On the number of Smarandache Zero-Divisors and Smarandache Weak Zero-Divisors in Loop Rings of the Loops $L_n(m)$:

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Abstract

In this paper we find the number of smarandache zero divisors (S-zero divisors) and smarandache weak zero divisors (S-weak zero divisors) for the loop rings $Z_2L_n(m)$ of the loops $L_n(m)$ over Z_2. We obtain the exact number of S-zero divisors and S-weak zero divisors when $n = p^2$ or p^3 or pq where p, q are odd primes. We also prove $ZL_n(m)$ has infinitely many S-zero divisors and S-weak zero divisors, where Z is the ring of integers. For any loop L we give conditions on L so that the loop ring Z_2L has S-zero divisors and S-weak zero diviosrs.

§ Introduction:

This paper has four sections. In the first section, we just recall the definitions of S-zero divisors and S-weak divisors and some of the properties of the new class of loops $L_n(m)$. In section two, we obtain the number of S-zero divisors of the loop rings $Z_2L_n(m)$ and show when $n = p^2$, p an odd prime, $Z_2L_n(m)$ has $p(1 + \sum_{r=2,r\ even}^{p-1} {}^{p+1}C_r)$ S-zero divisors. Also when $n = p^3$, p an odd prime, $Z_2L_n(m)$ has $p(1+\sum_{r=2,r\ even}^{p^2-1} {}^{p^2+1}C_r)+p^2(1+\sum_{r=2,r\ even}^{p-1} {}^{p+1}C_r)$ S-zero divisors. Again when $n = pq$, p, q are odd primes, $Z_2L_n(m)$ has $p + q + p(\sum_{r=2,r\ even}^{q-1} {}^{q+1}C_r) + q(\sum_{r=2,r\ even}^{p-1} {}^{p+1}C_r)$ S-zero divisors. Further we prove $ZL_n(m)$ has infinitely many S-zero divisors.

In section three, we find the number of S-weak zero divisors for the loop ring $Z_2L_n(m)$ and prove that when $n = p^2$, p an odd prime, $Z_2L_n(m)$ has $2p(1+\sum_{r=2,r\ even}^{p-1} {}^{p+1}C_r)$ S-weak zero divisors. Also when $n = p^3$, p an odd prime, $Z_2L_n(m)$ has $2p(\sum_{r=2,r\ even}^{p^2-1} {}^{p^2+1}C_r)+ 2p^2(\sum_{r=2,r\ even}^{p-1} {}^{p+1}C_r)$ S-weak zero divisors. Again when $n = pq$, p, q are odd primes, $Z_2L_n(m)$ has $2[p(\sum_{r=2,r\ even}^{q-1} {}^{q+1}C_r) + q(\sum_{r=2,r\ even}^{p-1} {}^{p+1}C_r)]$ S-weak zero divisors. We prove $ZL_n(m)$ has infinitely many S-weak zero divisors. The final section gives some unsolved problems and some conclusions based on our study.

§ 1 : Basic Results

Here we just recollect some basic results to make this paper a self contained one.

Definition 1.1 [4]: Let R be a ring. An element $a \in R \setminus \{0\}$ is said to be a S-zero divisor if $a.b = 0$ for some $b \neq 0$ in R and there exists $x, y \in R \setminus \{0, a, b\}$such that

$$i.\ \ a.x = 0 \quad or \quad x.a = 0$$
$$ii.\ \ b.y = 0 \quad or \quad y.b = 0$$
$$iii.\ \ x.y \neq 0 \quad or \quad y.x \neq 0.$$

Definition 1.2 [4]**:** Let R be a ring. An element $a \in R \setminus \{0\}$ is a S-weak zero divisor if there exists $b \in R \setminus \{0, a\}$ such that $a.b = 0$ satisfying the following conditions: There exists $x, y \in R \setminus \{0, a, b\}$ such that

$$i.\ \ a.x = 0 \quad or \quad x.a = 0$$
$$ii.\ \ b.y = 0 \quad or \quad y.b = 0$$
$$iii.\ \ x.y = 0 \quad or \quad y.x = 0.$$

Definition 1.3 [3]: Let $L_n(m) = \{e, 1, 2, 3..., n\}$ be a set where $n > 3, n$ is odd and m is a positive integer such that $(m, n) = 1$ and $(m - 1, n) = 1$ with $m < n$. Define on $L_n(m)$, a binary operation $'.'$ as follows:

$$i.\ \ e.i = i.e = i \qquad for\ all \quad i \in L_n(m) \setminus \{e\}$$
$$ii.\ \ i^2. = e \qquad for\ all \quad i \in L_n(m)$$
$$iii.\ \ i.j = t, \quad where\ \ t \equiv (mj - (m-1)i)(mod\ n) \quad for\ all\ i, j \in L_n(m),\ \ i \neq e\ and\ j \neq e.$$

Then $L_n(m)$ is a loop. This loop is always of even order; further for varying m, we get a class of loops of order $n + 1$ which we denote by L_n.

Example 1.1 [3]: Consider $L_5(2) = \{e, 1, 2, 3, 4, 5\}$. The composition table for $L_5(2)$ is given below:

.	e	1	2	3	4	5
e	e	1	2	3	4	5
1	1	e	3	5	2	4
2	2	5	e	4	1	3
3	3	4	1	e	5	2
4	4	3	5	2	e	1
5	5	2	4	1	3	e

This loop is non-commutative and non-associative and of order 6.

Theorem 1.1 [3]: Let $L_n(m) \in L_n$. For every $t|n$ there exists t subloops of order $k+1$, where $k = n/t$.

Theorem 1.2 [3]: Let $L_n(m) \in L_n$. If H is a subloop of $L_n(m)$ of order $t+1$ then $t|n$.

Remark 1.2 [3]: Lagrange's theorem is not satisfied by all subloops of the loop $L_n(m)$, i.e there always exists a subloop H of $L_n(m)$ which does not satisfie the Lagrange's theorem, i.e $o(H) \nmid o(L_n(m))$.

§ 2 : Determination of the number of S-zero divisors in $Z_2L_n(m)$ and $ZL_n(m)$.

In this section, we give the number of S-zero divisors in $Z_2L_n(m)$. We prove $ZL_n(m)$ (*where* $n = p^2$ or pq, p and q are odd primes), has infinitely many S-zero divisors. Further we show any loop L of odd (or even) order if it has a proper subloop of even (or odd) order then the loop ring $Z_2L_n(m)$ over the field Z_2 has S-zero divisors. We first show if L is a loop of odd order and L has a proper subloop of even order, then $Z_2L_n(m)$ has S-zero divisors.

Theorem 2.1: Let L be a finite loop of odd order. $Z_2 = \{0, 1\}$, the prime field of characteristic 2. Suppose H is a subloop of L of even order, then Z_2L has S-zero divisors.

Proof: **Let $|L| = n$; n odd. Z_2L be the loop ring of L over Z_2. H be the subloop of L of order m, where m is even. Let $X = \sum_{i=1}^{n} g_i$ and $Y = \sum_{i=1}^{m} h_i$, then**

$$X.Y = 0.$$

Now

$$(1 + g_t)X = 0, \qquad g_t \in L \backslash H$$

also

$$(1 + h_i + h_j + h_k)Y = 0, \qquad h_i, h_j, h_k \in H$$

so that

$$(1 + g_t)(1 + h_i + h_j + h_k) \neq 0.$$

Hence the claim.

Corollary 2.1: If L is a finite loop of even order n and H is a subloop of odd order m, then the loop ring Z_2L has S-zero divisors.

It is important here to mention that Z_2L may have other types of S-zero divisors. This theorem only gives one of the basic conditions for Z_2L to have S-zero divisors.

Example 2.1 Let $Z_2L_{25}(m)$ be the loop ring of the loop $L_{25}(m)$ over Z_2, where $(m, 25) = 1$ and $(m-1, 25) = 1$. As $5|25$, so $L_{25}(m)$ has 5 proper subloops each of order 6. Let H be one of the proper subloops of $L_{25}(m)$.
Now take

$$X = \sum_{i=1}^{26} g_i, \quad Y = \sum_{i=1}^{6} h_i \quad g_i \in L_{25}(m), h_i \in H,$$

then

$$(1 + g_i)X = 0, \quad g_i \in L_{25}(m)\backslash H$$
$$(1 + h_i)Y = 0, \quad h_i \in H$$

but

$$(1 + g_i)(1 + h_i) \neq 0.$$

So X and Y are S-zero divisors in $Z_2L_{25}(m)$.

Theorem 2.2: Let $L_n(m)$ be a loop of order $n+1$ (n an odd number, $n > 3$) with $n = p^2$, p an odd prime. Z_2 be the prime field of characeristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$p(1 + \sum_{r=2, r\ even}^{p-1} {}^{p+1}C_r)$$

S-zero divisors.

Proof: **Given $L_n(m)$ is a loop of order $n+1$, where $n = p^2$ (p an odd prime). Let $Z_2L_n(m)$ be the loop ring of the loop $L_n(m)$ over Z_2. Now clearly the loop $L_n(m)$ has exactly p subloops of order $p + 1$. The number of S-zero divisors in $Z_2L_n(m)$ for $n = p^2$ can be enumerated in the following way:**
Let

$$X = \sum_{i=1}^{n+1} g_i \text{ and } Y = \sum_{i=1}^{p+1} h_i$$

where $g_i \in L_n(m)$ **and** $h_i \in H_j$
for this

$$X.Y = 0$$

choose

$$a = (1+g), \quad g \in L_n(m) \backslash H_j$$

$$b = (h_i + h_j), \quad h_i, h_j \in H_j$$

then

$$a.X = 0 \text{ and } b.Y = 0$$

but

$$a.b \neq 0.$$

So X **and** Y **are S-zero divisors. There are** p **such S-zero divisors , as we have** p **subloops** H_j $(j = 1, 2, ..., p)$ **of** $L_n(m)$**.**

Next consider, S-zero divisors of the form

$$(h_1 + h_2)\sum_{i=1}^{n+1} g_i = 0, \text{ where } h_1, h_2 \in H_j,\ g_i \in L_n(m)$$

put

$$X = (h_1 + h_2), \quad Y = \sum_{i=1}^{n+1} g_i$$

we have ${}^{p+1}C_2$ **such S-zero divisors. This is true for each of the subloops. Hence there exists** ${}^{p+1}C_2 \times p$ **such S-zero divisors. Taking four elements** h_1, h_2, h_3, h_4 **from** H_j **at a time, we get**

$$(h_1 + h_2 + h_3 + h_4)\sum_{i=1}^{n+1} g_i = 0$$

so we get ${}^{p+1}C_4 \times p$ **such S-zero divisors.**
Continuing in this way, we get

$$(h_1 + h_2 + ... + h_{p-1})\sum_{i=1}^{n+1} g_i = 0 \text{ where } h_1, h_2, ..., h_{p-1} \in H_j.$$

So we get ${}^{p+1}C_{p-1} \times p$ **such S-zero divisors. Adding all these S-zero divisors we get**

$$p(1 + \sum_{r=2, r\ \ even}^{p-1} {}^{p+1}C_r)$$

number of S-zero divisors in the loop ring $Z_2L_n(m)$.
Hence the claim.

Example 2.2: Let $Z_2L_{49}(m)$ be the loop ring of the loop $L_{49}(m)$ over Z_2, where $(m, 49) = 1$ and $(m - 1, 49 = 1)$. Here $p = 7$, so from Theorem 2.2, $Z_2L_{49}(m)$ has

$$7(1 + \sum_{r=2,\ r\ even}^{6} {}^{7+1}C_r)$$

S-zero divisors i.e $7(1 + \sum_{r=2,r\ \ even}^{6}\ {}^{8}C_r) = 889$ S-zero divisors.

Theorem 2.3: Let $L_n(m)$ be a loop of order $n+1$ (n an odd number, $n > 3$) with $n = p^3$, p an odd prime. Z_2 be the prime field of characeristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$p(1 + \sum_{r=2,r\ \ even}^{p^2-1} {}^{p^2+1}C_r) + p^2(1 + \sum_{r=2,r\ \ even}^{p-1} {}^{p+1}C_r)$$

S-zero divisors.

Proof: **We enumerate all the S-zero divisors of $Z_2L_n(m)$ in the following way:**

Case I: As $p|p^3$, $L_n(m)$ has p proper subloops H_j each of order $p^2 + 1$. In this case I, we have $p^2 - 1$ types of S-zero diviosrs. We just index them by type I_1, type I_2,...type I_{p^2-1}

Type I_1: Here

$$\sum_{i=1}^{n+1} g_i \sum_{i=1}^{p^2+1} h_i = 0, \quad g_i \in L_n(m), \quad h_i \in H_j, \ (j = 1, 2..., p).$$

So we will get p S-zero divisors of this type.

Type I_2 :

$$(h_1 + h_2) \sum_{i=1}^{n+1} g_i = 0, \quad h_1, h_2 \in H_j \ (j = 1, 2, ..., p).$$

As in Theorem 2.2, we will get ${}^{p^2+1}C_2 \times p$ S-zero divisors of this type.

Type I_3:

$$(h_1+h_2+h_3+h_4)\sum_{i=1}^{n+1} g_i = 0 \quad h_1,h_2,h_3,h_4 \in H_j \ (j=1,2,...,p).$$

We will get ${}^{p^2+1}C_4 \times p$ S-zero divisors of this type.
Continuing this way,

Type I_{p^2-1}:

$$(h_1+h_2+...+h_{p^2-1})\sum_{i=1}^{n+1} g_i = 0, \quad h_i \in H_j.$$

We will get ${}^{p^2+1}C_{p^2-1} \times p$ S-zero divisors of this type.
Hence adding all this types of S-zero divisors we will get

$$p(1+\sum_{r=2, r\ even}^{p^2-1} {}^{p^2+1}C_r)$$

S-zero divisors for case I.

Case II : Again $p^2|p^3$, so there are p^2 subloops H_j each of order $p+1$. Now we can enumerate all the S-zero divisors in this case exactly as in case I above. So there are

$$p^2(1+\sum_{r=2, r\ even}^{p-1} {}^{p+1}C_r)$$

S-zero divisors. Hence the total number of S-zero divisors in $Z_2L_n(m)$ is

$$p(1+\sum_{r=2, r\ even}^{p^2-1} {}^{p^2+1}C_r) + p^2(1+\sum_{r=2, r\ even}^{p-1} {}^{p+1}C_r).$$

Hence the claim.

Example 2.3: Let $Z_2L_{27}(m)$ be the loop ring of the loop $L_{27}(m)$ over Z_2, where $(m,27)=1$ and $(m-1,27)=1$. Here $p=3$, so from Theorem 2.3, $Z_2L_{27}(m)$ has

$$3(1+\sum_{r=2,\ r\ even}^{8} {}^{3^2+1}C_r) + 3^2(1+\sum_{r=2,\ r\ even}^{2} {}^{4}C_r)$$

S-zero divisors i.e $3(1+\sum_{r=2, r\ even}^{8} {}^{10}C_r) + 9(1+\sum_{r=2,\ r\ even} {}^{4}C_r) = 1533$ **S-zero divisors.**

Theorem 2.4: Let $L_n(m)$ be a loop of order $n+1$ (n an odd number, $n > 3$) with $n = pq$, p, q are odd primes. Z_2 be the prime field of characeristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$p + q + p(1 + \sum_{r=2, r\ even}^{q-1} {}^{q+1}C_r) + q(1 + \sum_{r=2, r\ even}^{p-1} {}^{p+1}C_r)$$

S-zero divisors.

Proof: We will enumerate all the S-zero divisors in the following way:

Case I : as $p|pq$, $L_n(m)$ has p subloops H_j each of order $q+1$. Proceeding exactly in the same way as in Theorem 2.3, we will get $p+p(1+\sum_{r=2, r\ even}^{q-1} {}^{q+1}C_r)$ S-zero divisors for case I.

Case II : Again $q|pq$, so $L_n(m)$ has q subloops H_j each of order $p+1$. Now as above we will get $q + q(1 + \sum_{r=2, r\ even}^{p-1} {}^{p+1}C_r)$ S-zero divisors for case II. Hence adding all the S-zero divisors in case I and case II, we get

$$p + q + p(1 + \sum_{r=2, r\ even}^{q-1} {}^{q+1}C_r) + q(1 + \sum_{r=2, r\ even}^{p-1} {}^{p+1}C_r)$$

S-zero divisors in $Z_2L_n(m)$.

Hence the claim.

Now we prove for the loop ring $ZL_n(m)$ where $n = p^2\ or\ p^3\ or\ pq$ where p, q are odd primes, $ZL_n(m)$ has infinitely many S-zero divisors.

Theorem 2.5: Let $ZL_n(m)$ be the loop ring of the loop $L_n(m)$ over Z, where $n = p^2\ or\ p^3\ or\ pq$, (p, q are odd primes), then $ZL_n(m)$ has infinitely many S-zero divisors.

Proof: Let $L_n(m)$ be a loop such that $n = p^2$. $L_n(m)$ has p subloops (say H_j) each of order $p + 1$.

Now the loop ring $ZL_n(m)$ has the following types of S-zero divisors:

$$X = a - bh_1 + bh_2 - ah_3 \text{ and } Y = \sum_{i=1}^{n+1} g_i$$

where $a, b \in Z$ and $h_i \in H_i,\ g_i \in L_n(m)$ such that

$$(a - bh_1 + bh_2 - ah_3)\sum_{i=1}^{n+1} g_i = 0.$$

Again

$$(1 - g_k)Y = 0, \quad g_k \in L_n(m)\backslash H_j$$

also

$$(a - bh_1 + bh_2 - ah_3)\sum h_i = 0, \quad h_i \in H_j$$

clearly

$$(1 - g_k)(\sum_{h_i \in H_j} h_i) \neq 0.$$

So X, Y are S-zero divisors in $ZL_n(m)$. Now we see there are infinitely many S-zero divisors of this type for a and b can take infinite number of values in Z. For $n = p^2$ *or* p^3 *or* pq we can prove the results in a similar way.
Hence the claim.

§ 3 : Determination of the number of S-weak zero divisors in $Z_2L_n(m)$ and $ZL_n(m)$:

In this section, we give the number of S-weak zero divisors in the loop ring $Z_2L_n(m)$ when n is of the form p^2, p^3 *or* pq where p and q are odd primes. Before that we prove the existance of S-weak zero divisors in the loop ring Z_2L whenever L has a proper subloop.

Theorem 3.1: Let be a finite loop of odd order. Suppose H is a subloop of of L of even order, then Z_2L has S-weak zero divisors.

Proof: Let $|L| = n$; n odd. Z_2L be the loop ring. H be the subloop of L of order m, where m is even. Let $X = \sum_{i=1}^{n} g_i$ and $Y = 1 + h_t$, $g_i \in L$ and $h_t \in H$, then

$$X.Y = 0.$$

Now

$$Y.\sum_{i=1}^{m} h_i = 0, \quad h_i \in H$$

also

$$X(1 + g_t) = 0, \quad g_t(\neq h_t) \in H$$

so that

$$(1+g_t).\sum_{i=1}^{m} h_i = 0.$$

Hence the claim.

Example 3.1 Let $Z_2L_{25}(m)$ be the loop ring of the loop $L_{25}(m)$ over Z_2, where $(m,25)=1$ and $(m-1,25)=1$. As $5|25$, so $L_{25}(m)$ has 5 proper subloops each of order 6.
Take

$$X=\sum_{i=1}^{26} g_i, \quad Y=1+h_t,\ g_i\in L_{25}(m),\ h_t\in H$$

then

$$X.Y=0$$

again

$$X(1+g_t)=0, \quad g_t(\neq h_t)\in H$$

$$Y\sum_{i=1}^{6} h_i=0, \quad h_i\in H$$

also

$$(1+g_t)\sum_{i=1}^{6} h_i=0.$$

So X and Y are S-weak zero divisors in $Z_2L_{25}(m)$.

Example 3.2 Let $Z_2L_{21}(m)$ be the loop ring of the loop $L_{21}(m)$ over Z_2, where $(m,21)=1$ and $(m-1,21)=1$. As $3|21$, so $L_{21}(m)$ has 3 proper subloops each of order 8.
Take

$$X=\sum_{i=1}^{8} h_i, \quad Y=1+h_t, \quad h_i,\ h_t\in H$$

then

$$X.Y=0$$

again

$$X(1+g_t)=0, \quad g_t(\neq h_t)\in H$$

$$Y\sum_{i=1}^{22} g_i=0, \quad g_i\in L_{21}(m)$$

also

$$(1+g_t)\sum_{i=1}^{22} g_i=0.$$

So X and Y are S-weak zero divisors in $Z_2L_{21}(m)$.

Theorem 3.2: Let $L_n(m)$ be a loop of order $n+1$ (n an odd number, $n > 3$) with $n = p^2$, p an odd prime. Z_2 be the prime field of characeristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$2p(\sum_{r=2,r\ even}^{p-1} {}^{p+1}C_r)$$

S-weak zero divisors.

Proof: **Clearly the loop $L_n(m)$ has p subloops H_j each of order $p+1$. As in case of Theorem 2.3, we index the $p-1$ types of S-weak zero divisors by $I_1, I_2, ...I_{p-1}$. Now the number of S-weak zero divisors in $Z_2L_n(m)$ for $n = p^2$ can be enumerated in the following way:**

Type I_1. Let

$$X = h_1 + h_2, \quad Y = \sum_{i=1}^{n+1} g_i$$

where $h_1, h_2 \in H_j$ and $g_i \in L_n(m)$ then

$$XY = 0$$

take

$$a = \sum_{i=1}^{p+1} h_i, \text{ and } \ b = h_3 + h_4 \text{ where } \ h_i \in H_j, \ (j = 1, 2, ..., p)$$

then

$$aX = 0, \quad bY = 0$$

also

$$ab = 0.$$

So for each proper subloop we will get ${}^{p+1}C_2$ S-weak zero divisors and as there are p proper subloops we will get ${}^{p+1}C_2 \times p$ such S-weak zero divisors.

Type I_2. Again let

$$X = h_1 + h_2, \quad Y = \sum_{i=1}^{p+1} h_i, \quad h_i \in H_j$$

then

$$XY = 0$$

take

$$a=\sum_{i=1}^{n+1} g_i, \quad g_i \in L_n(m), \quad b=h_1+h_2, \quad h_1,h_2 \in H_j$$

then

$$aX=0, \quad bY=0$$

also

$$ab=0.$$

Here also we will get ${}^{p+1}C_2 \times p$ **S-weak zero divisors of this type.**

Type I_3**.**

$$(h_1+h_2+h_3+h_4)\sum_{i=1}^{n+1} g_i=0, \quad h_i \in H_j \textbf{ and } g_i \in L_n(m).$$

As above we can say there are ${}^{p+1}C_4 \times p$ **such S-weak zero divisors.**

Type I_4**.**

$$(h_1+h_2+h_3+h_4)\sum_{i=1}^{p+1} h_i=0, \quad h_i \in H_j.$$

There are ${}^{p+1}C_4 \times p$ **such S-weak zero divisors.**
Continuing this way,

Type I_{p-2}**.**

$$(h_1+h_2+...+h_{p-1})\sum_{i=1}^{n+1} g_i=0, \quad h_i \in H_j, \quad g_i \in L_n(m).$$

There are ${}^{p+1}C_{p-1} \times p$ **such S-weak zero divisors.**

Type I_{P-1}**.**

$$(h_1+h_2+...+h_{p-1})\sum_{i=1}^{n} h_i=0, \quad h_j \in H_i.$$

Again there are ${}^{p+1}C_{p-1} \times p$ **S-weak zero divisors of this type.**
Adding all these S-weak zero divisors we will get the total number of S-weak zero divisors in $Z_2L_n(m)$ **as**

$$2p(\sum_{r=2,r \ \ even}^{p-1} {}^{p+1}C_r).$$

Hence the claim.

Theorem 3.3: Let $L_n(m)$ be a loop of order $n+1$ (n an odd number,$n > 3$) with $n = p^3$, p an odd prime. Z_2 be the prime field of characeristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$2p(\sum_{r=2,r\ \ even}^{p^2-1} {}^{p^2+1}C_r) + 2p^2(\sum_{r=2,r\ \ even}^{p-1} {}^{p+1}C_r)$$

S-weak zero divisors.

Proof: **We enumerate all the S-zero divisors of $Z_2L_n(m)$ in the following way:**

Case I: As $p|p^3$, $L_n(m)$ has p proper subloops H_j each of order p^2+1. Now as in Theorem 3.2
Type I_1 :

$$(h_1 + h_2)\sum_{i=1}^{n+1} g_i = 0, \quad g_i \in L_n(m),\ h_i \in H_j.$$

So we will get ${}^{p^2+1}C_2 \times p$ S-weak zero divisors of type I_1.
Type I_2 :

$$(h_1 + h_2)\sum_{i=1}^{p^2+1} h_i = 0, \quad h_i \in H_j.$$

So we will get ${}^{p^2+1}C_2 \times p$ S-weak zero divisors of type I_2.
Continuing in this way
Type I_{p^2-2} :

$$(h_1 + h_2 + ... + h_{p^2-1})\sum_{i=1}^{n+1} g_i = 0.$$

So we will get ${}^{p^2+1}C_{p^2-1} \times p$ S-weak zero divisors of this type .
Type I_{p^2-1} :

$$(h_1 + h_2 + ... + h_{p^2-1})\sum_{i=1}^{p^2+1} h_i = 0.$$

So we will get ${}^{p^2+1}C_{p^2-1} \times p$ S-weak zero divisors of type I_{p^2-1}.
Adding all this S-weak zero divisors, we will get the total number of S-weak zero divisors (in case I) in $Z_2L_n(m)$ as $2p(\sum_{r=2,r\ \ even}^{p^2-1} {}^{p^2+1}C_r)$.

Case II: Again $p^2|p^3$, so there are p^2 proper subloops H_j each of order $p+1$. Now we can enumerate all the S-weak zero divisors in this case exactly as in case I above. So there are

$$2p^2(\sum_{r=2,r\ even}^{p-1} {}^{p+1}C_r)$$

S-weak zero divisors in case II.
Hence the total number of S-weak zero divisors in $Z_2L_n(m)$ is

$$2p(\sum_{r=2,r\ even}^{p^2-1} {}^{p^2+1}C_r) + 2p^2(\sum_{r=2,r\ even}^{p-1} {}^{p+1}C_r).$$

Hence the claim.

Theorem 3.4: Let $L_n(m)$ be a loop of order $n+1$ (n an odd number, $n > 3$) with $n = pq$, p, q are odd primes. Z_2 be the prime field of characteristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$2[p(\sum_{r=2,r\ even}^{q-1} {}^{q+1}C_r) + q(\sum_{r=2,r\ even}^{p-1} {}^{p+1}C_r)]$$

S-weak zero divisors.

Proof: We will enumerate all the S-weak zero divisors in the following way:

Case I: As $p|pq$, $L_n(m)$ has p subloops H_j each of order $q+1$. Proceeding exactly same way as in Theorem 3.3, we will get $2p(\sum_{r=2,r\ even}^{q-1} {}^{q+1}C_r)$ S weak zero divisors in case I.

Case II: Again as $q|pq$, $L_n(m)$ has q proper subloops H_j each of order $p+1$. So as above we will get $2q(\sum_{r=2,r\ even}^{p-1} {}^{p+1}C_r)$ S-weak zero divisors in case II.
Hence adding all the S-zero divisors in case I and case II, we get

$$2[p(\sum_{r=2,r\ even}^{q-1} {}^{q+1}C_r) + q(\sum_{r=2,r\ even}^{p-1} {}^{p+1}C_r)]$$

S-weak zero divisors in $Z_2L_n(m)$.
Hence the claim.

Now we prove for the loop ring $ZL_n(m)$ where $n = p^2$ *or* p^3 *or* pq, $(p, q$ are odd primes), $ZL_n(m)$ has infinitely many S-weak zero divisors.

Theorem 3.5: Let $ZL_n(m)$ be the loop ring of the loop $L_n(m)$ over Z, where $n = p^2$ *or* p^3 *or* pq $(p, q$ are odd primes). Then $ZL_n(m)$ has infinitely many S-weak zero divisors.

Proof: **Let $L_n(m)$ be a loop such that $n = p^2$. $L_n(m)$ has p subloops (say H_j) each of order $p + 1$. Now the loop ring $ZL_n(m)$ has the following types of S-weak zero divisors:**

$$X = a - bh_1 + bh_2 - ah_3, \text{ and } Y = \sum_{i=1}^{n+1} g_i$$

where $a, b \in Z, g_i \in L_n(m)$ and $h_1, h_2, h_3 \in H_j$ are such that

$$XY = 0.$$

Again

$$X \sum_{i=1}^{p+1} h_i = 0, \quad h_i \in H_j$$

also

$$(1 - g_t)Y = 0, \quad g_t(\neq h_t) \in H_j$$

clearly

$$(1 - g_t)(\sum_{i=1}^{p+1} h_i) = 0.$$

So X, Y are S-weak zero divisors in $ZL_n(m)$. Now we see there are infinitely many S-weak zero divisors of this type for a and b can take infinite number of values in Z.
For $n = p^2$ *or* p^3 *or* pq, we can prove the results in a similar way.

Hence the claim.

§ 4 Conclusions:

In this paper we find the exact number of S-zero divisors and S-weak zero divisors for the loop rings $Z_2L_n(m)$ in case of the special type of loops $L_n(m) \in L_n$ over Z_2, when $n = p^2$ or p^3 or pq $(p, q$ are odd primes). We also

prove for the loop ring $ZL_n(m)$ has infinite number of S-zero divisors and S-weak zero divisors. We obtain conditions for any loop L to have S-zero divisors and S-weak zero divisors. We suggest it would be possible to enumerate in the similar way the number of S-zero divisors and S-weak zero divisors for the loop ring $Z_2L_n(m)$ when $n = p^s,\ s > 3; p$ a prime or when $n = p_1p_2...p_t$ where $p_1, p_2, ...p_t$ are odd primes. However we find it difficult when we take Z_p instead of Z_2, where p can be an odd prime or a composite number such that $(p, n+1) = 1$ or $(p, n+1) = p$ and n is of the form $n = p_1^{t_1}p_2^{t_2}...p_r^{t_r},\ t_i > 1, n$ is odd and $p_1, p_2, ..., p_r$ are odd primes.

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