) is open in Y [since every r-open set is open] $\Rightarrow g(f$ (A)) is $\nu-$open in $Z \Rightarrow g \circ f\,(A)$ is $\nu-$open in $Z \Rightarrow g \circ f$ is $\nu-$open.

Corollary 3.6. If $f\colon X \to Y$ is rg-open; $g : Y \to Z$ is $\nu-$open [r-open] and Y is $r - T_{\frac{1}{2}}$, then $g \circ f$ is semi-open and hence $\beta-$open.

Theorem 3.11. If $f : X \to Y$; $g : Y \to Z$ be two mappings such that $g \circ f$ is $\nu-$open [r-open]. Then the following are true.

(i) If f is continuous [r-continuous] and surjective, then g is $\nu-$open.

(ii) If f is g-continuous, surjective and X is $T_{\frac{1}{2}}$, then g is $\nu-$open.

(iii) If f is g-continuous [rg-continuous], surjective and X is $r - T_{\frac{1}{2}}$, then g is $\nu-$open.

Proof. (i) Let A be open in $Y \Rightarrow f^{-1}$ (A) is open in $X \Rightarrow g \circ f\,(f^{-1}(A))$ is $\nu-$open in $Z \Rightarrow g\,(A)$ is $\nu-$open in $Z \Rightarrow g$ is $\nu-$open.

Similarly we can prove the remaining parts and so omitted.

Corollary 3.7. If $f : X \to Y$; $g : Y \to Z$ be two mappings such that $g \circ f$ is $\nu-$open [r-open]. Then the following are true.

(i) If f is continuous [r-continuous] and surjective, then g is semi-open and hence $\beta-$open.

(ii) If f is g-continuous, surjective and X is $T_{\frac{1}{2}}$, then g is semi-open and hence $\beta-$open.

(iii) If f is g-continuous [rg-continuous], surjective and X is $r - T_{\frac{1}{2}}$, then g is semi-open and hence $\beta-$open.

Theorem 3.12. If X is $\nu-$regular, $f : X \to Y$ is r-closed, nearly-continuous, $\nu-$open surjection and $A^o = A$ for every $\nu-$open set in Y, then Y is $\nu-$regular.

Proof. Let $p \in U \in \nu O(Y)$, $\exists x \in X \ni f\,(x) = p$ by surjection. Since X is $\nu-$regular and f is nearly-continuous, $\exists V \in RC(X) \ni x \in V^o \subset V \subset f^{-1}(U)$ which implies

$$p \in f\,(V^o) \subset f\,(V) \subset U. \tag{3}$$

Since f is $\nu-$open, $f\,(V^o) \subset U$ is $\nu-$open. By hypothesis

$$\{f\,(V^o)\}^o = f\,(V^o) \text{ and } \{f\,(V^o)\}^o = \{f\,(V)\}^o. \tag{4}$$

Combaining (3) and (4) $p \in f\,(V)^o \subset f\,(V) \subset U$ and f (V) is r-closed. Hence Y is $\nu-$regular.

Corollary 3.8. If X is $\nu-$regular, $f : X \to Y$ is r-closed, nearly-continuous, $\nu-$open surjection and $A^o = A$ for every r-open set in Y, then Y is $\nu-$regular.

Theorem 3.13. If $f : X \to Y$ is $\nu-$open [r-open] and A is open set of X, then $f_A : (X, \tau_{(A)}) \to (Y, \sigma)$ is $\nu-$open.

Proof. Let F be open set in A. Then $F = A \cap E$ for some open set E of X and so F is open in X which implies f (A) is $\nu-$open in Y. But $f(F) = f_A(F)$ and therefore f_A is $\nu-$open.

Corollary 3.9. If $f : X \to Y$ is $\nu-$open [r-open] and A is open set of X, then $f_A : (X, \tau_{(A)}) \to (Y, \sigma)$ is semi-open and hence $\beta-$open.

Theorem 3.14. If $f\colon X \to Y$ is $\nu-$open [r-open], X is $T_{\frac{1}{2}}$ and A is g-open set of X, then $f_A : (X, \tau_{(A)}) \to (Y, \sigma)$ is $\nu-$open.

Corollary 3.10. If $f\colon X \to Y$ is $\nu-$open [r-open], X is $T_{\frac{1}{2}}$ and A is g-open set of X, then $f_A : (X, \tau_{(A)}) \to (Y, \sigma)$ is semi-open and hence $\beta-$open.

Theorem 3.15. If $f_i : X_i \to Y_i$ be $\nu-$open [r-open] for $i = 1, 2$. Let $f\colon X_1 \times X_2 \to Y_1 \times Y_2$ be defined as follows: $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Then $f\colon X_1 \times X_2 \to Y_1 \times Y_2$ is $\nu-$open.

Proof. Let $U_1 \times U_2 \subset X_1 \times X_2$ where U_i is open in X_i for $i = 1, 2$. Then $f(U_1 \times U_2) = f_1(U_1) \times f_2(U_2)$ a $\nu-$open set in $Y_1 \times Y_2$. Thus $f(U_1 \times U_2)$ is $\nu-$open and hence f is $\nu-$open.

Corollary 3.11. If $f_i : X_i \to Y_i$ be $\nu-$open [r-open] for $i = 1, 2$. Let $f\colon X_1 \times X_2 \to Y_1 \times Y_2$ be defined as follows: $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Then $f\colon X_1 \times X_2 \to Y_1 \times Y_2$ is semi-open and hence $\beta-$open.

Theorem 3.16. Let $h : X \to X_1 \times X_2$ be $\nu-$open [r-open]. Let $f_i : X \to X_i$ be defined as: $h(x) = (x_1, x_2)$ and $f_i(x) = x_i$. Then $f_i : X \to X_i$ is $\nu-$open for $i = 1, 2$.

Proof. Let U_1 is open in X_1, then $U_1 \times X_2$ is open in $X_1 \times X_2$, and $h(U_1 \times X_2)$ is $\nu-$open in X. But $f_1(U_1) = h(U_1 \times X_2)$, therefore f_1 is $\nu-$open. Similarly we can show that f_2 is also $\nu-$open and thus $f_i : X \to X_i$ is $\nu-$open for $i = 1, 2$.

Corollary 3.12. Let $h : X \to X_1 \times X_2$ be $\nu-$open [r-open]. Let $f_i : X \to X_i$ be defined as: $h(x) = (x_1, x_2)$ and $f_i(x) = x_i$. Then $f_i : X \to X_i$ is semi-open and hence $\beta-$open for $i = 1, 2$.

Conclusion. We studied some properties and interrelations of $\nu-$open mappings.

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